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The Flag-Transitive C_3-Geometries of Finite Order

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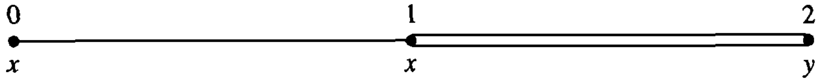
Abstract. It is shown that a flag-transitive C_3-geometry of finite order (x, y) with $x \geq 2$ is either a finite building of type C_3 (and hence the classical polar space for a 6-dimensional symplectic space, a 6-dimensional orthogonal space of plus type, a 6- or 7-dimensional hermitian space, a 7-dimensional orthogonal space, or an 8-dimensional orthogonal space of minus type) or the sporadic A_7-geometry with 7 points.

Keywords: incidence geometry, C_3-geometry, flag-transitivity, generalized quadrangle

1. Introduction

A *C_3-geometry $\mathcal{G}$ of finite order* (x, y) is a residually connected incidence geometry on $I = \{0, 1, 2\}$, in which the residue of an element of type i is isomorphic to a generalized quadrangle of finite order (x, y), to a generalized digon, or to a projective plane of order x, respectively for $i = 0, 1$, or 2.



The remarkable theorem of Tits [18] says that a residually connected geometry with generalized polygons as rank 2 residues is covered by a building, if its residues of type C_3 or H_3 are covered by buildings. Thus in attempting to classify a class of diagram geometries with C_3- or H_3-residues, we immediately meet problems over which we have no control. It would be nice if we had the classification of H_3- and C_3-geometries. However, it seems hopeless to classify them in general, because we can construct locally infinite H_3- and C_3-geometries by some kind of free construction [18] 1.6. Thus locally finite C_3- and H_3-geometries may be reasonable objects to consider. As for locally finite H_3-geometries, we can show that they are the icosahedron and the halved icosahedron (see [14] 13.2), since they are thin by Feit-Higman theorem. Locally finite non-building C_3-geometries are much more difficult to classify, because there is a finite thick non-building C_3-geometry (called the sporadic A_7-geometry) together with non-thick finite C_3-geometries. Hence the classification of localy finite C_3-geometries can be thought of as one of the central problems in diagram geometry.

It has been conjectured that a C_3-geometry of finite order (x, y) with $x \geq 2$ is either a finite building of type C_3 or the sporadic A_7-geometry, if it admits a flag-transitive automorphism group. M. Aschbacher [1] proved this conjecture assuming that the residues of planes (elements of type 2) and points (elements of type 0) are desarguesian projective planes and classical generalized quadrangles (associated with symplectic, hermitian or orthogonal forms), respectively. A. Pasini and G. Lunardon investigated the general case and derived several important results [11, 12, 13], which are summarized in [10]. In particular, the conjecture was proved assuming the residues of planes are desarguesian [12], and the flag-transitive flat C_3-geometries were classified [9]. Recently, the conjecture was proved in the case where the 2-order y is even [20].

In this paper, the conjecture is finally established.

Theorem *A flag-transitive C_3-geometry of finite order (x, y) with $x \geq 2$ is either a finite building of type C_3 or the sporadic A_7-geometry.*

Together with the works by Tits, Meixner, Brouwer and Cohen, and Pasini, this theorem completes the last case remaining open on the question of locally finite thick flag-transitive geometries belonging to Coxeter diagrams of rank at least 3 (see the discussion in the introduction and Theorem 5 in [10]). Note that the locally finiteness implies the finiteness for these geometries (see the last remark in [14] Section 14 as for those of type E_6, E_7, E_8 and F_4).

Corollary *A locally finite thick flag-transitive geometry belonging to a Coxeter diagram of rank at least* 3 *(that is, one of A_n, C_n, D_n for $n \geq 3$, F_4, H_3, H_4, E_6, E_7 and E_8) is either a finite building or the sporadic A_7-geometry.*

The main ingredient of the proof of Theorem is the classification of finite simple groups. At the present stage, this seems natural for the following reason.

Given a flag-transitive C_3-geometry of finite order, the residue of a plane is a flag-transitive finite projective plane. Since the residues of planes are desarguesian for the buildings and the A_7-geometry, in order to establish the conjecture, we have to eliminate any flag-transitive C_3-geometry of finite order with non-desarguesian flag-transitive projective planes as residues of planes. Thus we need some results on non-desarguesian flag-transitive projective planes. The best result available so far is the theorem by Kantor [7] (see Theorem 2.2.1), saying that a flag-transitive non-desarguesian finite projective plane of order x admits an action of a Frobenius group F_p^{x+1} with $p = x^2 + x + 1$ a prime. The proof of this result depends on the classification of finite simple groups.

In fact, it is conjectured that any flag-transitive finite projective plane is desarguesian. If this conjecture is solved affirmatively, any flag-transitive C_3-geometry has desarguesian planes as residues of planes, and hence the theorem above follows from the above-mentioned result of Pasini [12] (see also Theorem 3.7.3). However, at the present stage, it seems unlikely to obtain the complete solution for the conjecture on flag-transitive projective planes. We can eliminate flag-transitive non-desarguesian finite projective planes of 'small' order x, using an interpretation of the conjecture into a problem of elementary number theory by Feit [5] (see Proposition 2.2.2). Unfortunately, this interpretation is not only difficult to accomplish in general, but also depends on the above result of Kantor.

Hence so far we cannot get control over the residues of planes in a flag-transitive C_3-geometry without relying on the classification of finite simple groups.

On the other hand, the present proof does not require so much of knowledge and information on C_3-geometries. Moreover, required facts can be proved in a very elementary way, although they are scattered across many books and papers. Thus I decided to write the paper as self-contained as possible, including reproductions of some known results. Except for the classification of finite simple groups, the paper relies only on [18, 17, 7, 5] (this is quite elementary), and some part of [9] (which is not so difficult to read through: see also the sketch given in [10] 5.3). Other facts used in the paper are either elementary or can be found in some textbooks (e.g. [15, 16] and [3]).

In particular, I did not use results, whose proofs essentially require the representation theory of the Hecke algebra for a geometry of type C_3 developed by Ott and Liebler. In [20] (Lemma 1 (5)(7)) we use results obtained from the representation theory and a detailed analysis on the substructure fixed by an involution. However, this paper does not require that. To make this point clear, in Section 3, I include the representation free proofs for what I need, and also make detailed comments to some arguments in [11] (see 4.7) and [9] (see 4.1.2).

The proof goes as follows. First, the number of maximal flags can be expressed in terms of the orders x, y and an important constant α (Lemma 3.6.2). Assume that the geometry in question is not a building nor the A_7-geometry. We will derive a contradiction.

This assumption allows us to introduce an equivalence relation $\approx$ on the points (the elements of type 0), and in fact the points form one $\approx$-class. Then we can establish the faithfullness of the action of the stabilizer of a point on the residue of the point (Lemma 4.2). It is worth mentioning that at this very early stage we need the assumption of flag-transitivity (compare the comments to Theorem A in [7] p. 15 and those to [9] given in [10] p. 27, Remark 1, 2).

Our assumption also implies that the residue of a plane is non-desarguesian or of order $x = 8$ with the aid of the result of Kantor [7] and Lunardon-Pasini [9] (Lemma 4.1.1). Using elementary arguments on generalized quadrangles, we can then bound the order of the stabilizer of a maximal flag, and so the order of the whole flag-transitive group A in terms of the prime p in Kantor's result (Lemma 4.4). This is the crucial point of the paper.

The remaining part of the proof mainly requires group theory. We can show that there is a unique component L of the flag-transitive automorphism group A (Lemma 4.8) with the aid of the classification of finite simple groups and small remarks on the substructure fixed by an involution (Lemma 4.5.1). In particular, the non-solvability of A first proved in [20] is also established at this stage.

Now the simple factor S of L satisfies rather restricted conditions on the order of S (see the paragraphs before Lemma 5.1), and using the classification of finite simple groups, in Section 5 we can eliminate each possibility for S. Straightforward estimation for the orders of explicit simple groups in terms of the prime p plays a central role for the elimination, which is of somewhat similar flavor to Part II of [7]. Here an elementary result concerning an action of a central extension of a Frobenius group on a vector space is very effective (see 2.4). The case $x = 8$ often requires some special consideration.

The paper is organized as follows. In Section 2, I collect the standard terminologies on geometry and fundamental results on projective planes, generalized quadrangles and an action of a group. Lemma 2.3.4 seems to be new, and will be used to establish Lemma 4.4.

Section 3 is a summary of the results on C_3-geometries which will be used in the paper together with their proofs. Section 4 and 5 are the main parts of the paper, where the conjecture will be completely proved.

2. Review and preliminary results

In this section, we review some terminology on geometries (specifically generalized quadrangles) and groups, and then state some lemmas which turn out to be useful in Sections 4, 5.

2.1. *Geometries*

In this paragraph, we briefly review some fundamental terminologies in incidence geometry. We basically follow those in [2].

An *incidence geometry* over an ordered set $I = \{0, \dots, r-1\}$ $(0 < 1 < \dots < r-1)$ is a sequence $(\mathcal{G}_0, \dots, \mathcal{G}_{r-1})$ of r mutually disjoint non-empty sets $\mathcal{G}_i$ $(i \in I)$ arranged in the order given in I together with a reflexive and symmetric relation $*$ on $\mathcal{G}_0 \cup \dots \cup \mathcal{G}_{r-1}$ such that for each $i \in I$ we have $x * y$ for $x, y \in \mathcal{G}_i$ if and only if $x = y$. We usually write $\mathcal{G} = (\mathcal{G}_0, \dots, \mathcal{G}_{r-1}; *)$ or simply use $\mathcal{G}$ to denote such an object. The cardinality r of I is the *rank* of the geometry $\mathcal{G}$.

The elements of $\mathcal{G}_0 \cup \dots \cup \mathcal{G}_{r-1}$ are referred to as *elements* (or *varieties*) of $\mathcal{G}$. Two elements x, y of $\mathcal{G}$ are called *incident* if $x * y$. A *flag* is a set of mutually incident elements of $\mathcal{G}$. Two flags F and F' are called *incident* if every element of F is incident to every element of F'. The type of a flag F (written as type(F)) is the set of indices $i \in I$ with $\mathcal{G}_i \cap F \neq \emptyset$.

The *incidence graph* of a geometry $\mathcal{G}$ is the graph (V, E) with the set of elements of $\mathcal{G}$ as V and $\{x, y\} \in E$ whenever $x * y$ and $x \neq y$. A geometry $\mathcal{G}$ is *connected* if its incidence graph is connected. The *collinearity graph* of $\mathcal{G}$ is a graph with the set $\mathcal{G}_0$ as the set of vertices such that two elements $x, y \in \mathcal{G}_0$ form an edge if and only if $x \neq y$ and there is an element $l \in \mathcal{G}_1$ incident to both x and y.

Two geometries $\mathcal{G}$ and $\mathcal{H}$ over the same ordered set I are called *isomorphic* if there is a bijective map f from $\cup_{i \in I} \mathcal{G}_i$ to $\cup_{i \in I} \mathcal{H}_{i \in I}$ sending $\mathcal{G}_i$ to $\mathcal{H}_i$ for each $i \in I$ such that two elements x, y of $\mathcal{G}$ are incident in $\mathcal{G}$ iff $f(x)$ and $f(y)$ are incident in $\mathcal{H}$.

For a flag F and $j \in J := I - \text{type}(I)$, we write $\mathcal{G}_j(F) := \{y \in \mathcal{G}_j \mid x * y (\forall x \in F)\}$. The sequence $(\mathcal{G}_{j_0}(F), \dots, \mathcal{G}_{j_m}(F))$ (arranged in the order on J inherited from I) together with the restriction of $*$ as the incidence relation forms a geometry over the set J, which is called the *residue of F in $\mathcal{G}$* and is denoted by $\text{Res}_{\mathcal{G}}(F)$ or simply by $\text{Res}(F)$. If $F = \{x\}$, we write $\text{Res}(F)$ by $\text{Res}(x)$. A connected geometry $\mathcal{G}$ is called *residually connected* if $|\mathcal{G}_i(F)| \geq 2$ for any $i \in I$ and for any flag F of type $I - \{i\}$, and if $\text{Res}(F)$ is connected for every flag F of $\mathcal{G}$ with $|I - \text{type}(F)| \geq 2$.

If there exists s_i (which is a natural number or the symbol ∞) depending only on $i \in I$ such that there are exactly exactly $s_i + 1$ maximal flags containing each flag of type $I - \{i\}$, s_i is called the i-th *order* of a geometry $\mathcal{G}$.

The isomorphisms from a geometry $\mathcal{G}$ to itself form a group with respect to the composition of maps, which is denoted by $\text{Aut}(\mathcal{G})$ and called the (special) *automorphism group* of

$\mathcal{G}$. If there is a homomorphism ρ from a group G to Aut($\mathcal{G}$), we say that G *acts on* $\mathcal{G}$ (or $\mathcal{G}$ *admits* G) and the kernel of ρ is called the *kernel* of the action. If a group G acts on $\mathcal{G}$, we denote by G_X the *stabilizer* of a flag X, that is, the subgroup of G of elements stabilizing X globally. Since isomorphisms of $\mathcal{G}$ preserve $\mathcal{G}_i$ for each $i \in I$, G_X acts on the geometry Res(X). The kernel of this action is denoted by K_X. That is, K_X is the normal subgroup of G_X fixing each variety contained in X, and hence G_X/K_X is isomorphic to a subgroup of Aut(Res(X)).

A group G is called *flag-transitive* on $\mathcal{G}$ if G acts transitively on the set of maximal flags. A geometry $\mathcal{G}$ is *flag-transitive* if it admits a flag-transitive group. If G is flag-transitive then the stabilizer G_X is flag-transitive on Res(X) and so G_X/K_X is a flag-transitive subgroup of Aut(Res(X)). Furthermore, if $\mathcal{G}$ is flag-transitive, the i-order of $\mathcal{G}$ can be defined for any $i \in I$.

2.2. *Generalized polygons and projective planes*

Let n, s, t be natural numbers with $n \geq 2$. A *generalized n-gon* is a connected incidence geometry $(\mathcal{P}, \mathcal{L}; *)$ of rank 2 (see 2.1) whose incidence graph is of diameter n and of girth $2n$. If the 0- and 1-orders of $\mathcal{G}$ can be defined, and they are s and t respectively, we refer to (s, t) as the *order* of $\mathcal{G}$.

The incidence graph of a generalized 2-gon (called *digon*) of order (s, t) is isomorphic to the complete bipartite graph with bipartite parts of sizes $s + 1$ and $t + 1$. It is easy to see that a connected incidence geometry $(\mathcal{P}, \mathcal{L}; *)$ of rank 2 is a generalized 3-gon if and only if it is a projective geometry (that is, for two distinct elements x, y of $\mathcal{P}$ (resp. $\mathcal{L}$) there is a unique element of $\mathcal{L}$ (resp. $\mathcal{P}$) incident to both x and y).

In the generalized 3-gon, or equivalently, a projective plane Π, we have $s = t$, which is simply called the *order* of Π. If $s = 1$, the elements of Π are just the vertices and the edges of an ordinary triangle.

The following result due to Kantor [7] (Theorem A and the proof of Lemma 6.5) on flag-transitive finite projective planes is based not only on the classification of finite simple groups but also on the classification of their primitive permutation representations of odd degrees.

Theorem 2.2.1 [7] *If* $\Pi = (\mathcal{P}, \mathcal{L}; *)$ *is a projective plane of finite order* x $(x > 1)$, *admitting a flag-transitive automorphism group* F, *then one of the following occurs.*

(1) Π *is desarguesian and* $F \geq PSL(3, x)$.

(2) Π *is non-desarguesian or desarguesian of order* $x = 2$ *or* 8. *The group* F *is a Frobenius group* $F^{(x+1)}_{(x^2+x+1)}$ *with the cyclic group of prime order* $p = x^2 + x + 1$ *as the kernel and a cyclic group of order* $x + 1$ *as a complement. The group* F *acts primitively both on* $\mathcal{P}$ *and* $\mathcal{L}$.

Note that in any case the group F above acts primitively both on $\mathcal{P}$ and $\mathcal{L}$.

It is conjectured that the Case (2) does not occur except for $x = 2$ and 8, but it seems difficult to prove this. In fact, many arithmetic properties for the prime $p = x^2 + x + 1$ are known, which are unlikely to hold. By an elementary argument, recently Feit [5] verified the following:

Proposition 2.2.2 [5] *Let* Π *be a flag-transitive non-desarguesian projective plane of finite order* x. *Then* x *is a multiple of* 8 *with* $x > 14,400,008$ *and* $p = x^2 + x + 1$ *is a prime with* $p > 207,360,244,800,073$.

2.3. *Generalized quadrangles*

A generalized 4-gon is also referred to as a *generalized quadrangle*, which will be abbreviated to GQ in this paper. It is easy to see that a connected incidence geometry $(\mathcal{P}, \mathcal{L}; *)$ of rank 2 is a GQ of order (s, t) if and only if the following conditions are satisfied, where we call elements of $\mathcal{P}$ and $\mathcal{L}$ *points* and *lines* respectively:

(1) Each line is incident to $s + 1$ lines and two distinct lines are incident to at most one point.
(2) Each point is incident to $t + 1$ lines and two distinct points are incident to at most one line.
(3) If P is a point and L is a line not incident to P, then there is a unique point Q incident to L and collinear with P.

Recall that two points are called collinear if they are incident to a line in common. Dually two lines are called *concurrent* if they are incident to a point in common.

Lemma 2.3.1 *For a GQ* $\mathcal{S} = (\mathcal{P}, \mathcal{L}; *)$ *of order* (s, t), *the following hold.*
(1) ([15] 1.2.1.) $|\mathcal{P}| = (s + 1)(st + 1)$ *and* $|\mathcal{L}| = (t + 1)(st + 1)$.
(2) ([15] 1.2.2.) $s + t$ *divides* $st(s + 1)(t + 1)$.
(3) ([15] 1.2.3, *The inequality of D.G. Higman.*) *If* $s > 1$, *then* $t \leq s^2$.
(4) ([15] 1.4.1.) *Let* $A = \{a_1, \ldots, a_m\} (m \geq 2)$ *and* $B = \{b_1, \ldots, b_n\}$ $(n \geq 2)$ *be disjoint sets of pairwise non-collinear points of* $\mathcal{S}$. *If* $s > 1$ *and each points of* A *is collinear with all the points of* B, *then* $(m - 1)(n - 1) \leq s^2$.

An *ovoid* of a GQ $\mathcal{S} = (\mathcal{P}, \mathcal{L}; *)$ is a subset $\mathcal{O}$ of $\mathcal{P}$ such that any line of $\mathcal{L}$ is incident to a unique point of $\mathcal{O}$. Any two distinct points of an ovoid are not collinear. We have $|\mathcal{O}| = st + 1$ by an elementary counting argument.

The GQ $\mathcal{S}' = (\mathcal{P}', \mathcal{L}'; *')$ of order (s', t') is called a *subquagrangle* of a GQ $\mathcal{S} = (\mathcal{P}, \mathcal{L}; *)$, if $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{L}' \subseteq \mathcal{L}$, and if $*'$ is the restriction of $*$ on the elements of $\mathcal{S}'$.

Lemma 2.3.2 *Let* $\mathcal{S} = (\mathcal{P}, \mathcal{L}; *)$ *be a GQ of order* (s, t), *having a subquadrangle* $\mathcal{S}' = (\mathcal{P}', \mathcal{L}'; *')$ *of order* (s, t'). *Assume that* $s > 1$ *and* $t > t'$. *Then the following hold.*
(1) ([15] 2.2.1) *For each point* Q *of* $\mathcal{P} - \mathcal{P}'$, *there are exactly* $st' + 1$ *points of* $\mathcal{P}'$ *collinear with* Q *which form an ovoid of* $\mathcal{S}'$.
(2) ([15] 2.2.2(vi)) *If* $\mathcal{S}'$ *has a subquadrangle* $\mathcal{S}''$ *of order* (s, t'') *with* $t'' < t'$, *then* $t'' = 1$, $t' = s$ *and* $t = s^2$.

(In Lemma 2.3.2(1) above, note that every point of $\mathcal{P} - \mathcal{P}'$ is an external point in the sense of [15] 2.2, since $\mathcal{S}'$ has order (s, t').)

In Section 4, we examine the substructure $\mathcal{S}^Z = (\mathcal{P}^Z, \mathcal{L}^Z; *')$ of a GQ $\mathcal{S} = (\mathcal{P}, \mathcal{L}; *)$ of order (s, t) stabilized by an automorphism group Z of $\mathcal{S}$. Here $\mathcal{P}^Z$ and $\mathcal{L}^Z$ are the sets of points and lines fixed by all elements of Z respectively, and $*'$ is the restriction of $*$ on $\mathcal{P}^Z \cup \mathcal{L}^Z$. The possible shapes of $\mathcal{S}^Z$ can be determined by the same argument as in [15] 2.4.1, where a *grid* means a geometry $(\mathcal{Q}, \mathcal{B}; *)$ of rank 2 with $\mathcal{Q} = \{x_{ij} \mid i = 0, \dots, s_1, j = 0, \dots, s_2\}$, $\mathcal{B} = \{L_i, M_j \mid i = 0, \dots, s_1, j = 0, \dots, s_2\}$ for some natural numbers s_1, s_2 with the incidence $*$ defined by $x_{ij} * L_k$ iff $i = k$ and $x_{ij} * M_k$ iff $j = k$, and a *dual grid* is a geometry $(\mathcal{G}_0, \mathcal{G}_1; *)$ such that $(\mathcal{G}_1, \mathcal{G}_0; *)$ is a grid.

Lemma 2.3.3 *The substructure $\mathcal{S}^Z = (\mathcal{P}^Z, \mathcal{L}^Z; *')$ of a GQ $\mathcal{S} = (\mathcal{P}, \mathcal{L}; *)$ of order (s, t) stabilized by an automorphism group Z of $\mathcal{S}$ is one of the following shapes:*

(1) *$\mathcal{L}^Z = \emptyset$ and any two distinct points of $\mathcal{P}^Z$ are not collinear.*

(1′) *$\mathcal{P}^Z = \emptyset$ and any two distinct lines of $\mathcal{L}^Z$ are not concurrent.*

(2) *$\mathcal{P}^Z$ contains a point P such that P is collinear with Q for every point $Q \in \mathcal{P}^Z$ and every line of $\mathcal{L}^Z$ is incident to P.*

(2′) *$\mathcal{L}^Z$ contains a line L such that L is concurrent with M for every line $M \in \mathcal{L}^Z$ and every point of $\mathcal{P}^Z$ is incident to L.*

(3) *$\mathcal{S}^Z = (\mathcal{P}^Z, \mathcal{L}^Z; *')$ is a grid.*

(3′) *$\mathcal{S}^Z = (\mathcal{P}^Z, \mathcal{L}^Z; *')$ is a dual grid.*

(4) *$\mathcal{S}^Z = (\mathcal{P}^Z, \mathcal{L}^Z; *')$ is a subquagrangle of $\mathcal{S}$ of order (s', t') for some $s' \geq 2$ and $t' \geq 2$.*

Combining the above results, we obtain the following new result on GQ's, which is crucial to establish the key lemma, Lemma 4.4, in this paper.

Lemma 2.3.4 *Assume that a group X acts on a GQ $\mathcal{S} = (\mathcal{P}, \mathcal{L}; *)$ of order (s, t) with $s > 2$ and $t > 1$, satisfying the following conditions.*

(i) *If an element $g \in X$ fixes a line $L \in \mathcal{L}$, all the points on L are fixed by g.*

(ii) *There are two non-concurrent lines of $\mathcal{L}$ fixed by X.*

Then $|X/K| < t$, where K is the kernel of the action of X on $\mathcal{S}$.

Proof: By the conditions (i), (ii), the substructure $\mathcal{S}^X = (\mathcal{P}^X, \mathcal{L}^X)$ of a GQ $\mathcal{S}$ fixed by X contains a pair of non-concurrent lines together with all the points on them. Then it follows from Lemma 2.3.3 that $\mathcal{S}^X$ is a subquadrangle of order (s, t'') for some $1 \leq t'' \leq t$. If $t'' = t$, X acts trivially on $\mathcal{S}$, and so $X = K$ and the claim follows in this case. Thus we may assume that $t'' < t$.

Then there is a point Q in $\mathcal{P} - \mathcal{P}^X$. Let $Y := X_Q$, the stabilizer of the point Q in X. Let $\mathcal{A}$ be the X-orbit on $\mathcal{P} - \mathcal{P}^X$ containing Q, and let $\mathcal{B}$ be the set of points of $\mathcal{P}^X$ collinear with Q. By Lemma 2.3.2(1), $\mathcal{B}$ is an ovoid of $\mathcal{S}^X$, and hence $\mathcal{B}$ consists of $st'' + 1$ pairwise non-collinear points. As $s > 1$, $|\mathcal{B}| > 1$. As X does not fix a point of $\mathcal{A}$, $|\mathcal{A}| > 1$. Since X fixes every point of $\mathcal{B}$ while acts transitively on $\mathcal{A}$, each point of $\mathcal{B}$ is collinear with all the points of $\mathcal{A}$. Suppose there are two distinct points S, $T = S^g$ $(g \in X)$ of $\mathcal{A}$ which are collinear, and let M be the unique line through S and T. Since S and T are two distinct points on the line M incident to a point P of $\mathcal{B}$, the line M goes through P by the definition of a GQ. Since M is incident to $P = P^g$, S

and $S^g = T$, M is the unique line through P and S, and also the unique line through $P = P^g$ and $S^g = T$. Thus the line M is fixed by g. Then by Condition (i) the point S is fixed by g, which contradicts the assumption that $S \neq T$. Hence the points of $\mathcal{A}$ are pairwise non-collinear. Since the assumptions of Lemma 2.3.1(4) are satisfied, we have $(|\mathcal{A}| - 1) \leq s^2/(st'' + 1 - 1) = s/t''$.

We can obtain another bound of $|\mathcal{A}|$ in terms of t as follows. Note that in the above we saw that each line through a point P of $\mathcal{P}^X$ is incident to at most one point of $\mathcal{A}$. Since the points on a line of $\mathcal{L}^X$ are fixed by X, $t'' + 1$ lines of $\mathcal{L}^X$ through P are not incident to a point of $\mathcal{A}$. Thus $|\mathcal{A}| \leq t - t'' < t$.

The substructure $\mathcal{S}^Y$ of $\mathcal{S}$ fixed by Y contains $\mathcal{P}^X \cup \{Q\}$ and $\mathcal{L}^X$. Thus it follows from Lemma 2.3.3 that $\mathcal{S}^Y$ is a subquadrangle of $\mathcal{S}$ of order (s, t') properly containing the subquadrangle $\mathcal{S}^X$. If $\mathcal{S}^Y = \mathcal{S}$, then $Y \leq K$ and $|X/K| \leq |X : Y| = |\mathcal{A}|$. Since $|\mathcal{A}| < t$, as we saw above, the claim follows in this case.

Hence we may assume that $\mathcal{S}^Y$ is properly contained in $\mathcal{S}$. Then it follows from Lemma 2.3.2(2) that $t'' = 1$, $t' = s$ and $t = s^2$. Pick a point R of $\mathcal{P} - \mathcal{P}^Y$. For the stabilizer $Z = Y_R$ of R in Y, the substructure $\mathcal{S}^Z$ fixed by Z contains $\mathcal{P}^Y \cup \{R\}$ and $\mathcal{L}^Y$, and hence $\mathcal{S}^Z$ is a subquadrangle of $\mathcal{S}$ properly containing $\mathcal{S}^Y$. Applying Lemma 2.3.2(2) to the sequence $(\mathcal{S}^Y, \mathcal{S}^Z, \mathcal{S})$ of GQs, we have $\mathcal{S}^Z = \mathcal{S}$, as $s > 1$. Hence $Z \leq K$.

We will bound $|X : Y| = |\mathcal{A}|$ and $|Y : Z| = |\mathcal{A}'|$ in terms of s, where $\mathcal{A}'$ is the Y-orbit on $\mathcal{P} - \mathcal{P}^Y$ containing R. We have already obtained the bound $|\mathcal{A}| \leq (s/t'') + 1 = s + 1$ in the above paragraph. Repeating exactly the same arguments in that paragraph for $\mathcal{A}'$ and the set $\mathcal{B}'$ of points of $\mathcal{S}^Y$ incident to R (and replacing X by Y), we conclude that $\mathcal{A}'$ and $\mathcal{B}'$ satisfy the assumptions of Lemma 2.3.1(4) and that $\mathcal{B}'$ is an ovoid of $\mathcal{S}^Y$ consisting of $(s^2 + 1)$ pairwise non-collinear points. Then we have $(|\mathcal{A}'| - 1) \leq s^2/(s^2 + 1 - 1) = 1$ and so $|\mathcal{A}'| = 2$. Hence $|X/K| \leq |X : Y||Y : Z| = |\mathcal{A}||\mathcal{A}'| \leq 2(s + 1)$ in the remaining case. As $s > 2$ by our assumption, $2(s + 1) < s^2 = t$ and the claim follows. □

2.4. *Groups*

In this paper, the notation in [4] will be basically used to denote particular simple groups. For the definitions and the standard properties of coprime action of a group on another group, the Frobenius groups, the components, and $E(G)$ and $F(G)$ of a finite group G, see [16]. An elementary lemma [6] 3.11, p. 166 on linear groups turns out to be useful in Sections 4, 5, which I include here for the convenience of the readers.

Lemma 2.4.1 *Let F be a group acting faithfully on an n-dimensional vector space over a finite field $GF(q)$. Assume that F has a cyclic normal subgroup P such that, as a $GF(q)P$-module, V is the direct sum of s mutually isomorphic irreducible $GF(q)P$-modules of dimension t. Then, identifying V as an s-dimensional vector space W over $GF(q^t)$, the permutation group F on V is equivalent to a subgroup of the group $\Gamma L(s, q^t)$ of semilinear transformations on W, where $\Gamma L(s, q^t)$ is a split extension of the linear group $GL(s, q^t)$ by the group of field automorphism isomorphic to the cyclic group $Gal(GF(q^t)/GF(q))$ of order t. Furthermore, $C_F(P)$ corresponds to a subgroup of the linear group $GL(s, q^t)$ on W.*

In particular, if F and P satisfy the assumption of the lemma above, $F/C_F(P)$ is isomorphic to a subgroup of the cyclic group $\mathrm{Gal}(GF(q^t)/GF(q))$ of order t. In Section 4, we frequently apply this lemma in the following form.

Lemma 2.4.2 *Let B be a finite group containing a normal subgroup C such that B/C is a Frobenius group with the kernel of prime order p and a cyclic complement of order m. Assume that B acts on an r-group R for a prime r distinct from p. Assume furthermore that there is a Sylow p-subgroup P of B of order p such that PC/C is the Frobenius kernel of B/C and $[P, R] \neq 1$. Then $|R| \geq r^m$.*

Proof: Since B/C is isomorphic to the Frobenius group F_p^m, PC is normal in B and hence $B = N_B(P)C$ by the Frattini argument. Then $B/PC \cong N_B(P)/PC_C(P)$ is a cyclic group of order m, and there is an element $w \in N_B(P)$ such that $N_B(P) = \langle w\rangle PC_C(P)$ and $z := w^m \in C_C(P)$. We set $F := P\langle w\rangle$. Then $Z(F) = \langle z\rangle$ and $F/Z(F) \cong F_p^m$.

The group P acts coprimely and non-trivially on R by the assumption. The kernel $C_F(R)$ of the action of F on R is a normal subgroup of F not containing P. As $F/Z(F) \cong F_p^m$, we have $C_F(R) \leq Z(F)$. Let K be the full inverse image of $O_r(F/C_F(R))$ in F. As K is a normal subgroup of F not containing P, $K \leq Z(F)$. Let $R = R_0 \supset R_1 \supset \cdots \supset R_{l-1} \supset R_l = 1$ be the chief F-series of R. Each chief factor R_{i-1}/R_i is an elementary abelian r-group, affording an irreducible representation of F over $GF(r)$. We can easily verify that K coincides with the kernel of the action of F on these chief factors: $K = \cap_{i=1}^{l} C_F(R_{i-1}/R_i)$. As $K \leq Z(F)$, P is not contained in K, and hence there is an F-irreducible module $V := R_{i-1}/R_i$ over $GF(r)$ with $P \not\subseteq C_F(V)$.

The kernel $C_F(V)$ of the action of F on V is a normal subgroup of F not containing P, and so $C_F(V) \subseteq Z(F)$. The group $\bar{F} := F/C_F(V)$ acts faithfully and irreducibly on the vector space V and $\bar{P} = PC_F(V)/C_F(V)$ is a cyclic normal group of $\bar{F}$ of order p. By the Clifford theorem, as a $\bar{P}$-module, V is the direct sum of irreducible $\bar{P}$-modules $V_1, \ldots, V_s$ on which $\bar{F}$ acts transitively. We set $n := \dim V$ and $k := n/s$. By Lemma 2.4.1, $\bar{F}$ can be identified with a group of semilinear transformations on V recognized as an s-dimensional space over $GF(r^k)$, in which the group of linear transformations corresponds to $C_{\bar{F}}(\bar{P})$. Hence $\bar{F}/C_{\bar{F}}(\bar{P})$ is isomorphic to a subgroup of the cyclic group $\mathrm{Gal}(GF(r^k)/GF(r)) \cong Z_k$. Since $F_p^m \cong F/Z(F)$, m divides $|\bar{F}/C_{\bar{F}}(\bar{P})|$ and so k. Thus we have $m \leq k \leq n$ and $r^m \leq r^n = |V| \leq |R|$. □

3. Properties of C_3-geometries

In this section, I give several known facts about C_3-geometries with some sketch of proofs, in order to make this paper as self-contained as possible. Especially, I quote some results from [12] with explicit proofs along with the original one, because they are very much important to start the proof of the main theorem. Here I thank Antonio Pasini for allowing me to do so. Note that the results in 3.5–3.7 do not require the flag-transitivity.

3.1. *Definition*

A residually connected geometry $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2; *)$ over $I = \{0, 1, 2\}$ is called a C_3-*geometry of order* (x, y) if the following hold:

(1) For each element $a \in \mathcal{G}_0$, the residue $\mathrm{Res}_{\mathcal{G}}(a)$ of a is a GQ of order (x, y),
(2) For each element $l \in \mathcal{G}_1$, the residue $\mathrm{Res}_{\mathcal{G}}(l)$ of l is a generalized digon, and
(3) For each element $u \in \mathcal{G}_2$, the residue $\mathrm{Res}_{\mathcal{G}}(u)$ of u is a projective plane of order x.

3.2. *Notation*

In the remainder of this paper, $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2; *)$ will always mean a C_3-geometry of finite order (x, y) with $x \geq 2$. The letters x and y are always used to denote the 0- and 2-order respectively. Furthermore, the letter A is always used to denote a flag-transitive automorphism group of $\mathcal{G}$, if $\mathcal{G}$ is flag-transitive.

Elements of $\mathcal{G}_i$ are called *points*, *lines* and *planes* respectively for $i = 0, 1, 2$. We usually use the letters a, l and u to denote a point, a line and a plane in a typical maximal flag. For a flag F and a type i not contained in the type of F, we use $\mathcal{G}_i(F)$ to denote the set of elements of type i incident to all the elements of F.

Two distinct points a, b are called *collinear* and denoted by $a \sim b$ if there is a line incident to both a and b. In general, there are several distinct lines incident to a and b. The number of lines incident to two distinct points a, b will be denoted by $n(a, b) := |\mathcal{G}_1(a) \cap \mathcal{G}_1(b)|$.

Two distinct lines are called *coplanar* if they are incident to a plane. If two distinct lines l and m are coplanar, they intersect at a point a in the projective plane $\mathrm{Res}(v)$, where v is a plane incident to l and m. If w is another plane incident to both l and m, we have two distinct "lines" v and w in the GQ $\mathrm{Res}(a)$ incident to two "points" of the GQ $\mathrm{Res}(a)$, which is a contradiction. Thus if two distinct lines l and m are coplanar, there is a unique plane incident to them.

Two distinct planes v, w are called *cocollinear* and denoted by $v \sim w$ if there is a line l incident to both v and w. If there is another line m incident to both v and w, two coplanar lines l and m are incident to distinct planes, which is not the case as we saw above. Hence if two distinct planes v and w are cocollinear, there is a unique line incident to both v and w, which will be denoted by $v \cap w$.

3.3. *Buildings of type C_3*

The typical examples of flag-transitive C_3-geometries of order (x, y) with finite x, y with $x \geq 2$ are the finite classical polar spaces of type C_3. Explicitly, they are the classical polar spaces for 6-dimensional symplectic spaces, 6-dimensional orthogonal spaces of plus type, 7-dimensional orthoganal spaces, 8-dimensional orthogonal spaces of minus type, and 6- and 7-dimensional hermitian spaces, which are described as follows.

Let (V_6, s_6) be a 6-dimensional vector space over a finite field $GF(q)$ with a non-degenerate symplectic form s_6, (V_6, f_6^+) a 6-dimensional vector space over a finite field $GF(q)$ with a non-singular quadratic form f_6^+ of plus type, (V_7, f_7) a 7-dimensional vector space over a finite field $GF(q)$ with a non-singular quadratic form f_7, (V_8, f_8^-) an 8-dimensional vector space over a finite field $GF(q)$ with a non-singular quadratic form f_8^- of minus type, and let (U_6, h_6) and (U_7, h_7) be 6- and 7-dimensional vector spaces over $GF(q^2)$ with non-degenerate hermitian forms h_6 and h_7 respectively. Let (W, f) be one of these spaces with forms. Note that maximal totally isotropic (or singular) subspaces of W are of dimension 3. Define $\mathcal{G}_0$, $\mathcal{G}_1$ and $\mathcal{G}_2$ to be the sets of 1-, 2- and 3-dimensional totally isotropic (or singular) subspaces of W. We define the incidence $*$ by inclusion. Then we may verify that the resulting geometry $\mathcal{G}(W, f) = (\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2; *)$ is a C_3-geometry admitting the flag-transitive action of associated classical groups. The order of $\mathcal{G}(W, f)$ is (q, q), $(q, 1)$, (q, q), (q, q^2), (q^2, q) or (q^2, q^3), if $(W, f) = (V_6, s_6)$, (V_6, f_6^+), (V_7, f_7), (V_8, f_8^-), (U_6, h_6), or (U_7, h_7), respectively.

These classical polar spaces are finite buildings of type C_3, which are characterized by Tits in terms of the (LL) condition [18] p. 543, Proposition 9. Here the Condition (LL) means that there is at most one line through two distinct points, which is equivalent to saying that $n(a, b) = 1$ for any pair of collinear points a, b. (Note that the Condition (O) in [18] Proposition 9 is equivalent to the Condition (LL), as $n = 3$. See also [14] 7.4.3 for an elementary proof of the following Theorem.)

Theorem 3.3.1 [18, 14] *If a C_3-geometry $\mathcal{G}$ satisfies the condition that $n(a, b) = 1$ for any pair of collinear points a, b, then $\mathcal{G}$ is a building of type C_3.*

Theorem 3.3.1 and [17] p. 106, 7.4 imply that a geometry $\mathcal{G}$ in Theorem above corresponds bijectively to a polar space $\mathcal{S}$ of rank 3. (Note that in [17], a building in our sense is called a weak building.) Assume, futhermore, that $\mathcal{G}$ has finite order (x, y) with $x \geq 2$. If $y = 1$, each line of $\mathcal{S}$ is contained in exactly two planes of $\mathcal{S}$, and $\mathcal{S}$ is uniquely determined by [17] p. 113, 7.13. In our case, as x is finite, $\mathcal{S}$ is a polar space for a non-singular orthogonal form of plus type on a 6-dimensional space over the finite field $GF(x)$. In particular, x is a prime power. If $y \geq 2$, the polar space $\mathcal{S}$ is thick (see [17] p. 105, line 1–3) and hence every plane of $\mathcal{S}$ is a Moufang projective plane by [17] p. 110, 7.11. Then each plane of $\mathcal{S}$ is coordinatized by an alternative division ring. Since a finite alternative division ring is a finite field by the theorem of Artin-Zorn, the finiteness of x implies that every plane of $\mathcal{S}$ is desarguesian. By [17] p. 167, 8.11, $\mathcal{S}$ is embeddable, which implies that $\mathcal{S}$ can be realized as $\mathcal{G}(W, f)$ for some vector space W having a symplectic, orthogonal or hermitian form f in the way described above. Hence we have the following.

Theorem 3.3.2 *If a C_3-geometry of finite order (x, y) with $x \geq 2$ satisfies the property that $n(a, b) = 1$ for any pair of collinear points a, b, then it is one of the above six families of finite classical polar spaces for some prime power $q = x$.*

3.4. The sporadic A_7-geometry

The sporadic A_7-geometry is described as follows: First, we set $\mathcal{G}_0$:= the 7 letters of $\Omega = \{1, 2, \ldots, 7\}$ and $\mathcal{G}_1$:= the 35 (unordered) triples of Ω. We consider a projective plane having Ω as the set of points. Such plane should be of order 2 and can be determined by specifying its 7 lines. For example, $\Pi = (\Omega, \mathcal{L})$ is a projective plane, where $\mathcal{L}$ consists of the lines 123, 145, 167, 246, 257, 347 and 356. Here we also denote a line by the triple of points on it. It can be verified that there are 30 such planes, which form two orbits of the same length 15 under the action of the alternating group A_7 on Ω. Two planes belong to the same A_7-orbit if and only if they have exactly one line in common. Now we define $\mathcal{G}_2$ as one of these two A_7-orbits, and determine $*$ by natural containment. The resulting geometry $(\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2; *)$ is called *the sporadic A_7-geometry*.

In general, a C_3-geometry is called *flat* if each point is incident to every plane. We can easily observe that the sporadic A_7-geometry is a flat C_3-geometry, admitting a flag-transitive action of A_7. In fact, the sporadic A_7-geometry can be characterized by this property.

Theorem 3.4.1 [9] *If $\mathcal{G}$ is a flag-transitive flat C_3-geometry of finite order (x, y) with $x \geq 2$, then $\mathcal{G}$ is isomorphic to the sporadic A_7-geometry.*

3.5. *Finiteness*

We can verify that the local finiteness (that is, the finiteness of order) of a thick C_3-geometry $\mathcal{G}$ implies the (global) finiteness of $\mathcal{G}$.

Lemma 3.5.1 *Let $\mathcal{G}$ be a C_3-geometry of finite order (x, y). Then for any point a and any plane u not through a, there is a plane v through a cocollinear with u.*

Proof: Since $\mathcal{G}$ is connected, the incidence graph of $\mathcal{G}$ is connected. As the residue of a point is a GQ, this implies that any plane u_0 through a can be joined to the plane u by a sequence $(u_0 = w, u_1, \ldots, u_n = u)$ of planes such that u_{i-1} is cocollinear with u_i for each $i = 1, \ldots, n$. Let n be the minimum length of sequences $(u_0, u_1, \ldots, u_n = u)$ with $a * u_0$, $u_{i-1} \sim u_i$ $(i = 1, \ldots, n)$. If $n \leq 1$, then the claim follows. Suppose $n \geq 2$.

Consider the lines $l := u_0 \cap u_1$ and $m := u_1 \cap u_2$ in the projective plane $\mathrm{Res}(u_1)$. If $l = m$, the sequence $(u_0, u_2, \ldots, u_n)$ of planes with $u_{i-1} \sim u_i$ has length $n - 1$ and joins w and u, which contradicts the minimality of n. Thus $l \neq m$, and hence they intersect at a unique point b on u_1. Let r be the unique line in the projective plane $\mathrm{Res}(u_0)$ joining a and b. Note that r does not lie on u_2, as a is not on u_2 by the minimality of n. Thus r and u_2 are non-incident elements in the GQ $\mathrm{Res}(b)$, and hence there is a plane v through r cocollinear with u_2. Then the sequence $(v, u_2, \ldots, u_n = u)$ of planes of length $n - 1$ joins a and u, and satisfies $v \sim u_2 \sim \cdots \sim u$. This is a contradiction. □

Corollary 3.5.2 *If $\mathcal{G}$ is a C_3-geometry of finite order (x, y), then $\mathcal{G}$ is a finite geometry.*

Proof: We fix a point a of $\mathcal{G}$. If b is a point not collinear with a, any plane u containing b is not incident to a, since any two distinct points can be joined by a line in the projective plane $\mathrm{Res}(u)$. By Lemma 3.5.1 there is a plane v through a cocollinear with u. Since a (resp. b) is collinear with any point on $l = u \cap v$ in the projective plane $\mathrm{Res}(v)$ (resp. $\mathrm{Res}(u)$), the collinearity graph Γ of $\mathcal{G}$ is of diameter at most 2. Since there are a finite number of lines through a point and each line is incident to a finite number of points, the neighbourhood of a point in Γ is a finite set. As the diameter of Γ is finite, $\mathcal{G}$ has a finite number of points. Since the residues of points are finite, $\mathcal{G}$ has finite number of lines and planes. $\square$

3.6. The Ott-Liebler number

For every point-plane flag (a, u), we denote by $\alpha(a, u)$ the number of planes $v(\neq u)$ through a cocollinear with u but a is not incident to $u \cap v$. As is shown in the Lemma below, $\alpha := \alpha(a, u)$ is a constant, not depending on the particular choice of a point-plane flag (a, u). The number α is called the *Ott-Liebler number* of $\mathcal{G}$, after the mathematicians who first investigated the meaning of this constant in terms of the representation theory of the Hecke algebra associated to $\mathcal{G}$. The following result was proved first with the aid of representation theory, but later Pasini provided an elementary and representation-free proof, ([11] p. 82–84) which I include here.

Lemma 3.6.1

(1) *The number $\alpha := \alpha(a, u)$ is a constant, not depending on the particular choice of a plane u and a point a on u.*

(2) *For any point b not on a plane u, there are exactly $\alpha + 1$ planes through b cocollinear with u.*

Proof: (1) For any point-plane flag (b, v), we write

$$\mathcal{A}(b, v) := \{w \in \mathcal{G}_2(b) \mid w \sim v, b \not* (v \cap w)\}.$$

Then $\alpha(b, v) = |\mathcal{A}(b, v)|$.

We will first show that $\alpha(a, u) = \alpha(a, u')$ for any plane u' incident to the point a. Since $\mathrm{Res}(a)$ is connected, it suffices to prove this claim for a plane u' cocollinear with u such that $a * (u \cap u')$. We set $l := u \cap u'$, and define a map $f : \mathcal{A}(a, u) \to \mathcal{A}(a, u')$ as follows.

For each $v \in \mathcal{A}(a, u)$, the line $u \cap v$ is distinct from l, as $a \not* (u \cap v)$. Then $u \cap v$ and l intersect at a unique point, say b, distinct from a in the projective plane $\mathrm{Res}(u)$. Let m be the line joining a and b in the projective plane $\mathrm{Res}(u)$. As $a \not* (u \cap v)$, m is not incident to u, in particular, $l \neq m$. Since l is the unique line of the projective plane $\mathrm{Res}(u')$ through a and b, we conclude that m is not incident to u'. Thus, in the GQ $\mathrm{Res}(b)$, there is a unique plane v' through m coplanar with u'. Clearly $a * v'$, but $u' \cap v'$ is distinct from l, and hence $u' \cap v'$ is not incident to a. Thus the plane v' uniquely determined by $v \in \mathcal{A}(a, u)$ lies in $\mathcal{A}(a, u')$. Define $(v)^f := v'$.

The map $f' : \mathcal{A}(a, u') \to \mathcal{A}(a, u)$ can be similarly defined, and it is immediate to see that f' is the inverse map of f. Thus f is a bijection and so $\alpha(a, u) = |\mathcal{A}(a, u)| = |\mathcal{A}(a, u')| = \alpha(a, u')$.

Next we will show that $\alpha(a, u) = \alpha(a', u)$ for any point a' on the plane u. We may assume that $a \neq a'$. Let l be the unique line on the projective plane $\mathrm{Res}(u)$ joining a and a'. We define a map $g : \mathcal{A}(a, u) \to \mathcal{A}(a', u)$ as follows.

For each plane $v \in \mathcal{A}(a, u)$, the line $u \cap v$ is distinct from l, as $a \not* (u \cap v)$. In particular, l is not incident to v. Then there is a unique plane w in the GQ $\mathrm{Res}(a)$ through l and cocollinear with v. As u and v are not coplanar in $\mathrm{Res}(a)$, $u \neq w$. Since l is incident to both w and u, we have $l = u \cap w$. As a is not on $u \cap v$, we have $(u \cap w) \neq (u \cap v)$, and hence they intersect at a unique point, say b, in the projective plane $\mathrm{Res}(v)$. If b lies on $l = u \cap w$, then u, v, w form a proper triangle in the GQ $\mathrm{Res}(b)$, which is a contradiction. Thus b is not on l, and in particular, $a' \neq b$. Let m' be the unique line in the projective plane $\mathrm{Res}(w)$ joining a' and b. If the line m' is also on u, then $m' = u \cap w = l$ and l is incident to b, which contradicts the above conclusion. Hence m' is not on u. Then there is a unique plane v' through m' and cocollinear with u in $\mathrm{Res}(b)$.

We define $v^g := v'$. As v' is incident to m', the point a' is on v', but the line $(v' \cap u)$ does not pass through a', since the unique line l on u through a and a' does not pass through b. Hence $v' \in \mathcal{A}(a', u)$. The similar map $g' : \mathcal{A}(a', u) \to \mathcal{A}(a, u)$ can be defined by exchanging a and a', and it is immediate to to check that g' is the inverse map of g. Thus g is a bijection and $\alpha(a, u) = \alpha(a', u)$.

Since $\mathcal{G}$ is residually connected, the conclusions above imply that $\alpha(a, u)$ is constant for any point-plane flag (a, u), and the Claim (1) is proved.

(2) By Lemma 3.5.1, there is a plane v through b cocollinear with u. We fix such a plane v, and set $\mathcal{B}(b, u) := \{w \in \mathcal{G}_2(b) \mid w \neq v, w \sim u\}$. We will define a bijective map f from $\mathcal{B}(b, u)$ to $\mathcal{A}(b, v)$, where $\mathcal{A}(b, v)$ means the same notation as in the proof of (1).

For each $w \in \mathcal{B}(b, u)$, consider the line $u \cap w$ on u. If $u \cap w = u \cap v$, $w \in \mathcal{A}(b, u)$, and we define $w^f := w$. Assume that $u \cap w$ is distinct from $u \cap v$, and let a be the unique point on the lines $u \cap v$ and $u \cap w$ in the projective plane $\mathrm{Res}(u)$. As $b \not\ast u$, a is distinct from b. Let m be the unique line joining a and b in the projective plane $\mathrm{Res}(w)$. If m is on v, $m = u \cap w$, and u, v, w form a proper triangle in the GQ $\mathrm{Res}(a)$, which is a contradiction. Thus m is not on v, and hence there is a unique plane w' in the GQ $\mathrm{Res}(a)$ through m and cocollinear with u. As $b * m * w'$, $b * w'$, but $b \not\ast (v \cap w')$, for otherwise $m = (v \cap w')$ is the unique line in the projective plane $\mathrm{Res}(v)$ joining two points a and b. Thus $w' \in \mathcal{A}(b, u)$. We define $w^f := w'$.

To show the bijectivity of f, we will give the inverse map g of f. For each $w' \in \mathcal{A}(b, v)$, let consider the line $w' \cap v$. If $w' \cap v = v \cap u$, then we set $(w')^g := w'$. Assume that $w' \cap v \neq v \cap u$. Then the lines $w' \cap v$ and $v \cap u$ intersect at a unique point, say a, in the projective plane $\mathrm{Res}(v)$. As b is not on u, $a \neq b$. Let m be the unique line of $\mathrm{Res}(w')$ joining a and b. As m is not on u, there is a unique plane w in the GQ $\mathrm{Res}(a)$ through m and cocollinear with u. Clearly $w \in \mathcal{B}(b, u)$. Define $(w')^g := w$. It is immediate to see that g gives the inverse map of f above. Then f is a bijection, and hence $\alpha = |\mathcal{A}(a, u)| = |\mathcal{B}(b, u)|$ is the number of planes through b cocollinear with u minus 1. The Claim (2) is proved. □

Lemma 3.6.2 *If $\mathcal{G}$ is a C_3-geometry of finite order (x, y), then $\mathcal{G}$ has $(x^2+x+1)(x^2y+1)/(\alpha+1)$ points, $(x^2+x+1)(x^2y+1)(xy+1)/(\alpha+1)$ lines, $(x^2y+1)(xy+1)(y+1)/(\alpha+1)$ planes, and $(x^2+x+1)(x^2y+1)(xy+1)(y+1)(x+1)/(\alpha+1)$ maximal flags.*

Proof: For a fixed point a, we will count the number of the following set in two ways:

$$\mathcal{X} = \{(v, l, u) \in \mathcal{G}_2(a) \times \mathcal{G}_1 \times (\mathcal{G}_2 - \mathcal{G}_2(a)) \mid v * l * u\}.$$

For each plane u not through a, there are $\alpha + 1$ planes through a cocollinear with u by Lemma 3.6.1(1). For each such plane v, $v \cap u = l$ is a unique line with $(v, l, u) \in \mathcal{X}$. Thus $|\mathcal{X}| = (|\mathcal{G}_2| - |\mathcal{G}_2(a)|)(\alpha + 1)$.

On the other hand, for each plane v through a, we have $(v, l, u) \in \mathcal{X}$ if and only if l is a line on v not incident to a, u is a plane through l not incident to a. There are x^2 lines l on v not through a, and for each such line l there are y planes $(\neq v)$ through l. Since there are exactly α planes w through a cocollinear with v but $l = v \cap w$ does not pass through a by Lemma 3.6.1(1), among x^2y such pairs of (l, u), there are exactly $x^2y - \alpha$ pairs (l, u) with $(v, l, u) \in \mathcal{X}$. Hence we have $|\mathcal{X}| = |\mathcal{G}_2(a)|(x^2y - \alpha)$.

Since $|\mathcal{X}| = (|\mathcal{G}_2| - |\mathcal{G}_2(a)|)(\alpha + 1) = |\mathcal{G}_2(a)|(x^2y - \alpha)$, we have

$$|\mathcal{G}_2| = |\mathcal{G}_2(a)|(x^2y + 1)/(\alpha + 1) = (xy + 1)(y + 1)(x^2y + 1)/(\alpha + 1).$$

Then $|\mathcal{G}_0|$ and $|\mathcal{G}_1|$ can be obtained from $|\mathcal{G}_0| = |\mathcal{G}_2|(x^2 + x + 1)/(xy + 1)(y + 1)$ and $|\mathcal{G}_1| = |\mathcal{G}_2|(x^2 + x + 1)/(y + 1)$. The number of maximal flags is obtained as $|\mathcal{G}_0|(xy + 1)(y + 1)(x + 1)$. □

3.7. A characterization

By elementary counting arguments involving the Ott-Liebler number α and Theorem 3.3.2, we can obtain a nice characterization [12] of the finite buildings of type C_3 and the sporadic A_7-geometry. Since this is very important to our proof, I repeat it for the convenience for the readers. We first need the following elementary lemma.

Lemma 3.7.1 *Let $\mathcal{G}$ be a C_3-geometry of finite order (x, y) with $x \geq 2$. Assume that there is a point b not on a plane u. Then the following holds:*

(1) *For any line m through b, we have*

$$\alpha = \sum_{c \in \mathcal{G}_0(m) - \{b\}} (n(b, c) - 1).$$

(2) *We have*

$$(x + 1)(\alpha + 1) = \sum_{a \in \mathcal{G}_0(u)} n(b, a).$$

(3) *Assume that there is a line l on u, which is not $u \cap v$ for any plane v through b cocollinear with u. Then we have*

$$\alpha + 1 = \sum_{a \in \mathcal{G}_0(l)} n(b, a).$$

(4) *Assume that there is a line-plane flag (l, v) such that v is incident to b but l does not pass through b. Then we have*

$$\alpha - x + x|\mathcal{G}_2(b) \cap \mathcal{G}_2(l)| = \sum_{a \in \mathcal{G}_0(l)} (n(b, a) - 1).$$

Proof: We use the double counting argument to prove each claim.

(1) Choose a plane v incident to m. Let $\mathcal{A}(b, v)$ be the set of planes w $(\neq v)$ incident to b and cocollinear with v but the line $v \cap w$ is not incident a. By Lemma 3.6.1(1), $\alpha = |\mathcal{A}(b, v)|$. We will count the cardinality of the following set in two ways.

$$\mathcal{X} := \{(w, l, c) \in \mathcal{A}(b, v) \times (\mathcal{G}_1(b) - \{m\}) \times (\mathcal{G}_0(m) - \{b\}) \mid w * l,\, c \in \mathcal{G}_0(l)\}$$

For each plane $w \in \mathcal{A}(b, v)$, the line $v \cap w$ on w is not incident to b. Then m and $v \cap w$ are distinct lines in the projective plane Res(w), and they intersect at the unique point c $(\neq b)$. Since b and c are distinct points on the projective plane Res(w), there is a unique line l $(\neq m)$ on w joining b and c. Thus $\alpha = |\mathcal{X}|$.

On the other hand, for each point c on m distinct from b, there are $n(b, c) - 1$ lines through b, c distinct from m. For each such line l, l is not incident to v in the GQ Res(b). For, otherwise, there are two distinct lines l, m through two distinct points in the projective plane Res(v). Then there is a unique plane w $(\neq v)$ of Res(b) incident to l and cocollinear with v. Since $w \in \mathcal{A}(b, v)$, we have $|\mathcal{X}| = \sum_{c\in\mathcal{G}_0(m)-\{b\}}(n(b, c) - 1)$.

(2) We count the cardinality of the following set in two ways:

$$\mathcal{Y} = \{(a, l, v) \in \mathcal{G}_0(u) \times \mathcal{G}_1(b) \times \mathcal{G}_2(b) \mid a * l * v,\, u \sim v\}.$$

Fix a point a on u and a line l through a and b. Since b is on l but not on u, l is a line not incident to u in the GQ Res(a). Then there is a unique plane v through l cocollinear with u. Thus $|\mathcal{Y}| = \sum_{a\in\mathcal{G}_0(u)} n(b, a)$.

On the other hand, there are $\alpha + 1$ planes v through b cocollinear with u by Lemma 3.6.1(2). For each such plane v, there are $x + 1$ points on $u \cap v$, and each point on $u \cap v$ can be joined to b by a unique line in the projective plane Res(v). Thus $|\mathcal{Y}| = (\alpha + 1)(x + 1)$.

(3) We count the cardinality of the following set in two ways:

$$\mathcal{Z} = \{(a, m, v) \in \mathcal{G}_0(l) \times \mathcal{G}_1(b) \times \mathcal{G}_2(b) \mid a * m * v,\, u \sim v\}.$$

For a point a on l and a line m through a and b, there is a unique plane v through m cocollinear with u, by the same reason as we saw in the former part of the proof of (2). Thus $|\mathcal{Z}| = \sum_{a\in\mathcal{G}_0(l)} n(b, a)$. On the other hand, for each plane v through b cocollinear with u $u \cap v$ intersects l at a unique point, as $u \cap v$ and l are distinct lines of the projective lane Res(u). Thus $|\mathcal{Z}| = (\alpha + 1)$ by Lemma 3.6.1(2).

(4) We count the cardinality of the following set in two ways:

$$\mathcal{W} = \{(a, m, w) \in \mathcal{G}_0(l) \times (\mathcal{G}_1(b) - \mathcal{G}_1(v)) \times (\mathcal{G}_2(b) - \{v\}) \mid a * m * w,\, w \sim u\}.$$

We have $|\mathcal{W}| = \sum_{a\in\mathcal{G}_0(l)}(n(b, c) - 1)$, because for each point a on l and each line m through a and b not on v, m and u are not incident in the GQ Res(a), and so there is a unique plane w $(\neq v)$ incident to m and cocollinear with u.

On the other hand, choose any one of α planes w $(\neq v)$ incident to b cocollinear with v but $v \cap w$ is not incident to b. If $l = v \cap w$, then for any point a on l, the unique line m through a and b in Res(w) is not incident to v and $(a, m, w) \in \mathcal{W}$. If $l \neq v \cap w$, then there is a unique point a on l incident to w (which is the point of intersection of l and

$v \cap w$), and $(a, m, w) \in \mathcal{W}$ for the uniuqe line m through a and b on w. Since there are $|\mathcal{G}_2(b) \cap \mathcal{G}_2(l)| - 1$ planes through b cocollinear with v and $v \cap w = l$, we have

$$\begin{aligned}|\mathcal{W}| &= (x+1)(|\mathcal{G}_2(b) \cap \mathcal{G}_2(l)| - 1) + 1 \cdot (\alpha + 1 - (|\mathcal{G}_2(b) \cap \mathcal{G}_2(l)| - 1)) \\ &= \alpha + x|\mathcal{G}_2(b) \cap \mathcal{G}_2(l)| - x,\end{aligned}$$

and the claim follows. □

By Lemma 3.7.1(1), the condition $\alpha = 0$ if and only if $n(a, b) = 1$ for any pair of collinear points a, b. Thus if $\alpha = 0$, then $\mathcal{G}$ is a building by Theorem 3.3.1.

Theorem 3.7.2 [12] *If $n(a, b)$ is constant for any pair of collinear points a, b in a C_3-geometry $\mathcal{G}$ of finite order (x, y) with $x \geq 2$, then either $\mathcal{G}$ is flat or $\alpha = 0$. In the latter case, $\mathcal{G}$ is a building as we remarked above, and hence one of the six families of finite classical polar spaces in* 3.3.

Proof: Assume that $\mathcal{G}$ is not flat. Then there is a point b and a plane u not through b. Let N be the constant $n(b, c)$ for any point c ($\neq b$) collinear with b, and let M be the number of points on u collinear with b.

Choose any line m through b. Since x points on m distinct from b are collinear with b, it follows from Lemma 3.7.1(1) that $N = 1 + (\alpha/x)$. Then the Lemma 3.7.1(2) implies that $M = (x+1)(\alpha+1)/N = (x+1)x(\alpha+1)/(\alpha+x)$. Since $x \geq 1$, $M \leq x(x+1) < x^2 + x + 1 = |\mathcal{G}_0(u)|$. Then there is a point, say a_0, on u not collinear with b. Any line l on u through a_0 is not of form $u \cap v$ for any plane v through b cocollinear with u. Applying Lemma 3.7.1(3) to such a line l, we conclude that the number of points on l collinear with b is equal to $(\alpha+1)/N = x(\alpha+1)/(\alpha+x)$. We set $L := x(\alpha+1)/(\alpha+x)$. Then L is a natural number less than x and $M = (x+1)L$.

Now there are exactly $\alpha + 1$ planes through b cocollinear with u by Lemma 3.6.1(2). As $\alpha \geq 0$, there is at least one of such plane v. Let v be one of such plane and set $l' := u \cap v$. As b is not on u, l' is not incident to b. Applying Lemma 3.7.1(4), we have $\alpha + 1 + xK = N(x+1)$, where we set $K = |\mathcal{G}_2(b) \cap \mathcal{G}_2(l')|$. As $N = (\alpha+x)/x$, we have $K = 1 + (\alpha/x^2)$. In particular, x^2 divides α, and hence $\alpha = 0$ or $\alpha \geq x^2$. Since $L = x(\alpha+1)/(\alpha+x)$ is a natural number less than or eqaul to x, $L \leq x - 1$ or $L = x$. In the latter case, we have $x = 1$, which contradicts our assumption. Thus $L \leq x - 1$. Then we have $x\alpha + x \leq x\alpha - \alpha + x^2 - x$ and so $\alpha \leq x^2 - 2x$. Hence $\alpha = 0$. This implies that $N = 1$, and therefore $\mathcal{G}$ is a building of type C_3 by Theorem 3.3.1. □

3.8. Imprimitivity blocks and planes

Here I give an elementary lemma, whose proof requires the assumption of flag-transitivity of A on $\mathcal{G}$.

Lemma 3.8.1 *Let $\mathcal{G}$ be a C_3-geometry of finite order (x, y) with $x \geq 2$, admitting a flag-transitive automorphism group A. If Ω is a system of imprimitivity blocks of $\mathcal{G}_0$ under the action of A, then either $|\mathcal{G}_0(u) \cap \Delta| \leq 1$ for each block Δ and each $u \in \mathcal{G}_2$ or $\Omega = \{\mathcal{G}_0\}$.*

Proof: Assume that $|\mathcal{G}_0(u) \cap \Delta| \geq 2$ for some block Δ in Ω. As $\Delta = \Delta^g$ or $\Delta \cap \Delta^g = \emptyset$ for $g \in A_u$, the set $\{\mathcal{G}_0(u) \cap \Delta^g \mid g \in A_u\}$ forms a system of imprimitivity blocks in $\mathcal{G}_0(u)$ under the action of A_u. By Theorem 2.2.1, in any case, A_u acts primitively on $\mathcal{G}_0(u)$. As $|\mathcal{G}_0(u) \cap \Delta| \geq 2$, we have $\mathcal{G}_0(u) \cap \Delta = \mathcal{G}_0(u)$, or $\mathcal{G}_0(u) \subseteq \Delta$.

Now choose any plane v ($u \neq v$) cocollinear with u. As $v = u^g$ for some $g \in A$, we have $\emptyset \neq \mathcal{G}_0(u \cap v) \subseteq \mathcal{G}_0(u) \cap \mathcal{G}_0(v) \subseteq \Delta \cap \Delta^g$. As Δ is a block under the action of A, we have $\Delta = \Delta^g \supset \mathcal{G}_0(v)$.

Since the incidence graph of $\mathcal{G}$ is connected, we can verify that any plane w can be joined to u by a sequence $u = u_0, \dots, u_m = w$ of planes such that u_{i-1} is cocollinear with u_i for each $i = 1, \dots, m$. Hence the above argument shows that Δ contains $\mathcal{G}_0(w)$. As w is an arbitrary plane, we conclude that $\mathcal{G}_0 = \Delta$. □

4. Some Lemmas

In the remainder of this paper, we assume that $\mathcal{G}$ is a C_3-geometry of finite order (x, y) with $x \geq 2$, admitting a flag-transitive automorphism group A. Furthermore, we assume that $\mathcal{G}$ is neither a building nor the sporadic A_7-geometry. In Sections 4, 5, we will derive a contradiction.

4.1. The structure of residues of planes

Lemma 4.1.1 *The following hold:*

(1) *For a plane u of $\mathcal{G}$, A_u/K_u is isomorphic to a Frobenius group F_p^{x+1} of order $p(x+1)$ with the cyclic kernel of prime order $p = x^2 + x + 1$ and a cyclic complement of order $x + 1$. Furthermore, either $x = 8$ and $p = 73$, or $x > 14,400,008$ and hence $p > 207,360,244,800,073$.*

(2) *The stabilizer $A_{a,l,u}$ of a maximal flag (a, l, u) of $\mathcal{G}$ coincides with the kernel K_u on the plane u.*

Proof: Since A_u acts flag-transitively on the projective plane $\mathrm{Res}(u)$ of order x, Theorem 2.2.1 implies that either A_u/K_u contains $PSL_3(x)$ (and $\mathrm{Res}(u)$ is desarguesian) or A_u/K_u has the shape described in the Claim (1).

Assume that A_u/K_u contains $PSL_3(x)$. Then A_u acts doubly transitive on the set of points on u. Hence $n(a, b)$ is constant for any pair (a, b) of distinct points on u. Since any two collinear points are incident to a line and so a plane, the transitivity of A on the planes implies that $n(a, b)$ is constant for any pair of collinear points. Then it follows from Theorem 3.7.2 that $\mathcal{G}$ is either flat or one of classical polar spaces of type C_3.

Since $\mathcal{G}$ is not a building by the asumption, $\mathcal{G}$ should be flat. However, Theorem 3.4.1 implies that $\mathcal{G}$ is the sporadic A_7-geometry, which again contradicts the assumption. Thus it follows from Theorem 2.2.1 that $\mathrm{Res}(u)$ is either non-desarguesian or the desarguesian plane of order $x = 2$ or 8. Moreover, $A_u/K_u \cong F_p^{x+1}$ in any case.

If $x = 2$, the group $A_u/K_u \cong F_7^3$ acts transitively on the set of $21 = \binom{7}{2}$ pairs of distinct points of $\mathrm{Res}(u)$. Then $n(a, b)$ is constant as we saw in the above paragraphs, and obtain a contradiction. Thus $x \neq 2$.

Now the Claim (1) follows from Theorem 2.2.1 and Proposition 2.2.2. Then Claim (2) follows from Claim (1), since the Frobenius group $A_u/K_u \cong F_p^{x+1}$ acts sharply transitively on the maximal flags of $\mathrm{Res}(u)$. □

Remark 4.1.2 In the above proof, a characterization of the A_7-geometry (Theorem 3.4.1, [9]) is required. However, we do not need the whole proof of Lemma 2 [9] for our purpose, since we may assume that the residues of planes are desarguesian. (Explicitly we can omit the proof from the line 5 p. 266 to the line 25 p. 267 [9], where the non-desarguesian case is treated.)

Moreover, it should be mentioned that the conclusion $x \leq y$ of Lemma 1 [9] (whose proof essentially requires some representation theory) is not needed to establish the main theorem of [9] in our situation. Indeed, this was used at only two places: one is at the last part of the paragraph mentioned above, and the other is to examine the case $x = 2$ (the second line p. 270 [9]). The former is related to the case we do not care about.

The latter case can be treated as follows, not relying on any deep results. The latter claim "$y \leq x^2 - x$" of Lemma 1 [9] follows from the fact that the residue of a point has an ovoid (see the proof of Lemma 5 [9]) by applying a standard result on GQs with ovoids ([15] 1.8.3). Then a flag-transitive flat C_3-geometry $\mathcal{G}$ of order $(x = 2, y)$ has order $(2, 1)$ or $(2, 2)$. Since each line is realized as $u \cap v$ for some two cocollinear planes u and v, we can easily observe that A is faithful on the set of planes. In particular, the kernel K_a for a point a is trivial, as a is incident to every plane. If $(x, y) = (2, 1)$, there are 6 planes by Lemma 3.6.2, and hence the flag-transitive group A is a subgroup of S_6 as A faithfully acts on $\mathcal{G}_2$. However, A is not transitive on 7 points. Thus $(x, y) \neq (2, 1)$. It is easy to verify that any GQ of order $(2, 2)$ is isomorphic to the GQ of 1- and 2-spaces of the 4-dimensional symplectic space and so it has the flag-transitive automorphism group A_6 or S_6. As $K_a = 1$ at a point a, the stabilizer A_a is isomorphic to A_6 or S_6, and so $A \cong A_7$ or S_7 as $|\mathcal{G}_0| = 7$. Since $|\mathcal{G}_2| = 15$, we have $A \cong A_7$, and now it is easy to see that $\mathcal{G}$ is uniquely determined.

Lemma 4.2 *The stabilizer A_a of a point a acts faithfully on the residue $Res_{\mathcal{G}}(a)$.*

Proof: For two points a, b of $\mathcal{G}$, we write $a \approx b$ if $a = b$ or there is a sequence $a = a_0, a_1, \ldots, a_m = q$ of points with $n(a_{i-1}, a_i) \geq 2$ for each $i = 1, \ldots, m$. (For the definition of $n(a, b)$, recall 3.2.) Clearly, the relation $\approx$ is an equivalence relation on the set $\mathcal{G}_0$ of points, and hence each equivalence class is a block of imprimitivity under the action of A.

If $n(a, b) = 1$ for any collinear points a, b, $\mathcal{G}$ is a building by Theorem 3.3.1. Hence each $\approx$-class contains at least two points. Choose points a, b with $n(a, b) \geq 2$, and let m be a line incident to a and b, and pick a plane u incident to m. Then $\mathcal{G}_0(u)$ contains at least two distinct points a, b in the $\approx$-class through a. Then it follows from Lemma 3.8.1 that $\mathcal{G}_0$ is one $\approx$-equivalence class.

Now we will show that $K_a = K_b$ for distinct points a, b with $n(a, b) \geq 2$. By the result above, this implies that $K_a = K_b$ for all the points b, and hence K_a fixes every element of $\mathcal{G}$ and $K_a = 1$, as we claimed.

Let m be any line through a and b, and let v be any plane through m. As the kernel K_a fixes every element in Res(a), K_a fixes a maximal flag (a, m, v), and hence $K_a \leq K_v = A_{a,m,v}$ by Lemma 4.1.1(2). In particular, K_a fixes the point b on m and every line through b on v.

Now we pick two distinct lines l, m through a and b. Note that there is no plane w incident to both l and m. For, otherwise, the distinct lines l and m in the projective plane Res(w) intersect in two disitinct points a and b. In particular, l and m are non-coplanar lines in the GQ Res(b) on which the group K_a acts. By the remark above, K_a fixes the non-coplanar lines l, m and each plane through l or m together with all the lines on it. By Lemma 2.3.3, the substructure of the GQ Res(b) fixed by K_a is a subquadrangle of order (x, y), and hence coincides with Res(b). Thus K_a acts trivially on Res(b), or equivalently $K_a \leq K_b$. We have $K_b \leq K_a$ by the same argument replacing a and b, and hence $K_a = K_b$. As we saw above, this implies $K_a = 1$, the faithfulness of the action of A_a on Res(a). □

Lemma 4.3 *For a prime $p = x^2 + x + 1$, a Sylow p-subgroup P of A is of order p and acts semiregularly both on $\mathcal{G}_0$ and $\mathcal{G}_1$.*

Proof: Let P be a Sylow p-subgroup of A. Suppose that there is a non-trivial element g of P fixing a point a. We may assume that g is of order p. The element g acts on the GQ Res(a) with $(x + 1)(xy + 1)$ lines.

If $p = x^2 + x + 1$ divides $xy + 1$, p divides $(-x^2)(xy + 1) + (xy - y + 1)(x^2 + x + 1)$ $= x + 1 - y$. If $y < x + 1$, then $p = x^2 + x + 1 \leq x + 1 - y$ and so $y + x^2 \leq 0$, which is a contradiction. If $y > x + 1$, then $p = x^2 + x + 1 \leq y - x - 1$ and so $x^2 < x^2$ $+2(x + 1) \leq y$, which contradicts Lemma 2.3.1(3). Thus $y = x + 1$. By Lemma 2.3.1(2), $x + y = 2x + 1$ divides $xy(x + 1)(y + 1) = x(x + 1)^2(x + 2)$. As $2x + 1$ is prime to x and $x + 1$, $2x + 1$ divides $x + 2$. However, this implies that $2x + 1 \leq x + 2$, which contradicts the assumption that $x \geq 2$.

Hence p is prime to $|\mathcal{G}_1(a)| = (x + 1)(xy + 1)$. Thus g fixes a line l through a, and acts on the set of $y + 1$ planes through l. Since $y + 1 < x^2 + x + 1 = p$ by Lemma 2.3.1(3), g fixes every plane through l. As $A_{a,l,u} = K_u$ by Lemma 4.1.1(2), g fixes every plane through l together with the lines on them. Now take any line m in Res(a) coplanar with l. Then g acts on the set of $y + 1$ planes through m and hence fixes them together with the lines on them, as $y + 1 < p$. Thus g fixes all the elements of Res(a), and so $g = 1$ by Lemma 4.2. We have proved that P acts semi-regularly on the set $\mathcal{G}_0$ of points of $\mathcal{G}$.

Suppose $|A|_p \geq p^2$. As $|\mathcal{G}_0| = p(x^2y + 1)/(\alpha + 1)$, then it follows from the above semi-regularity of P on $\mathcal{G}_0$ that p divides $x^2y + 1$. Then p divides $(x^2y + 1)x - (x^2 + x + 1)$ $(xy - y) = x + y$. This implies that $p = x^2 + x + 1 \leq x + y \leq x^2 + x$ (see Lemma 2.3.1(3)), which is a contradiction. Thus $|A|_p = p$.

If P fixes a line l, P fixes each point of l, as $x + 1 = |\mathcal{G}_0(l)| < x^2 + x + 1 = p$. Since this contradicts the semi-regularity of P on $\mathcal{G}_0$, P is also semi-regular on $\mathcal{G}_1$. □

Lemma 4.4 *We have $|K_u| < \min\{\alpha y, y^2\}$ if $y > 1$, and $K_u = 1$ if $y = 1$. In both cases, we have $|A| < p^7$ for a prime $p = x^2 + x + 1$.*

Proof: By the definition of the Ott-Liebler number α, for each point-plane flag (b, u), there are exactly α planes v $(\neq u)$ incident to b and cocollinear with u but $b \notin \mathcal{G}_0(u \cap v)$. As we remarked before Theorem 3.7.2, the condition $\alpha = 0$ is equivalent to $n(a, b) = 1$ for any collinear points a, b. Since we assume that $\mathcal{G}$ is not a building, we have $\alpha \neq 0$ by Theorem 3.3.1. Thus there is at least one plane v $(\neq u)$ incident to b and cocollinear with u but b is not incident to $u \cap v$. Let v be one of such planes, and set $l = u \cap v$.

The kernel K_u fixes the line l and the plane u, and hence acts on the set $\mathcal{G}_2(l) - \{u\}$ of y planes through l distinct from u. As $K_u \cap A_v$ fixes a line-plane flag (l, v) together with all the points on l, it follows from Lemma 4.1.1(2) that $K_u \cap A_v \leq A_{a,l,v} = K_v$ for a point a on l. Thus $K_u \cap A_v = K_u \cap K_v$.

In particular, $|K_u : K_u \cap K_v| = |v^{K_u}| \leq |\mathcal{G}_2(l) - \{u\}| = y$. On the other hand, note that v is an element of the set $\mathcal{A}(b, u) := \{w \in \mathcal{G}_2(b) \mid w \sim u, b \not* (u \cap w)\}$ of cardinality $\alpha \neq 0$ (see 3.6.1(1)), on which K_u acts. Thus the above conclusion also implies that $|K_u : K_u \cap K_v| = |v^{K_u}| \leq |\mathcal{A}(b, u)| = \alpha$. Hence $|K_u : K_u \cap K_v| \leq \min\{\alpha, y\}$.

Assume that $y = 1$. Then each line m on u is incident to exactly two planes u and w, say. As K_u fixes m and u, it also fixes the plane w. Then for a point b on m, we have $K_u \leq A_{b,m,w} = K_w$ by Lemma 4.1.1(2). Thus $K_u = K_w$ for any plane w cocollinear with u. By the connectivity of $\mathcal{G}$, this implies that K_u fixes every element of $\mathcal{G}$, and hence $K_u = 1$ if $y = 1$.

Now assume that $y > 1$. We set $X := K_u \cap K_v$. The group X fixes the point b, and so acts on the GQ Res(b). We will verify that X satisfies the conditions (i), (ii) in Lemma 2.3.4. First note that $x > 2$ by Lemma 4.1.1(1).

Assume that an element $g \in X$ fixes a plane w of Res(b). If $w = u$, $g \in K_u$ fixes every line of Res(b) on w. Assume that $w \neq u$. If w intersects u at a line m in Res(b), say, then $g \in K_u$ fixes a maximal flag (b, m, w). By Lemma 4.1.1(2), g fixes every line on w. If w does not intersect u in Res(b), each line m on w is cocollinear with the unique line m' on u, as Res(b) is a GQ. Since g fixes both w and m', g fixes the line m, and hence g fixes every line on w. Thus the property (i) holds.

Next, suppose u and v are cocollinear in Res(b). Then there is a line m of Res(b) incident to both u and v, and hence the lines m and l intersect at a unique point c in the projective plane Res(u). However, in Res(c), u and v are planes incident to two distinct lines l and m, and hence $u = v$, a contradiction. Thus u and v are non-cocollinear planes of Res(b), and the Condition (ii) is verified.

It now follows from Lemma 2.3.4 that $|X| < y$, as X acts on Res(b) faithfully by Lemma 4.2. Hence we have $|K_u| = |K_u : X||X| < \min\{\alpha, y\} \cdot y = \min\{\alpha y, y^2\}$, if $y > 1$.

Since $y \leq x^2$ by Lemma 2.3.1(3) and $x^2 < x^2 + x + 1 = p$, we now have $|K_u| < \alpha y < \alpha p$ for any y. Moreover, $y + 1 \leq p$, $x^2 y + 1 \leq x^4 + 1 < p^2$, and $(xy + 1)(x + 1) \leq (x^3 + 1)(x + 1) < p^2$. Since A is transitive on the set of maximal flags of $\mathcal{G}$ with the stabilizer $K_u = A_{a,l,u}$ of a maximal flag (a, l, u) (see Lemma 4.1.1(2)), it follows from Lemma 3.6.2 that

$$|A| = p\left(\frac{x^2 y + 1}{\alpha + 1}\right)(xy + 1)(y + 1)(x + 1)|K_u|.$$

Now substituting the above inequalities into this formula, we obtain

$$\begin{aligned}|A| &< p \cdot (x^2y + 1) \cdot (xy + 1)(x + 1) \cdot (y + 1) \cdot (\alpha/(\alpha + 1)) \cdot p \\ &< p \cdot p^2 \cdot p^2 \cdot p \cdot 1 \cdot p = p^7.\end{aligned}$$

□

4.5. *Substructure fixed by an involution*

Assume that $|A|$ is even, and let $\mathcal{G}^i$ be the substructure of $\mathcal{G}$ fixed by an involution i of A. The substructure $\mathcal{G}^i$ is extensively investigated in [11, 10]. However, there is a gap in the proof of [11] Lemma 2.6 line -14 to -7 p. 176. (The formula for the number t was miscopied from [15] 2.2.1: This should be $t = (1 + x)(1 + xz)(y - z)$, but not $t = (1 + z)(1 + xz)(y - z)$, as stated.) Since the conclusion $x = z$ in that paragraph was derived from this error, the original proof does not hold as it is. This is not a serious gap, because we can afford a new proof of this fact by modifying the arguments in [11] 2.7.

In fact, in order to establish the main theorem in this paper, we only need the following information on the substructure $\mathcal{G}^i$.

Lemma 4.5.1 *If y is odd, any involution i of A does not fix a plane.*

Proof: Assume that y is odd, and suppose there is an involution $i \in A$ fixing a plane u. Since A_u/K_u is of odd order $p(x + 1)$ by Lemma 4.1.1(1), i is contained in K_u. Let a be any point on u. The involution i fixes a and so acts on the GQ Res(a). The plane $u \in$ Res(a) is fixed by i together with all the lines on u. Then for each line m in Res(a) on u, the involution i acts on the set of y planes through m distinct from u. As y is odd, i fixes at least one of such planes, say v. Since $i \in A_{a,m,v} = K_v$ by Lemma 4.1.1(2), i fixes every line on v. Then it follows from Lemma 2.3.3 that the substructure of the GQ Res(a) fixed by i is a sub GQ of order (x, z) for some $1 \leq z \leq y$.

If $z = y$, i acts trivially on Res(a) and so $i \in K_a$, which contradicts Lemma 4.2. Thus $1 \leq z < y$ and there is a line of Res(a) not fixed by i. Pick a line $m \in Res(a)$ not fixed by i. By Lemma 2.3.2(1), the set $\mathcal{O}$ of lines of Res(a) which are fixed by i and are coplanar with m consists of $xz + 1$ lines. The lines m and m^i are coplanar with each line of $\mathcal{O}$. If m is coplanar with m^i, the unique plane v incident to both m and m^i is stablized by the involution i. As A_v/K_v is of odd order, i acts trivially on Res(v). In particular, $m = m^i$, which is a contradiction. Thus m is not coplanar with m^i. Since Res(a) is of order (x, y), there are exactly $y + 1$ lines of Res(a) coplanar with m and m^i, among which exactly $xz + 1$ lines of $\mathcal{O}$ are fixed by i. However, as $y - xz$ is odd, the involution i should fix at least one line not in $\mathcal{O}$, which is a contradiction. □

Lemma 4.6 *Any Sylow p-subgroup P of A is not contained in the Fitting subgroup $F(A)$ of A and centralizes $F(A)$.*

Proof: Let P be a Sylow p-subgroup of A_u for a plane u. We may assume that $F(A) \neq 1$. As $p = |A|_p$ and $F(A) \trianglelefteq A$, $F(A)$ contains a Sylow p-subgroup of A iff $F(A)$ contains all Sylow p-subgroups of A iff p divides $|F(A)|$. If p divides $|F(A)|$, then P is the unique Sylow p-subgroup of the nilpotent group $F(A)$, and hence $P \lhd A$. As A is transitive on the planes, P fixes all the planes of $\mathcal{G}$. However, this implies that P fixes a line $u \cap v$ for a plane v cocollinear with u, which does not occur by Lemma 4.3. Thus $p \not| |F(A)|$, equivalently $P \not\subseteq F(A)$.

Now assume that P acts non-trivially on $F(A)$. Since $F(A)$ is nilpotent, there is a prime r such that P acts non-trivially on $R := O_r(F(A))$. We have $r \neq p$, by Lemma 4.3 and the claim in the above paragraph. Since $A_u/K_u \cong F_p^{x+1}$, we may apply Lemma 2.4.2 for $B = A_u$, $C = K_u$, P and $m = x + 1$, and conclude that $r^{x+1} \leq |R|$.

On the other hand, we can obtain an upper bound of $|R|$ as follows. Since $P \not\subseteq R = O_r(A)$ and $(R \cap A_u)K_u/K_u \trianglelefteq A_u/K_u \cong F_p^{x+1}$, $R \cap A_u \subseteq K_u$. Then $|R \cap A_u| < \alpha y \leq \alpha x^2$ by Lemma 4.4 and Lemma 2.3.1(3). We have $|R : R \cap A_u| \leq |A : A_u| = (x^2y + 1)(xy + 1)(y + 1)/(\alpha + 1) < (x^2x^2 + 1)(x \cdot x^2 + 1)(x^2 + 1)/(\alpha + 1)$ by Lemma 3.6.2 and Lemma 2.3.1(3). Since $x^i + 1 < 2x^i$ for $i = 2, 3, 4$, $|R| = |R : R \cap A_u||R \cap A_u| < 8x^{4+3+2}x^2 = 8x^{11}$.

It follows from the above bounds for $|R|$ that $2^{x+1} \leq r^{x+1} \leq |R| < 8x^{11}$, and so $2^{x-2} < x^{11}$. Then $x < 79$, and so $x = 8$ and $p = 73$ by Lemma 4.1.1(1).

In this case, we may verify that the possible values for y are 4, 6, 8, 10, 13, 16, 20, 24, 28, 34, 40, 48, 55, 56 and 64 by Lemma 2.3.1(2). If y is even, as x is even by Proposition 2.2.2, K_u contains a Sylow 2-subgroup of A by Lemma 3.6.2. In particular, the largest normal 2-subgroup $O_2(A)$ of A is contained in K_w for each plane w, and hence $O_2(A) \leq \cap_{w \in \mathcal{G}_2} K_w = 1$. If y is odd, $y = 13$ or 55. It follows from Lemma 4.5.1 and Lemma 3.6.2 that $|A|_2 = (y + 1)_2 = 2$ or 2^3. If a Sylow p-subgroup P of A_u acts non-trivially on $O_2(A)$, then $|O_2(A)| \geq 2^{x+1} = 2^9$ by Lemma 2.4.2. Hence P acts trivially on $O_2(A)$. Thus in any case, we have $[P, O_2(A)] = 1$.

Suppose $F(A) \cap K_u$ contains an r-subgroup on which P acts non-trivially. Then r is odd by the above paragraph, and $3^{x+1} \leq r^{x+1} \leq |K_u| < y^2 \leq x^4$ by Lemma 2.4.2 and Lemma 4.4. However, $x = 8$ does not satisfy this inequality. Thus $F(A) \cap K_u \leq C_{F(A)}(P)$.

Now, as we saw in the first paragraph, P acts coprimely on $F(A)$. Then $[F(A), P]$ is a group of order $|F(A)|/|C_{F(A)}(P)|$, which is an odd number dividing $|F(A)|/|F(A) \cap K_u|$ as we saw above. Thus $|[F(A), P]|$ divides $[A : K_u]$. By Lemma 2.4.2, for each prime divisor r of $|[F(A), P]|$, $r^{x+1} = r^9$ divides $|[F(A), P]|$ and so $[A : K_u]$. However, for each possible value for y above we can compute $[A : K_u]$ by Lemma 3.6.2 and conclude that there is no such prime r. Hence we have a final contradiction, and obtain that $[P, F(A)] = 1$. □

Lemma 4.7 *Assume that y is even and $x \neq 8$. Then a Sylow p-subgroup P centralizes a Sylow 2-subgroup T of A contained in K_u.*

Proof: As y is even, the number of maximal flags is odd by Lemma 3.6.2 (note that x is even by Proposition 2.2.2). Then $K_u = A_{a,l,u}$ (see Lemma 4.1.1(2)) contains a Sylow 2-subgroup of A. Since $|A|_p = p$ by Lemma 4.3, a Sylow p-subgroup P contained in A_u acts coprimely on K_u, and hence there is a P-invariant Sylow 2-subgroup T of K_u.

Assume that $[P, T] \neq 1$. Then we can apply Lemma 2.4.2 for $B = A_u$, $C = K_u$, P, $m = x + 1$, $R = T$ and $r = 2$. Then we have $|T| \geq 2^{x+1}$. On the other hand, as T is a subgroup of K_u, we have $y^2 > |T|$ by Lemma 4.4. Thus we have $y^2 > 2^{x+1}$. If $x \neq 8$, it follows from Lemma 4.1.1(1) and Lemma 2.3.1(3) that $2^{x+1} > x^4 \geq y^2$, which is against the inequality $y^2 > 2^{x+1}$. □

Lemma 4.8 *There is exactly one component L of A. The group L contains all the Sylow p-subgroups of A, and $C_A(L)$ is of odd order. Moreover, L is transitive on $\mathcal{G}_0$.*

Proof: Let $F^*(A) = F(A)E(A)$ be the generalized Fitting subgroup of A, and let P be a Sylow p-subgroup of A contained in A_u for a plane u. Suppose that $F(A) = F^*(A)$. By Lemma 4.6, $P \leq C_A(F^*(A)) \leq F^*(A) = F(A)$ by the fundamental property of the generalized Fitting subgroup. However, this contradicts the fact that $P \not\leq F(A)$ remarked in Lemma 4.6. Thus $E(A)$ contains at least one component.

Suppose $E(A)$ does not contain P. Then P acts coprimely on $E(A)$. If P does not normalize some component L, $E(A)$ contains at least p isomorphic components. As each component has at least $60 = |A_5|$ elements, we have $|A| \geq |E(A)|p \geq 60^p p$. By Lemma 4.4, this implies that $p^6 > 60^p$, which contradicts Lemma 4.1.1(1). Thus each component L of A is normalized by P. If P centralizes all the components of A, then $P \leq C_A(F^*(A)) \leq F^*(A)$, as $[P, F(A)] = 1$ by Lemma 4.6. However, this implies that p divides $E(A)$ and hence $P \leq E(A)$, which contradicts the assumption.

Hence there is at least one component L on which P acts coprimely and non-trivially. Then $S := L/Z(L)$ is a non-abelian simple group such that Out(S) contains a subgroup of odd prime order p, which is greater than 71 by Lemma 4.1.1(1), and $|S| < |A|/p < p^6$ by Lemma 4.4. Note that $2^p > p^6$ for such an integer p.

We now use the classification of finite simple groups to verify that there is no such simple group. Clearly, S is not an alternating group or a sporadic group. Thus $S = X(q)$ is of Lie type for some Dynkin diagram X and a prime power $q = r^f$ (here we follow the notation in ATLAS [4], Section 3, Page xvi). Then we can easily observe that either p divides f, or p divides $(n+1, q \pm 1)$ and $S \cong A_n(q)$ or ${}^2A_n(q)$. In the first case, we have $q = r^f \geq 2^p > p^6$, and hence $|S| \geq q > p^6$, a contradiction. In the latter case, $p \leq n+1$, and so $|S| \geq q^{n(n+1)/2} > 2^p > p^6$, a contradiction.

Hence $E(A)$ contains P. As $|A|_p = p$, there is a unique component L with $|L|_p = p$. Thus L is a normal subgroup of A and contains all the Sylow P-subgroups of A.

Suppose that L is not transitive on $\mathcal{G}_0$. Then the L-orbits on $\mathcal{G}_0$ form a proper (may be the trivial) system of imprimitivity blocks under the action of A. By Lemma 3.8.1, $a^L \cap \mathcal{G}_0(u) = \{a\}$ for any point a on a plane u. For any $g \in L \cap A_u$, a^g lies in $a^L \cap \mathcal{G}_0(u) = \{a\}$, and so $a^g = a$. Thus $L \cap A_u$ fixes every point on u. This implies that $L \cap A_u \leq K_u$ and so $P \not\leq A_u \cap L$, which is a contradiction. Thus L is transitive on $\mathcal{G}_0$.

Suppose that there is an involution i of $C_A(L)$. Since L contains every Sylow p-subgroup of A, i centralizes each of them. Assume y is odd. As x is even by Proposition 2.2.2, there are an odd number of lines of $\mathcal{G}$. Thus there is a line l fixed by i. As i does not fix any plane by Lemma 4.5.1, any plane v through l is cocollinear with v^i and $l = v \cap v^i$. Now take a Sylow p-subgroup Q of A contained in A_v. Since $[Q, i] = 1$, Q should fix a line $l = v \cap v^i$, which contradicts Lemma 4.3.

Assume y is even. Since $K_u = A_{a,l,u}$ for a plane u contains a Sylow 2-subgroup of A by Lemma 4.1.1(2), there is a plane, say u, such that $i \in K_u$. If there is a line l on u and a plane v through l not fixed by i, then $v \cap v^i = l$ is fixed by a Sylow p-subgroup Q of A contained in A_v, which contradicts Lemma 4.3. Thus every plane cocollinear with u is fixed by i. In particular, for any point a on u, the action of i on the GQ Res(a) fixes u and every plane cocollinear with u. Since A_v/K_v is of odd order $p(x+1)$ for any plane v, the involution i fixes all the lines on each plane cocollinear with u. By Lemma 2.3.3, this imples that $i \in K_a$. However, $K_a = 1$ by Lemma 4.2.

Hence, in any case, we have a contradiction. Thus $C_A(L)$ is of odd order. In particular, L is the unique component of A. □

5. Proof of the Theorem

We use the notation in Section 4. Let L be the unique component of A (see Lemma 4.8), and $S := L/Z(L)$. By Lemma 4.4, we have $|S| \leq |A| < p^7$. Moreover, we can estimate the order of a Sylow 2-subgroup of S in terms of p.

If y is even, the number of maximal flags is odd by Lemma 3.6.2, and so $A_{a,l,u} = K_u$ contains a Sylow 2-subgroup of A. By Lemma 4.4 and Lemma 2.3.1(3), we have $|A|_2 < y^2 < (x^2+x+1)^2 = p^2$. Moreover, a Sylow p-subgroup P of A centralizes a Sylow 2-subgroup of A by Lemma 4.6.

If y is odd, A_u is of odd order for any plane u by Lemma 4.5.1. Then the 2-part $|A|_2$ of $|A|$ divides $(x^2y+1)(xy+1)(y+1)/(\alpha+1) = |\mathcal{G}_2| = |A : A_u|$ (see Lemma 3.6.2). As x is even by Proposition 2.2.2, $|A|_2$ divides $y+1$. In particular, $|A|_2 \leq y+1 < x^2+x+1 = p$ by Lemma 2.3.1(3). Furthermore, if $y = 1$, we have $|A|_2 = 2$ and hence A is a solvable group, which contradicts the existence of L in Lemma 4.8. Thus $y > 1$.

Hence, it follows from the above conclusions and Lemmas 4.1.1, 4.3, 4.4 that the non-abelian simple group S satisfies the following properties:

(i) $p = x^2+x+1 = |S|_p$,
(ii) $|S| < p^7$,
(iii) $p = x^2+x+1$ is a prime with $p > 207,360,244,800,073$ and $x > 14,400,008$, or $p = 73$ and $x = 8$.
(iv) We have $y > 1$ and $|S|_2 < y^2 < p^2$. If y is odd, $|S|_2 \leq y+1 < x^2+x+1 = p$. Moreover, if y is even and $x \neq 8$, a Sylow 2-subgroup of S is centralized by an element of order p in S.

We first make a list of simple groups S with these properties, using the classification of finite simple groups and some calculations. (In fact, we can eliminate the case y even and $x \neq 8$ if we observe that no finite simple group satisfies the Condition (iv) by examining their subgroup structures. However, we do not need such examination, because the other conditions are strong enough to eliminate every possibility for S.)

In view of Atlas [4], there is no sporadic simple group S with a prime divisor p satisfying (iii). If S is the alternating group of degree m, then $p \leq m$ and $|S| = m!/2 \geq p\cdot(p-1)!/2 \geq p \cdot 2^{p-3}$, as each of the $p-2$ integers in the interval $[2, p-1]$ is at least 2. Then the

Condition (ii) above implies that $p^6 > 2^{p-3}$, and therefore $p = x^2 + x + 1 \leq 33$, which contradicts the Condition (iii).

Thus S must be a finite simple group of Lie type. We write $S = X_n(q)$, where X_n shows the associated Dynkin diagram of rank n (with the order of graph automorphism if S is of twisted type) and q is the size of the defining field. Here we follow the Atlas notation [4], and for example, we use ${}^2A_n(q)$ (not ${}^2A_n(q^2)$) to denote $PSU_n(q^2) = U_n(q)$.

Lemma 5.1 *Assume that $S = X_n(q)$ is a simple group of Lie type satisfying the properties* (i) (ii) (iii) (iv). *Then one of the following holds, where $q = r^f$ is a power of an odd prime r distinct from p, except in the Case* (1) *and* (7).

(1) $S \cong A_1(q) \cong L_2(q)$ *for some* $q = r^f$ *with an odd prime* r.
(2) $S \cong A_n(q) \cong L_{n+1}(q)$ *or* $S \cong {}^2A_n(q) \cong U_{n+1}(q)$ *for some* $n = 2, \ldots, 6$.
(3) $S \cong B_2(q) = C_2(q) \cong S_4(q) = O_5(q)$.
(4) $S \cong {}^2D_4(q) \cong O_8^-(q)$.
(5) $S \cong {}^3D_4(q)$ *for some* $q = r^f$.
(6) $S \cong G_2(q)$ *for some* $q = r^f$ *with* $q > 2$.
(7) $S \cong {}^2G_2(q)$ *for some* $q = 3^{2k+1}$ *with* $k \geq 1$.

Proof: We write $q = r^f$ for a prime r. For $\varepsilon = \pm 1$, we use the symbol ${}^\varepsilon A_n$ to denote A_n and ${}^2A_n$ by ${}^1A_n$ and ${}^{-1}A_n$ respectively. We also use the similar convention ${}^\varepsilon D_n$ for D_n and ${}^2D_n$. The order $|X_n(q)|$ is given as follows ([4]): For short, here we instead give $d|X_n(q)|$ and the order d of the center of some covering group of $X_n(q)$.

$X_n(q)$	$d\|X_n(q)\|$	d
${}^\varepsilon A_n(q)$ $(n \geq 1 \text{ if } \varepsilon = 1, n \geq 2 \text{ if } \varepsilon = -1)$	$q^{n(n+1)/2}\Pi_{i=1}^n(q^{i+1} - \varepsilon^{i+1})$	$(n+1, q-\varepsilon)$
$B_n(q)$ $(n \geq 2)$	$q^{n^2}\Pi_{i=1}^n(q^{2i} - 1)$	$(2, q-1)$
$C_n(q)$ $(n \geq 3)$	$q^{n^2}\Pi_{i=1}^n(q^{2i} - 1)$	$(2, q-1)$
${}^2B_2(q)$ $(q = 2^{2k+1})$	$q^2(q^2+1)(q-1)$	1
${}^\varepsilon D_n(q)$ $(n \geq 4)$	$q^{n(n-1)}(q^n - \varepsilon)\Pi_{i=1}^{n-1}(q^{2i} - 1)$	$(4, q^n - \varepsilon)$
${}^3D_4(q)$	$q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)$	1
$G_2(q)$ $(q > 2)$	$q^6(q^6-1)(q^2-1)$	1
${}^2G_2(q)$ $(q = 3^{2k+1}, k \geq 1)$	$q^3(q^3+1)(q-1)$	1
$F_4(q)$	$q^{24}(q^{12}-1)(q^8-1)(q^6-1)(q^2-1)$	1
${}^2F_4(q)$ $(q = 2^{2k+1}, k \geq 0)$	$q^{12}(q^6+1)(q^4-1)(q^3+1)(q-1)$	1
$E_6(q)$	$q^{36}(q^{12}-1)(q^9-1)(q^8-1)$ $(q^6-1)(q^5-1)(q^2-1)$	$(3, q-1)$
${}^2E_6(q)$	$q^{36}(q^{12}-1)(q^9+1)(q^8-1)$ $(q^6-1)(q^5+1)(q^2-1)$	$(3, q+1)$
$E_7(q)$	$q^{63}(q^{18}-1)(q^{14}-1)(q^{12}-1)$ $(q^{10}-1)(q^8-1)(q^6-1)(q^2-1)$	$(2, q-1)$
$E_8(q)$	$q^{120}(q^{30}-1)(q^{24}-1)(q^{20}-1)(q^{18}-1)$ $(q^{14}-1)(q^{12}-1)(q^8-1)(q^2-1)$	1

If $p = r$, the r-part of $|S|$ is just r, which is realized only when $X_n = A_1$ and $q = r$, that is, $S \cong L_2(p)$. This is contained in the Case (1) of the claim. Thus we may assume that $p \neq r$.

We will now show that r is odd, except possibly for the Case (1). Assume that $r = 2$. First, consider the case y is even. In this case, S is a simple group of Lie type in characteristic 2, and so the normalizer of a Sylow 2-subgroup U of S is a Borel subgroup, which is a semidirect product of H by a Cartan subgroup T. If S is of untwisted type, each non-trivial element of H corresponds to a non-trivial character χ of the integral lattice generated by roots into the defining field, which sends under conjugation an element $x_r(t)$ of a root subgroup of U to $x_r(\chi(r)t)$ (see e.g. [3] p. 100). In particular, each non-trivial element of T acts non-trivially on U. For S twisted, H and U are subgroups of an untwisted group of Lie type defined on a larger field, and we can obtain the similar formula (see e.g. [3] p. 194), and hence the similar observation holds. Thus in any case there is no non-trivial element of odd order of S centralizing a Sylow 2-subgroup of S.

Then it follows from the Condition (iv) that $x = 8$ and so $p = 73$, if y is even. Moreover, we may assume that a Sylow p-subgroup P of S does not centralize a Sylow 2-subgroup T of A contained in K_u ($u \in \mathcal{G}_2$). As we saw in the proof of Lemma 4.7, then it follows from Lemma 2.4.2 that T is of order at least $2^{x+1} = 2^9$. As $|T| \leq |A|_2 < p^2 = 73^2$, we have $2^9 \leq |T| \leq 2^{13}$. Observing the list of orders $|X_n(2^f)|$ above, we can determine the groups $X_n(2^f)$ with $2^9 \leq |X_n(2^f)|_2 \leq 2^{13}$ and $p = 73 \mid |X_n(2^f)|$. In fact, the group ${}^1A_1(2^9) = L_2(2^9)$ is the unique such group, and this belongs to the Case (1).

Hence we may assume that y is odd. Now notice that p is an odd prime dividing the square free part of $|S|$ by the Condition (i). If $S \cong {}^{\varepsilon}A_n(q)$ for $\varepsilon = \pm 1$, then p divides $q^{i+1} - \varepsilon^{i+1}$ for some i ($1 \leq i \leq n$), but not $q - \varepsilon$, as $(q - \varepsilon)^2$ divides $|S|$. Thus the prime p divides $(q^{i+1} - \varepsilon^{i+1})/(q - \varepsilon)$. Then $p \leq (q^{n+1} - \varepsilon^{n+1})/(q - \varepsilon) < q^{n+1}$, and so $p < q^{n(n+1)/2} = |S|_2$. This contradicts the Condition (iv). Hence r is an odd prime if $S \cong {}^{\varepsilon}A_n(q)$.

If $S \cong B_n(q)$ or $C_n(q)$, p divides $q^i - 1$ or $q^i + 1$ for some $i \leq n$, and so $p \leq q^n + 1$. As y is odd, we have $|S|_2 = q^{n^2} < p \leq q^n + 1$ by the Condition (iv). However, as $q^n \geq 2$, this is impossible. Similarly, we can eliminate all the cases, except when $X_n(q) = {}^2B_2(q)$ and $p = q^2 + 1$, and $X_n(q) = {}^2G_2(q)$ and $p = q^3 + 1$. As $p - 1 = x(x + 1)$ ($x > 1$) is not a power of a prime, these cases do not occur.

Thus we conclude that r is an odd prime, except possibly for the Case (1). In particular $X_n \neq {}^2B_2, {}^2F_4$.

We can also eliminate the cases $X_n = F_4, E_6, {}^2E_6, E_7$ and E_8, as follows. For example, consider the case $S = F_4(q)$. Then $|S| = q^{24}A^2B$ with $A = (q^6 - 1)(q^2 - 1)(q^2 + 1)$ and $B = (q^4 + 1)(q^2 - q + 1)(q^2 + q + 1)$. As $|S|_p = p$, p divides B, and hence $p \leq q^4 + 1$. As $q^4 + 1$ is an even integer, $p \neq q^4 + 1$ and $p < q^4$, as $p \neq q^4$. In particular, $q^{24} > p^6$ and $B > p$. Thus $|S| > p^7$, which contradicts the Condition (ii). By the same argument, we can immediately eliminate the above cases.

It remains to restrict n in the cases $X_n = A_n, {}^2A_n, B_n, C_n, D_n$ and ${}^2D_n$. Consider the case $S \cong {}^{\varepsilon}A_n(q)$ for some $n \geq 2$ and $q = r^f$ with a prime r. Since $|S|_r = q^{n(n+1)/2} \geq q^2$ for $n \geq 2$, the condition $|S|_p = p$ implies that p is a prime distinct from r dividing exactly one of $q^{i+1} - \varepsilon^{i+1}$ ($i = 1, \ldots, n$). In particular, p is prime to $q - \varepsilon$. If $i + 1$

is even or $\varepsilon = 1$, $q^{i+1} - \varepsilon^{i+1} = q^{i+1} - 1$ and hence p divides $(q^{i+1} - 1)/(q - 1) = q^i + \cdots + q + 1 < 2q^i$. If $i + 1$ is odd and $\varepsilon = -1$, $q^{i+1} - \varepsilon^{i+1} = q^{i+1} + 1$ and p divides $(q^{i+1} + 1)/(q + 1) = q^i - q^{i-1} \cdots - q + 1 < 2q^i$. In both cases, we have $p < 2q^i \leq 2q^n$. Thus $2^7 q^{7n} > p^7 > |S| = |{}^{\varepsilon}A_n(q)|$ by the Condition (ii). On the other hand, as $q^{i+1} - \varepsilon^{i+1} > 2q^i$ for $i = 1, \ldots, n$, we have $\Pi_{i=1}^{n}(q^{i+1} - \varepsilon^{i+1}) > 2^n q^{n(n+1)/2}$. Hence

$$(2^{7/n} q^7)^n > |S| > q^{n(n+1)/2} \cdot q^{n(n+1)/2} \cdot 2^n/(n + 1) > (q^{n+1})^n.$$

Then we obtain $2^{7/n} > q^{n-6}$. If $n \geq 7$, $2 \geq 2^{7/n} > q$, while q is an odd prime power. Hence $n \leq 6$ and we obtain the Case (2) in the claim.

Next consider the case $S \cong B_n(q)$ or $C_n(q)$ with $n \geq 2$ for a power $q = r^f$ of a prime r. In this case, we have $|S| = |B_n(q)| = |C_n(q)| = q^{n^2} \Pi_{i=1}^{n}(q^{2i} - 1)/(2, q - 1)$. As $|S|_r = q^{n^2}$ is divided by r^2, $p \neq r$ and hence p divides exactly one of $q^i - 1$ or $q^i + 1$ $(i = 1, \ldots, n)$. In particular, $p \leq q^n + 1$. As $q^i \pm 1$ is even, while p is odd, we have $p \leq q^i \leq q^n$. On the other hand, we note that $q^{2i} - 1 > 2q^{2i-1}$ for $i = 1, \ldots, n$, as $q^{2i-1}(q - 2) > 1$. Then $|S| > q^{n^2} 2^n q^{1+3+\cdots+(2n-1)}/2 = 2^{n-1} q^{n^2+n^2}$. Thus it follows from the Condition (ii) that

$$q^{7n} > p^7 > |S| > 2^{n-1} q^{2n^2},$$

or $1 > 2^{n-1} q^{2n^2-7n}$. Then $(2n - 7)n < 0$, and so $n \leq 3$.

If $n = 3$, the square free part of $|S|$ divides $(q^2 - q + 1)(q^2 + q + 1)(q^2 + 1)$. As p divides the square free part of $|S|$, p divides $q^2 + 1$ or $q^2 \pm q + 1$. Then we can obtain a better bound $p \leq q^2 + q + 1 < 2q^2$ in this case. By the Condition (ii), we have

$$2^7 q^{14} > p^7 > |S| > 2^2 q^{18}.$$

Then we have $2^5 > q^4$, which implies $q = 2$. However, then we have $p < 2q^2 = 8$, which contradicts the Condition (iii). Thus $n = 2$ and we obtain the Case (3) in the claim.

Finally consider the case $S \cong {}^{\varepsilon}D_n(q)$ for some $n \geq 4$ and a power $q = r^f$ of a prime r. As $|S|_p = p$, p is a prime distinct from r dividing exactly one of $q^n - \varepsilon$, $q^i + 1$ and $q^i - 1$ $(i = 1, \ldots, n - 1)$. In particular, p is prime to $q - \varepsilon$, and $p \leq q^n + 1$. Then $p \neq q^n + 1$ as $q^n + 1$ is even, and hence $p \leq q^n$. Note that $q^{2i} - 1 > 2q^{2i-1}$ and $\Pi_{i=1}^{n-1}(q^{2i} - 1) > 2^{n-1} q^{1+3+\cdots+(2n-3)} = 2^{n-1} q^{(n-1)^2}$, as we saw in the above paragraph. It now follows from the Condition (ii) that

$$q^{7n} > p^7 > |S| > q^{n(n-1)}(q^n - \varepsilon) \cdot 2^{n-1} q^{(n-1)^2}/4,$$

and hence $1 > q^{2n^2-10n+1}(q^n - \varepsilon)$. This implies that $2n^2 - 10n + 1 < 0$ or equivalently $n \leq 4$. Thus $n = 4$.

If $\varepsilon = 1$ and $n = 4$, $|S| = q^{12}(q^2 - 1)^3(q^2 + 1)^2(q^2 + q + 1)(q^2 - q + 1)$. Then $p \leq q^2 + q + 1 < 2q^2$ and we have

$$2^7 q^{14} > p^7 > |S| > q^{12}(q^4 - 1)2q^9,$$

or $2^6 > q^7(q^4 - 1)$, which is a contradiction. Hence we obtain the Case (4) in the claim, and we exhausted all the remaining cases. □

In order to eliminate the remaining cases in Lemma 5.1, we will estimate $N_A(P)/C_A(A)$ for a Sylow p-subgroup P of L (and so A) contained in the stabilizer A_u for a plane u, and then to obtain the lower bound of $|S|$ in terms of x and so p. Then we obtain the contradiction by the Condition (ii).

For that purpose, we consider the covering (general) linear group G of S acting on its natural module V over an extension field of $GF(q)$. In the Case (2) in Lemma 5.1, we have $G \cong GL_{n+1}(q)$ acting on the $(n + 1)$-space over $GF(q)$, or $G \cong GU_{n+1}(q)$ acting on the $(n+1)$-unitary space over $GF(q^2)$. Let $\tilde{P}$ ($\cong P$) be the commutator subgroup of the inverse image of $PZ(L)/Z(L)$ in G. In the Case (2), we can show that the commutator space $[V, \tilde{P}]$ is an irreducible modules for $\tilde{P}$ (see the proof of Lemma 5.2). Applying Lemma 2.4.1, then we conclude that $N_G(\tilde{P})/C_G(\tilde{P})$ is a subgroup of the cyclic group of order $\dim([V, \tilde{P}])$. Since $A/LC_A(L)$ is a subgroup of the outer automorphism group $\mathrm{Out}(S)$, we can conclude that $N_A(P)/C_A(P)$ is a subgroup of the cyclic group of order $\dim[V, \tilde{P}]$ (which corresponds to $N_G(\tilde{P})/C_G(\tilde{P})$) extended by the field automorphism group and the graph automorphism group. On the other hand, as A_u/K_u is a Frobenius group F_p^{x+1}, A_u/PK_u is isomorphic to a subgroup of $N_A(P)/C_A(P)$. Thus $x + 1$ divides the odd part of $|N_A(P)/C_A(P)|$, and therefore, we obtain a bound $r^{(x+1)/\dim V} \leq q$, where r is the prime divisor of q. This gives a lower bound of $|S|$ in terms of x and so $p = x^2 + x + 1$, which together with the Condition (ii) will be enough to eliminate the Cases (2). Similarly we can eliminate the Cases (3)–(7) in Lemma 5.1. The Case (1) will be treated separately in Lemma 5.3.

Lemma 5.2 *The cases* (2)–(7) *in Lemma* 5.1 *do not occur.*

Proof: We also use the convention used in the proof of Lemma 5.1. As we descibed above, we first take the covering general linear group G for S and the action on its natural module V. We also use $\tilde{S}$ and $\tilde{P}$ to denote the inverse image in G of S and the commutator subgroup of the inverse image of $PZ(L)/Z(L)$, respectively. Explicitly, G and V are given as follows, where the orthogonal space for $GO_8^-(q)$ is of minus type, but of plus type (over $GF(q^3)$) for ${}^3D_4(q)$, and through the representation on the 7-dimensional orthogonal space for $G_2(q)$ and ${}^2G_2(q)$, we have ${}^2G_2(q) \leq G_2(q) \leq GO_7(q)$.

Case in 5.1	S	G	V
(2)	$A_n(q)$ $(n = 2, \ldots, 6)$	$GL_{n+1}(q)$	$(n + 1)$-dim. space over $GF(q)$
(2)	${}^2A_n(q)$ $(n = 2, \ldots, 6)$	$GU_{n+1}(q^2)$	$(n + 1)$-dim. unitary space over $GF(q^2)$
(3)	$C_2(q)$	$Sp_4(q)$	4-dim. symplectic space over $GF(q)$
(4)	${}^2D_4(q)$	$GO_8^-(q)$	8-dim. orthogonal space over $GF(q)$
(5)	${}^3D_4(q)$	${}^3D_4(q)$	8-dim. orthogonal space over $GF(q^3)$
(6)	$G_2(q)$	$G_2(q)$	7-dim. orthogonal space over $GF(q)$
(7)	${}^2G_2(q)$	${}^2G_2(q)$	7-dim. orthogonal space over $GF(q)$

If the Case (2) holds, $G \cong GL_{n+1}(q)$ and $\tilde{S} \cong SL_{n+1}(q)$, or $G \cong GU_{n+1}(q^2)$ and $\tilde{S} \cong SU_{n+1}(q^2)$. Let $W := [V, \tilde{P}]$ be the commutator subspace of V under the action of $\tilde{P}$. As $\tilde{P}$ acts coprimely on V, $V = W \oplus C_V(\tilde{P})$ and W is the direct sum of irreducible $\tilde{P}$-modules on which $\tilde{P}$ acts fixed-point freely. In particular, for any non-trivial $\tilde{P}$-submodule of dimension $i+1$ of W ($1 \leq i \leq \dim W \leq n+1$), p divides $q^{i+1} - 1$ if $\varepsilon = 1$ and p divides $q^{2(i+1)} - 1$ if $\varepsilon = -1$. Now the prime p divides exactly one of $q^{j+1} - \varepsilon^{j+1}$ ($j = 1, \dots, n$), as we saw in the proof of Lemma 5.1. Hence $\tilde{P}$ acts irreducibly on the commutator space W if $\varepsilon = 1$.

We will show that $\tilde{P}$ also acts irreducibly on W for the case $\varepsilon = -1$. First note that p is prime to $q^2 - 1$. For, otherwise, p divides $q-1$ or $q+1$, and so $p \leq (q+1)/2 < q$ as $q \pm 1$ is even by Lemma 5.1. Since $n \geq 2$, $|S| = |U_{n+1}(q)|$ is a multiple of $q^3(q^2-1)(q^3+1)/(q+1)$. However, as $q^3(q-1)(q^3+1) \geq p^3 \cdot p \cdot p^3 = p^7$, this contradicts the Condition (ii). Thus p is prime to $q^2 - 1$. In particular, there is no $\tilde{P}$-submodule of W of dimension 1.

Next, suppose that W has $\tilde{P}$-subspaces of dimension i and $i+1$ for some $2 \leq i \leq \dim W - 1$. Then p divides both $q^{2i} - 1$ and $q^{2(i+1)} - 1$. However, this implies that p divides g.c.d.$(q^{2i} - 1, q^{2(i+1)} - 1) = q^2 - 1$, which contradicts the above remark.

Now, suppose that W contains the direct sum of $\tilde{P}$-subspaces of dimension 2 and i for some $i \geq 4$, or the direct sum of two $\tilde{P}$-subspaces of dimension 3. In the former case, p divides $q^4 - 1$, and hence p divides $q^2 + 1$ by the above remark. Then $p \leq (q^2+1)/2 < q^2$ as q is odd by Lemma 5.1. In the latter case, p divides $q^6 - 1$, and hence divides $q^2 - q + 1$ or $q^2 + q + 1$. Then in any case $p < 2q^2$. However, since we have $\dim(W) \geq 2 + i \geq 6$ in both cases, $|S| = |U_{n+1}(q)|$ is a multiple of $q^{21} = q^7 \cdot q^{14} > (q/2)^7 p^7 \geq p^7$, which contradicts the Condition (ii). Thus W does not contain the direct sum of such $\tilde{P}$-submodules.

Since $\dim W \leq n + 1 \leq 7$ by Lemma 5.1, it follows from the remarks in the above paragraphs that W is an irreducible $\tilde{P}$-module, as we claimed.

Thus the normalizer $N := N_G(\tilde{P})$ preserves the decomposition $V = W \oplus C_V(\tilde{P})$ and the group $N/C_N(W)$ is a linear group on W with a cyclic normal group $\tilde{P}C_N(W)/C_N(W)$ acting irreducibly on W. Hence $N_G(\tilde{P})/C_G(\tilde{P})$ is isomorphic to a subgroup of the cyclic group of order $\dim(W)$ by Lemma 2.4.1.

Now, note that $A/C_A(L)$ is a subgroup of $\mathrm{Aut}(S)$, which is an extension of $G/Z(G)$ by the field automorphism corresponding to $\mathrm{Aut}(GF(q)) \cong Z_f$ and possibly the graph automorphism group of order 2. Thus $N_A(P)/C_A(P)$ is at most the extension of $N_G(\tilde{P})/C_G(\tilde{P})$ by the above automorphism group of S. In particular, $|N_A(P)/C_A(P)|$ divides $\dim W \cdot f \cdot 2$. On the other hand, as A_u/K_u is the Frobenius group F_p^{x+1} with the kernel PK_u/K_u, $N_A(P)/C_A(P)$ contains a cyclic group of odd order $x+1$. Thus $x+1$ divides $\dim W \cdot 2f$ and so $\dim W \cdot f$.

Now consider the case $x = 8$. Since $2 \leq \dim W \leq n + 1 \leq 7$ by Lemma 5.1, either $x + 1 = 9$ divides f or $\dim W = 3$ and f is a multiple of 3. We first consider the latter case. We write $q = s^3$. Since $p = 73$ divides $q^3 - 1 = s^9 - 1$, $s \neq 3, 5$ nor 7. Since s is odd by Lemma 5.1, $s > 8 = x$ and so $q = s^3 > x^3$. In the former case, $q = r^f \geq 3^9 > 8^3 = x^3$. Thus we have $q > x^3$ if $x = 8$.

Now consider the case $x \neq 8$. Since $x + 1$ divides $\dim W \cdot f$, as we saw above, we have $x + 1 \leq (n+1)f$. Since $n \leq 6$ by Lemma 5.1, this implies that $(x+1)/7 \leq f$. Then $q = r^f \geq r^{(x+1)/7} \geq 3^{(x+1)/7} > x^3$ by the Condition (iii).

Hence in any case we have $q > x^3$. As $p = x^2 + x + 1 < 2x^2 = 2(x^3)^{(2/3)}$, this implies that $p < 2q^{2/3}$. It follows from the Condition (ii) and the lower bound for $|S|$ in the proof of Lemma 5.1 that $2^7 q^{(14/3)} > p^7 > q^{n(n+1)}$. As $n \geq 2$, we have $2^7 > q^{6-(14/3)} = q^{4/3}$, and so $2^6 > q$. However, as $q > x^3$, we have $2^2 > x$, which contradicts the Condition (iii), and hence the Case (2) is eliminated.

The other cases can also be eliminated by repeating the similar arguments to those in the paragraph above. Note that the Schur multiplier of S is of even order or 1, in the remaining cases, and so $S = L$ by Lemma 4.8.

If the Case (3) holds, then p divides the square free part of $|S|$, which is $q^2 + 1$. As $(q^2 + 1, q^i - 1) = 2$ for $i = 1, 2, 3$, $\tilde{P}$ acts irreducibly on the 4-dimensional space V over $GF(q)$. Thus $N_G(\tilde{P})/C_G(\tilde{P})$ is a subgroup of the cyclic group of order 4 by Lemma 2.4.1. Since Aut(S) is the extension of $G/Z(G)$ by the field automorphism corresponding to Aut$(GF(q)) \cong Z_f$ (as q is odd), $N_A(P)/C_A(P)$ is at most the extension of $N_G(\tilde{P})/C_G(\tilde{P})$ by Z_f. Since $N_A(P)/C_A(P)$ contains a cyclic group of odd order $x + 1$ induced from A_u/PK_u, $x + 1$ divides f. Then $q = r^{x+1} \geq 3^{x+1} > x^3$ as $x > 2$, and $2^7 q^{14/3} > p^7$ as $2x^2 > p = x^2 + x + 1$. Since $p^7 > |S| > 2q^8$ as we saw in the proof of Lemma 5.1, we have $2^6 > q^{10/3}$ and so $2^{18} > q^{10} > x^{30}$. However, x is an integer.

If the Case (4) holds, p divides $q^4 + 1$ or $q^2 \pm q + 1$, the divisors of the square free part of $|S|$. In the latter case, we obtain a contradiction by the argument in the proof of Lemma 5.1. If the former case holds, as $(q^4 + 1, q^i - 1) = 2$ for $i = 1, \dots, 7$, $\tilde{P}$ is irreducible on the 8-dimensional space V over $GF(q)$. Then by the same argument as above, we can conclude that $x + 1$ divides $|N_G(\tilde{P})/C_G(\tilde{P})|$, which is a divisor of $8f$. In particuler, $x + 1$ divides f, and so $q \geq 3^{x+1} > x^3$. Then we have a contradiction $2^7 q^{14/3} > p^7 > q^{12}(q^4 + 1)2q^9$.

If the Case (5) holds, p divides $q^4 - q^2 + 1$. As $(q^4 - q^2 + 1, q^{3i} - 1) = 2$ for $i = 1, \dots, 7$ except $i = 4$, either $\tilde{P}$ is irreducible on V or V is the direct sum of the two $\tilde{P}$-modules of dimension 4, at least one of which is an irreducible $\tilde{P}$-module. In each possible case, we can conclude that $N_G(\tilde{P})/C_G(\tilde{P})$ is a 2-group by Lemma 2.4.1. (The information on the normalizers of cyclic subgroups of S is available in [8]). As Out(S) is the field automorphism, the same argument as above shows that $x + 1$ divides $3f$. If $x \neq 8$, we have $q \geq 3^f \geq 3^{(x+1)/3} > x^3$ as $x > 16$ by the Condition (iii). Then we have a contradiction $2^7 q^{14/3} > p^7 > |S| > q^{12}$. If $x = 8$, f is a multiple of 3, and hence $q \geq 3^3$. $3^{28} > 73^7 = p^7 > |S| > q^{12} > 3^{36}$, which is a contradiction.

If the Case (6) holds, p divides $q^2 \pm q + 1$. When p divides $q^2 - q + 1$, $\tilde{P}$ acts irreducibly on a 6-subspace W of V and stabilizing the complementary 1-subspace U, since $(q^2 - q + 1, q^i - 1) = 2$ for $i = 1, \dots, 5$. When p divides $q^2 + q + 1$, obsere that G is a subgroup of $GO_7(q)$ and that there is a subgroup of $GO_7(q)$ isomorphic to $SL_3(q)$ stabilizing mutually disjoint maximal isotropic subspaces W_1, W_2 and the complementary 1-subspace R. Then by Sylow's theorem, we may assume that $\tilde{P}$ is contained in this subgroup isomorphic to $SL_3(q)$. As p is prime to $q^j - 1$ for $j = 1, 2$, $\tilde{P}$ acts irreducibly on W_1, W_2 and acts trivially on R. Thus in both cases, $N_G(\tilde{P})/C_G(\tilde{P})$ is a subgroup of the cyclic group of order 6 by Lemma 2.4.1. As Out(S) is the field automorphism group extended possibly by the graph automorphism of order 2, $x + 1$ divides $3f$. If $x \neq 8$, $q \geq 3^f \geq 3^{(x+1)/3} > x^3$ by the Condition (iii). For the case $x = 8$, we may write $q = s^3$, as $9 = x + 1$ divides $3f$. Now note that $p = 73$ divides $q^2 - q + 1$ or $q^2 + q + 1$, as we saw above. We may verify that $s \neq 3, 5$ nor 7. Thus $s > 8 = x$ and so $q = s^3 > x^3$, as q is odd by Lemma 5.1. Thus in any case, we have $q > x^3$. Then we have $2^7 q^{14/3} > p^7 > |S| > q^3(q^3 + 1)$ by the Condition (ii), which is a contradiction.

If the Case (7) holds, p divides $q \pm 1$ or $q \pm 3^{k+1} + 1$. As G has a maximal subgroup isomorphic to $Z_2 \times L_2(q)$, we can verify that the normalizer $N_G(\tilde{P})$ is contained in this group if p divides $q \pm 1$, and hence $|N_G(\tilde{P}) : C_G(\tilde{P})| = 2$. If the latter case holds, $N_G(\tilde{P})/C_G(\tilde{P})$ is a cyclic group of order 6. (The information on the normalizers of cyclic subgroups of S is available in [19].) As $\mathrm{Out}(S) \cong Z_{2k+1}$, the odd number $x + 1$ divides $3(2k + 1)$ by the same argument as above. Thus if $x \neq 8$, $q = 3^{2k+1} \geq 3^{(x+1)/3} > x^3$ by the Condition (iii). If $x = 8$, we may write $q = 3^{2k+1} = 3^{3\cdot(2l+1)}$ for some $l \geq 0$, as $x + 1 = 9$ divides $3(2k + 1)$. Since $p = 73$ divides $q \pm 1$ or $q \pm 3^k + 1$, we have $l \geq 1$, and so $q \geq 3^{3\cdot 3} > x^3 = 8^3$. Hence we always have $q > x^3$. Then it follows from the Condition (ii) that $(2x^2)^7 > p^7 > |S| = q^3(q^3 + 1)(q - 1) > x^9 x^9 x^3$, or equivalently $2 > x$, which is a contradiction. $\square$

Lemma 5.3 *The Case* (1) *in Lemma* 5.1 *does not occur.*

Proof: Assume that $r = p$. Then $p = q$ as $|S|_p = p$, and $S \cong L_2(p)$. As $Z(L)$ is of odd order by Lemma 4.8, $S = L$ and $LC_A(L) = L \times C_A(L)$. The group $A/(L \times C_A(L))$ is a subgroup of $\mathrm{Out}(L) = \mathrm{Out}(L_2(p))$ of order 2. Since a Sylow p-subgrop of $L_2(p)$ is self-centralizing in $\mathrm{Aut}(L_2(p))$, $C_A(P)$ is contained in $L \times C_A(L)$, and therefore $C_A(P) = C_L(P) \times C_A(L) = P \times C_A(L)$. Since a Sylow p-subgroup of $L_2(p)$ does not normalize any non-trivial subgroup of $L_2(p)$, the subgroup $L \cap K_u$ of L normalized by P ($\leq L \cap A_u$) is the trivial subgroup. Then $[K_u, P] \leq K_u \cap [K_u, L] \leq K_u \cap L = 1$ as $L \trianglelefteq A$. Thus K_u is a p'-subgroup of $C_A(P) = P \times C_A(L)$, and hence $K_u \leq C_A(L)$.

If y is odd, A_u is of odd order by Lemma 4.5.1. Then we have $A_u \leq L \times C_A(L)$ as $[A : L \times C_A(L)] \leq 2$. If y is even, K_u contains a Sylow 2-subgroup of A by Lemma 3.6.2. Then $A = (L \times C_A(L))K_u$, as $[A : L \times C_A(L)] \leq 2$, and hence $A = L \times C_A(L)$, as $K_u \leq C_A(L)$ by the above paragraph. In any case, we have $A_u \leq L \times C_A(L)$. Since $K_u \leq C_A(L)$ and $A_u/K_u \cong F_p^{x+1}$ by Lemma 4.2.1(1), we have $K_u = A_u \cap C_A(L)$. Thus $A_u = (L \cap A_u) \times K_u$ and $L \cap A_u = N_L(P) \cap A_u \cong F_p^{x+1}$. In particular, $L \cap A_{a,u}$ is a cyclic subgroup of $L_2(p)$ of order $x + 1$. Since L is transitive on $\mathcal{G}_0$ by Lemma 4.8, $L \cap A_a$ is a subgroup of $L \cong L_2(p)$ of index $p(x^2y + 1)/(\alpha + 1) = |\mathcal{G}_0|$ by Lemma 3.6.2, which is a multiple of p by Lemma 4.3. Observing a list of maximal subgroups of $L_2(p)$, p a prime, [16] Chap. 3, Section 6, the subgroup $L \cap A_a$ of L containing $L \cap A_{a,u} \cong Z_{x+1}$ is contained in a dihedral group of order $p - 1 = x(x + 1)$ (note that $x + 1$ is enough large by Condition (iii), and hence $L \cap A_a \ncong A_4, S_4, A_5$).

Thus $|L \cap A_a| = (x + 1)(x/t)$ for some integer t dividing x. Then $|\mathcal{G}_0| = p(x^2y + 1)/(\alpha + 1) = |L : L \cap A_a| = pt(p + 1)/2$ by the transitivity of L on $\mathcal{G}_0$ (see Lemma 4.8). As t divides x, $(t, x^2y + 1) = 1$. Thus $t = 1$ and

$$(x^2y + 1)/(\alpha + 1) = (p + 1)/2.$$

The dihedral group $L \cap A_a$ of order $x(x + 1)$ acts on $\mathrm{Res}(a)$ with $(xy + 1)(y + 1)$ planes. As $L \cap A_a \trianglelefteq A_a$, all $(L \cap A_a)$-orbits on $\mathcal{G}_2(a)$ have the same length $|L \cap A_a : L \cap A_{a,u}| = x$.

Thus x divides $(xy+1)(y+1)$, and so x divides $y+1$. We may write $y = xk - 1$ for some natural number k. Then $x + y = (1+k)x - 1$ is prime to x and so to y. By Lemma 2.3.1(2), $x+y$ divides $(x+1)(y+1)$, and so $xy + 1 = kx^2 - x + 1$. Then $x + y = (1+k)x - 1$ divides $-(1+k)(xy+1) + (kx)(x+y) = x - (1+k)$. As $x - (1+k) < x < x + y$, $x - (1+k) \leq 0$. If $x < 1 + k$, $x + y$ divides the natural number $(1+k) - x$, and hence $x + y = (1+k)x - 1 \leq (1+k) - x$. Then $(2+k)x \leq (2+k)$, which contradicts the assumption $x \geq 2$. Thus $x = 1 + k$ and $y = x^2 - x - 1$. However, it then follows from the equality above that $(x^2(x^2 - x - 1) + 1)/(\alpha + 1) = (x^2 + x + 2)/2$, and so

$$\alpha + 1 = 2(x^2 - 2x - 1) + \frac{2(5x+3)}{x^2 + x + 2}.$$

As α and x are integers, this implies that $x \neq 8$ and $2(5x+3) > x^2 + x + 2$, which contradicts the Condition (iii).

Hence $p \neq r$. Next we show that r is an odd prime. If $r = 2$, then the odd prime $p = x^2 + x + 1$ divides $q+1$ or $q-1$. If $p = q+1$, $x(x+1) = q = 2^f$, which is not the case, as $x > 1$. As p is odd, $p \neq q$. Thus $p \leq q - 1 < |S|_2 = q$, contradicting the Condition (iv), if y is odd. If y is even and $x \neq 8$, P centralizes a Sylow 2-subgroup of $L \cong L_2(2^f)$ by the Condition (iv), which is a contradiction.

In the remaining case, y is even and $x = 8$. As we saw in the proof of Lemma 5.1, in this case $S \cong L$ is isomorphic to $L_2(2^9)$ of order $2^9 \cdot 3^3 \cdot 7 \cdot 19 \cdot 73$. Since the odd part of $|K_u|$ is at most $p^2/2^9$, it is at most 10. In particular, the prime divisor 19 of $|S|$ divides $[A : K_u] = p(x^2y+1)(xy+1)(y+1)(x+1)/(\alpha+1)$ (see Lemma 3.6.2). As we remarked in the proof of Lemma 4.6, the possible values of y can be obtained by Lemma 2.3.1(2), among which $y = 8$, 56 or 64 are the only values satisfying the condition $19 \mid [A : K_u]$. As $2^9 \leq |K_u| < y^2$ by the Condition (iv), $y \neq 8$. If $y = 56$, the odd part of $|K_u|$ is at most $[56^2/2^9] = 6$. However, the prime divisor 7 (>6) of $|S|$ does not divide $[A : K_u] = 73 \cdot (3 \cdot 5 \cdot 239) \cdot 449 \cdot (3 \cdot 19) \cdot 3^2/(\alpha + 1)$, which is a contradiction. Thus $y = 64$. Then $|\mathcal{G}_0| = 73 \cdot (17 \cdot 241)/(\alpha + 1)$. Since $\mathcal{G}$ is not a building nor flat, $\alpha + 1 \neq 1$ nor $17 \cdot 241$. Thus $|\mathcal{G}_0| = 73 \cdot 17$ or $73 \cdot 241$, both of which do not divide $|L| = |L_2(2^9)|$. This contradicts the transitivity of L on $\mathcal{G}_0$ (see Lemma 4.8). Thus we established that r is an odd prime distinct from p.

Then p divides $(q \pm 1)/2$, and we use the arguments in Lemma 5.2. The normalizer $N_L(P)$ is a dihedral group of order $(q \pm 1)$ with the cyclic normal group of order $(q \pm 1)/2$. Since $\mathrm{Out}(L)$ is the extension of the diagonal automorphism group of order 2 by the field automorphism of order f, the odd part of $|N_A(P)/C_A(P)|$ divides f. In particular, $x+1$ divides f. If $x \neq 8$, we have $q = r^f \geq 3^{x+1} > (x^2 + x + 1)^3 = p^3$ by the Condition (iii). Then it follows from the Condition (ii) that $q^{7/3} > p^7 > |S| = q(q^2 - 1)/2$, which is a contradiction.

Assume that $x = 8$ and $p = 73$. If $q = s^9$ for $s \geq 5$, then $|S| = q(q^2-1)/2 > 5^9 5^{17} > 125^{21} > p^7$, which contradicts the Condition (ii). Thus $q = 3^9$. However, $L \cong L_2(3^9)$ is of order prime to 73, a contradiction. □

Now we eliminated all the possibilities for the non-abelian simple group S appeared as the factor of the unique component L of A (see Lemma 4.8). Thus there is no flag-transitive C_3-geometry of finite order (x, y) with $x \geq 2$, and therefore Theorem is established.

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