

A Characterization of Distance-Regular Graphs with Diameter Three

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Abstract. We characterize the distance-regular graphs with diameter three by giving an expression for the number of vertices at distance two from each given vertex, in terms of the spectrum of the graph.

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1. Introduction

It was proven by the second author [6] that if a graph G has the spectrum of a distance-regular graph with diameter three, and has the right number of vertices at distance two from each given vertex, then G is itself distance-regular. Here we drop the assumption that G has the spectrum of a distance-regular graph. We only assume that G is regular with four distinct eigenvalues and prove that G is distance-regular if and only if the number of vertices at distance two from each given vertex satisfies an expression in terms of the spectrum of G .

To obtain our results we use quotient matrices of the adjacency matrix of the graph with respect to some partition of the vertices. The *neighbourhood partition* of some vertex x is the partition of the vertices into $\{x\}$, the set of neighbours of x , and the set of all other vertices. The *distance partition* of x is the partition of the vertices into X_i , $i = 0, 1, \dots$, where X_i is the set of vertices at distance i from x . The *quotient matrix* of the adjacency matrix with respect to some partition of the vertices is obtained by symmetrically partitioning the adjacency matrix according to the partition of the vertices and then taking the average row sums in the blocks of the partition. The partition is called *regular* if every block has constant row sum. We shall use the fact that the eigenvalues of the quotient matrix interlace the eigenvalues of the adjacency matrix. If the interlacing is tight then the partition is regular (cf. [5] or [1]).

As general reference for distance-regular graphs we use the book of Brouwer et al. [1].

2. The characterization

To prove the main result we first need a characterization of strongly regular graphs.

Lemma *Let G be a connected regular graph on v vertices with eigenvalues $k > \lambda_2 \geq \dots \geq \lambda_v$. Let B be the quotient matrix with respect to the neighbourhood partition of an*

arbitrary vertex x . Suppose B has eigenvalues $k \geq \mu_2 \geq \mu_3$. If for every vertex x one of the equalities $\lambda_2 = \mu_2$ and $\lambda_v = \mu_3$ holds, then G is strongly regular.

Proof: Let G have adjacency matrix A . Fix an arbitrary vertex x and suppose one of the equalities holds, say μ_i is also an eigenvalue of A . Let V be the 3-dimensional subspace of \mathbb{R}^v of vectors that are constant over the parts of the neighbourhood partition of x . Then A has an eigenvector $u = (u_0, u_1, \dots, u_1, u_2, \dots, u_2)^T$ with eigenvalue μ_i in V (cf. [5, Thm. 2.1.ii] or [1, Section 3.3]). Also the all-one vector j is an eigenvector (with eigenvalue k) of A in V . Furthermore $A(1, 0, \dots, 0)^T = (0, 1, \dots, 1, 0, \dots, 0)^T \in V$. Since u, j and $(1, 0, \dots, 0)^T$ are linearly independent vectors in V (otherwise $u_1 = u_2$, and applying A gives $u_0 = u_2$, which gives a contradiction), we have that $AV \subseteq V$. So we have three linearly independent eigenvectors of A in V , and it follows that the neighbourhood partition of x is regular. So the number of common neighbours of x and a vertex y adjacent to x is independent of y . This holds for every x and since G is connected, it follows that this number is also independent of x , and so for every vertex the neighbourhood partition is regular with the same quotient matrix, proving that G is strongly regular. \square

Theorem Let G be a connected regular graph on v vertices with four distinct eigenvalues, say with spectrum $\Sigma = \{[k]^1, [\lambda_2]^{m_2}, [\lambda_3]^{m_3}, [\lambda_4]^{m_4}\}$. Let p be the polynomial given by $p(x) = (x - \lambda_2)(x - \lambda_3)(x - \lambda_4) = x^3 + p_2x^2 + p_1x + p_0$ and let λ be given by $\lambda = (k^3 + m_2\lambda_2^3 + m_3\lambda_3^3 + m_4\lambda_4^3)/vk$. Then G is distance-regular if and only if for every vertex x the number of vertices k_2 at distance two from x equals

$$f(\Sigma) = \frac{k(k - 1 - \lambda)^2}{(k - \lambda)(\lambda + p_2) - k - p_1 + p_0}.$$

Proof: Suppose that G is distance-regular. Consider the quotient matrix C with respect to the distance partition of some arbitrary vertex x . Then

$$C = \begin{pmatrix} 0 & k & 0 & 0 \\ 1 & \lambda & k - 1 - \lambda & 0 \\ 0 & c & k - c - b & b \\ 0 & 0 & k - a & a \end{pmatrix},$$

for some a, b and c . Note that $\lambda = \text{trace}(A^3)/vk$ equals the number of common neighbours of two adjacent vertices. Since C has eigenvalues k, λ_2, λ_3 and λ_4 , it follows that the characteristic polynomial of C equals

$$\begin{aligned} (x - k)p(x) &= \det(xI - C) = \det \begin{pmatrix} x & -k & x - k & 0 \\ -1 & x - \lambda & x - k & 0 \\ 0 & -c & x - k & -b \\ 0 & 0 & x - k & x - a \end{pmatrix} \\ &= (x - k)(x^3 + (b + c - \lambda - a)x^2 + (\lambda a - ca - b\lambda - k + c)x + ka - ca - bk). \end{aligned}$$

Now it follows that

$$c = \frac{(k - \lambda)(\lambda + p_2) - k - p_1 + p_0}{k - 1 - \lambda},$$

and since $k_2c = k(k - 1 - \lambda)$, we have that $k_2 = f(\Sigma)$.

Suppose now that $k_2 = f(\Sigma)$ for every vertex x . Let a, b and c be given by $c = k(k - 1 - \lambda)/f(\Sigma)$, $a = k - (\lambda k + p_2k + p_0)/c$ and $b = a + \lambda - c + p_2$. Then the matrix C as given above has eigenvalues k, λ_2, λ_3 and λ_4 (again, this follows by inspecting the characteristic polynomial).

First suppose that $f(\Sigma) < v - 1 - k$, i.e., G has diameter three. We shall prove that the quotient matrix B with respect to the distance partition of x equals C , thus proving that G is distance-regular around x . Without loss of generality we assume that $k > \lambda_2 > \lambda_3 > \lambda_4$. Suppose that B has eigenvalues $k \geq \mu_2 \geq \mu_3 \geq \mu_4$. Since the eigenvalues of B interlace the eigenvalues of the adjacency matrix A of G , it follows that $\lambda_2 \geq \mu_2$ and $\mu_4 \geq \lambda_4$.

Since G is a connected regular graph with four distinct eigenvalues, the number of triangles through x equals $\Delta = \frac{1}{2}k\lambda$ (cf. [2]). From this it follows that $B_{22} = \lambda$, and consequently $B_{23} = k - 1 - \lambda$ and $B_{32} = c$. So $B = C + E$, where E equals

$$E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & -\varepsilon \\ 0 & 0 & -\delta & \delta \end{pmatrix}$$

for some ε and δ . To use inequalities for eigenvalues we want symmetric matrices. Therefore we multiply B, C and E from the left by $K^{1/2}$ and from the right by $K^{-1/2}$, where $K = \text{diag}(1, k, f(\Sigma), v - 1 - k - f(\Sigma))$, to get \tilde{B}, \tilde{C} and \tilde{E} , respectively. Now the eigenvalues have not changed and \tilde{B} is symmetric, but to show that \tilde{C} (and consequently \tilde{E}) is symmetric, we have to prove that $(v - 1 - k - f(\Sigma))(k - a) = f(\Sigma)b$. This follows since

$$\begin{aligned} & ((v - 1 - k - f(\Sigma))(k - a) - f(\Sigma)b)c \\ &= (v - 1 - k)(k - a)c - f(\Sigma)c(k - a + b) \\ &= (v - 1 - k)(\lambda k + p_2k + p_0) - k(k - 1 - \lambda)(k + \lambda + p_2) \\ &\quad + k((k - \lambda)(\lambda + p_2) - k - p_1 + p_0) \\ &= v(\lambda k + p_2k + p_0) - (k^3 + p_2k^2 + p_1k + p_0) = 0. \end{aligned}$$

The last equation follows by taking the trace of the equation $p(A) = (p(k)/v)J$, where A is the adjacency matrix of G , and J is the all-one matrix (cf. [2, 7]).

Let $w_1 = K^{1/2}(1, 1, 1, 1)^T$, then it is an eigenvector of both \tilde{B} and \tilde{C} with eigenvalue k . Let w_i be an eigenvector of \tilde{C} with eigenvalue $\lambda_i, i = 2, 3, 4$, such that $\{w_1, w_2, w_3, w_4\}$ is orthogonal. Let $v_i = K^{-1/2}w_i$, then v_i is eigenvector of C with eigenvalue $\lambda_i, i = 1, 2, 3, 4$. Now we shall prove that $\tilde{E} = O$ or, equivalently, that $\varepsilon = 0$. Suppose that $\varepsilon > 0$. Now \tilde{E}

is positive semidefinite, and so

$$\mu_2 \geq \frac{w_2^T \tilde{B} w_2}{w_2^T w_2} = \lambda_2 + \frac{w_2^T \tilde{E} w_2}{w_2^T w_2} \geq \lambda_2,$$

and since $\lambda_2 \geq \mu_2$ it follows that $\mu_2 = \lambda_2$ and $\tilde{E} w_2 = o$. Then also $E v_2 = o$, and so $v_{23} = v_{24}$. Using that v_2 is eigenvector of C with eigenvalue λ_2 , we find that $v_2 = o$, which is a contradiction. Similarly we find that $v_4 = o$ by assuming that $\varepsilon < 0$. So $\varepsilon = 0$ and $B = C$. Thus the distance partition is regular, and since this holds for every vertex, we find that G is distance-regular.

Next suppose that $f(\Sigma) = v - 1 - k$, i.e., G has diameter two. We shall show that this cannot occur. Again, consider the matrix C as given above. From the equation $(v - 1 - k - f(\Sigma))(k - a) = f(\Sigma)b$ it follows that $b = 0$. Let B be the quotient matrix with respect to the neighbourhood partition of an arbitrary vertex x , then

$$B = \begin{pmatrix} 0 & k & 0 \\ 1 & \lambda & k - 1 - \lambda \\ 0 & c & k - c \end{pmatrix}.$$

Let B have eigenvalues k, μ_2 and μ_3 , then on one hand the eigenvalues of C are k, μ_2, μ_3 and a , and on the other hand they are k, λ_2, λ_3 and λ_4 and so we have one of the equalities needed to apply our lemma. So G is strongly regular, which is a contradiction with the fact that G has four distinct eigenvalues. □

This theorem proves a conjecture by the second author [6]. Although he uses a different expression for $f(\Sigma)$, it is easily proven that the expressions are the same for any regular graph with four eigenvalues (cf. [4]), that is,

$$f(\Sigma) = \frac{k(k - 1 - \lambda)^2}{\theta_4 - \lambda^2 - k}, \quad \text{where } \theta_4 = \frac{1}{vk} (k^4 + m_2 \lambda_2^4 + m_3 \lambda_3^4 + m_4 \lambda_4^4).$$

Proposition *With the hypothesis of the previous theorem, if $f(\Sigma)$ is integral and $f(\Sigma) \leq v - 1 - k$, then $k_2 \geq f(\Sigma)$.*

Proof: Suppose for some vertex x we have $k_2 < f(\Sigma)$. Consider the distance partition of x and change this partition by moving $f(\Sigma) - k_2$ vertices from the set of vertices at distance three to the set of vertices at distance two. By repeating the second part of the proof of the previous theorem, we find that the partition is regular, which is a contradiction. □

In a distance-regular graph μ is the number of common neighbours of two vertices at distance two. Here we find an easy way to see from the spectrum of a graph that it is distance-regular with diameter three and $\mu = 1$.

Corollary *With the hypothesis of the theorem, G is distance-regular with $\mu = 1$ if and only if $k - 1 - \lambda = (k - \lambda)(\lambda + p_2) - k - p_1 + p_0 \leq (v - 1 - k)/k$.*

Proof: If G is distance-regular with $\mu = 1$, then $c = 1$ in the proof of the theorem, and the equation follows and $k(k - 1 - \lambda) = k_2 \leq v - 1 - k$. On the other hand, if $c = 1$ then $f(\Sigma) = k(k - 1 - \lambda)$, which is integral and by assumption at most $v - 1 - k$. So the proposition states that $k_2 \geq f(\Sigma)$. Since every vertex at distance two from a given vertex has at least one common neighbour with that vertex, we have that $k_2 \leq k(k - 1 - \lambda)$. So $k_2 = f(\Sigma)$ and it follows from the theorem that G is distance-regular, and now $\mu = c = 1$. \square

We conjecture that the proposition is also true without the conditions for $f(\Sigma)$, i.e., that for every connected regular graph with four distinct eigenvalues we have that the number of vertices k_2 at distance two from a given vertex is at least $f(\Sigma)$.

More evidence for the conjecture is given by a bound by the first author [3], which for some cases coincides with $f(\Sigma)$, and in many other cases is just slightly worse. If G is a connected regular graph with four distinct eigenvalues $k > \lambda_2 > \lambda_3 > \lambda_4$ then for the number of vertices k_2 at distance two from a given vertex x we have that

$$k_2 \geq v - 1 - k - \frac{v}{1 + \frac{\gamma^2}{v-1}}, \quad \text{where } \gamma = \sum_{j \neq 1} \prod_{i \neq 1, j} \frac{|k - \lambda_i|}{|\lambda_j - \lambda_i|}.$$

The bound is tight for every vertex if and only if G is distance-regular such that the distance three graph G_3 of G is a strongly regular $(v, k_3, \lambda^*, \lambda^*)$ graph for some λ^* .

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