Journal of Algebraic Combinatorics **7** (1998), 91–107

Unimodality of Differences of Specialized Schur Functions*

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Received May 18, 1995; Revised November 26, 1996

Abstract. The centered difference of principally specialized Schur functions

$$s_{\tilde{\lambda}}(1, q, \ldots, q^n) - q^n s_\lambda(1, q, \ldots, q^n)$$

is shown to be a symmetric, unimodal polynomial in q with non-negative coefficients for certain choices of $\tilde{\lambda}$, λ, and n, in which $\tilde{\lambda}$ is always obtained from λ by adding two cells, and n is chosen to be odd or even depending on $\tilde{\lambda}$, λ. The basic technique is to find an injection of representations for the symplectic or orthogonal Lie algebras, and interpret the above difference as the principal specialization of the formal character of the quotient. As a special case, a difference of q-binomial coefficients is shown to be unimodal.

Keywords: unimodality, Peck, principal specialization

1. Introduction

It is well known [25, Theorem 13] that the principal specialization of a Schur function,

$$s_\lambda(1, q, \ldots, q^{n-1}),$$

is a symmetric, unimodal polynomial in q with non-negative coefficients. If $\lambda = 1^k$ is a single column, we have

$$s_{1^k}(1, q, \ldots, q^{n-1}) = q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q .$$

For $\lambda = k$ a single row,

$$s_k(1, q, \ldots, q^{n-1}) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q .$$

thus proving the q-binomial coefficient is a symmetric unimodal polynomial in q. In this paper we prove (Theorems 1, 5, 8) that certain differences of principal specializations of

*This work was supported by NSF Postdoctoral Research Fellowship DMS-9206371 and NSF grant DMS-9400510.

Schur functions

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are symmetric and unimodal. Our basic technique is to realize the differences as principal specializations of formal characters of representations of the symplectic or orthogonal Lie algebras.

We consider two special cases of $\lambda, \tilde{\lambda}$ related to some interesting posets in Sections 2 and 3, and consider more general $\lambda, \tilde{\lambda}$ in Section 4. Some poset conjectures are given in Sections 2 and 3, while unimodality conjectures are given in Section 5. All notation is taken from Macdonald [18].

2. The motivating special case

Empirically, the following theorem was discovered.

Theorem 1 *If n is an odd positive integer and $2k \leq n + 1$, then*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q - \begin{bmatrix} n \\ k-1 \end{bmatrix}_q$$

is a symmetric, unimodal polynomial in q with non-negative coefficients.

The first main goal of this section is to prove Theorem 1. It is straightforward to check that the above difference is symmetric as a polynomial in q, and Andrews [5] and Fishel [9] gave explicit sets of partitions for which the difference in Theorem 1 is the generating function, thus proving non-negativity. Moreover, the difference is known [6] to be the *Kostka polynomial* $K_{(n-k,k),1^n}(q)$ [18, p. 130], which has non-negative coefficients. However, none of these results gives unimodality.

First we note that one can rewrite the above difference using the q-Pascal's triangle recurrences (see [5, p. 21]) as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q - \begin{bmatrix} n \\ k-1 \end{bmatrix}_q = q^k \left(\begin{bmatrix} n-1 \\ k \end{bmatrix}_q - q^{n-2k+1} \begin{bmatrix} n-1 \\ k-2 \end{bmatrix}_q \right). \tag{2.1}$$

The advantage to this rewriting is that the two terms inside the parentheses on the right-hand side of (2.1) are now not only symmetric, but also centered at the same power of q. This suggests an algebraic interpretation as the sl_2-character of some quotient module, and we will construct such an sl_2-module as the *principal specialization* of the irreducible representation of sp_m corresponding to its kth fundamental weight. Our construction of these representations follows [4, Chap. VIII, Section 13] (see [10], Sections 17.3 and 24.2 for a discussion of these same representations in a *dualized* form using *Weyl's construction*).

Set $m = n - 1$, which is an even number, and let $m = 2l$. Let V be an m-dimensional $\mathbb{C}$-vector space with a *symplectic* form $\langle \cdot, \cdot \rangle$, i.e., $\langle \cdot, \cdot \rangle$ is a non-degenerate skew-symmetric bilinear form on V. Let $Sp(V)$ be the *symplectic group* inside of $GL(V)$ which consists of all invertible transformations preserving $\langle\cdot,\cdot\rangle$, and sp_m its Lie algebra. Then $Sp(V)$ and sp_m act on V, and hence on the exterior powers $\wedge^k V$ with formal character

$$\mathrm{char}_{sp_m}(\wedge^k V) = e_k\big(x_l, x_{l-1}, \dots, x_2, x_1, x_1^{-1}, x_2^{-1}, \dots, x_{l-1}^{-1}, x_l^{-1}\big)$$

where $e_k(z_1,\dots,z_m)$ is the kth *elementary symmetric function* in the variables $z_1,\dots,z_m$. Inside sp_m is a distinguished subalgebra isomorphic to sl_2 known as a *principal three-dimensional subalgebra* (*TDS*), which is unique up to conjugacy (see [22]). Restricting a representation of sp_m to this TDS yields an sl_2-module whose formal character is obtained from char_{sp_m} by the specialization $x_i = q^{2i-1}$, so for $\wedge^k V$ we obtain

$$\begin{aligned}\mathrm{char}_{sl_2}(\wedge^k V) &= e_k(q^{2l-1}, q^{2l-3}, \dots, q^{3-2l}, q^{1-2l})\\ &= q^{k(k-m)}\begin{bmatrix} m \\ k \end{bmatrix}_{q^2}.\end{aligned}$$

This then suggests that perhaps one could prove Theorem 1 by demonstrating an sp_m-equivariant injection $\phi : \wedge^{k-2}V \to \wedge^k V$, so that the quotient $\wedge^k V/\phi(\wedge^{k-2}V)$ would have formal characters

$$\begin{aligned}\mathrm{char}_{sp_m}(\wedge^k V/\phi(\wedge^{k-2}V)) &= e_k\big(x_l, x_{l-1}, \dots, x_{l-1}^{-1}, x_l^{-1}\big)\\ &\quad - e_{k-2}\big(x_l, x_{l-1}, \dots, x_{l-1}^{-1}, x_l^{-1}\big)\\ \mathrm{char}_{sl_2}(\wedge^k V/\phi(\wedge^{k-2}V)) &= q^{k(k-m)}\begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2} - q^{(k-2)(k-2-m)}\begin{bmatrix} n-1 \\ k-2 \end{bmatrix}_{q^2}\\ &= q^{k(k-m)}\left(\begin{bmatrix} m \\ k \end{bmatrix}_{q^2} - (q^2)^{n-2k+1}\begin{bmatrix} m \\ k-2 \end{bmatrix}_{q^2}\right)\\ &= q^{k(k-m)}\left(\begin{bmatrix} n \\ k \end{bmatrix}_{q^2} - \begin{bmatrix} n \\ k-1 \end{bmatrix}_{q^2}\right).\end{aligned}$$

Since such sl_2-characters are known to be symmetric, unimodal Laurent polynomials in q centered about q^0 (see [25], Theorem 15), this would imply that the difference in Theorem 1 is symmetric and unimodal as a polynomial in q.

In [4, Chap VIII, Section 13, no. 3], such a map ϕ is constructed by identifying the skew-symmetric form $\langle\cdot,\cdot\rangle$ with a skew-symmetric 2-tensor $w \in \wedge^2(V)$, and letting $\phi(v) = v \wedge w$. The fact that ϕ is sp_m-equivariant is immediate from the fact that $Sp(V)$ preserves $\langle\cdot,\cdot\rangle$ and hence sp_m annihilates w. Injectivity of ϕ is guaranteed by the following proposition (essentially proven in [4, p. 203]) whose statement we include here for later use in Section 4.

Proposition 2 *Let V be a m-dimensional vector space over $\mathbb{C}$. Fix $w \in \wedge^2(V)$, fix k, $2k \le m+2$, and define $\phi(v) = w \wedge v$. Then ϕ is an injection from $\wedge^{k-2}V$ to $\wedge^k V$, if, and only if, w corresponds to a non-degenerate (skew-symmetric) bilinear form.*

This completes the proof of Theorem 1. We note that Bourbaki also proves that the sp_m-representation $\wedge^k V/\phi(\wedge^{k-2}V)$ is irreducible and corresponds to the kth fundamental weight $\omega_1 + \cdots + \omega_k$ of sp_m. As such, one could compute a product formula for the difference in Theorem 1 using the q-Weyl dimension formula (see e.g., [22]), however, this yields no more in this case than the formula one gets by combining common factors in the product formulas for the individual q-binomial coefficients. Hughes [12] gave a different combinatorial formula for $\mathrm{char}_{sl_2}(\wedge^k V/\phi(\wedge^{k-2}V))$ based upon Freudenthal's multiplicity formula.

In the remainder of this section we wish to discuss how the partitions considered by Andrews in [5] naturally index a basis for the quotient space $\wedge^k V/\phi(\wedge^{k-2}V)$. We will explain how this basis coincides with another known basis for the irreducible representations of sp_m corresponding to fundamental weights.

There is an obvious bijection between partitions μ whose Ferrers diagrams fit inside a $k \times (m-k)$ rectangle and a basis for $\wedge^k V$. Namely, consider μ as a multiset of size k of integers in $\{0, 1, \ldots, m-k\}$, and add i to the ith smallest element of μ, to obtain a k-subset S of $\{1, \ldots, m\}$. These subsets become basis elements under the identification

$$v_S = \bigwedge_{i \in S} v_i$$

where $\{v_1, \ldots, v_m\}$ is a basis for V.

The quotient space $\wedge^k V/\phi(\wedge^{k-2}V)$ has sl_2-character equal to the left-hand side of (2.1), up to rescaling. Andrews [5] gave an explicit set of partitions inside a $k \times (m-k)$ rectangle whose generating function is given by (2.1) (NB: one might expect these partitions to lie inside a $k \times (n-k)$ rectangle, but in fact they lie in the smaller $k \times (m-k)$ rectangle). His description uses the *Frobenius notation* for a partition μ (see [5]): if the Durfee square of μ has size r^2, let $a_i = \mu_i - i$, $b_i = \mu'_i - i$, and

$$\mu = \begin{pmatrix} a_1 a_2 \cdots a_r \\ b_1 b_2 \cdots b_r \end{pmatrix}.$$

Proposition 3 (Andrews) *The generating function for all partitions λ whose Frobenius notation satisfies $a_1 \le m-k-1$, $b_1 \le k-1$, and $a_i - b_i \le m-2k$ is*

$$q^{-k}\left(\begin{bmatrix} n \\ k \end{bmatrix}_q - \begin{bmatrix} n \\ k-1 \end{bmatrix}_q\right) = \begin{bmatrix} m \\ k \end{bmatrix}_q - q^{m-2k+2}\begin{bmatrix} m \\ k-2 \end{bmatrix}_q.$$

If a partition μ inside a $k \times (m-k)$ rectangle satisfies the conditions of Proposition 3, we will say μ (or its corresponding subset S or its corresponding basis vector v_S) is *Andrews* and otherwise that it is *non-Andrews*.

Theorem 4 *If m is even and $2k \le m+2$, then the images of the Andrews partitions form a basis for the quotient $\wedge^k V/\phi(\wedge^{k-2}V)$.*

Proof: We originally had a direct proof of this, similar to the proof of Theorem 6, using as a key lemma a result of independent interest which we have relegated to the Appendix. We later found out that the theorem can be deduced from work of Berele [3], as we now explain.

Directly translating the condition for a partition μ to be non-Andrews via the correspondence with subsets, one can check that a k-subset S' is non-Andrews if and only if there exists some i for which the ith largest element a in S' and the ith smallest element b in $\{1, 2, \ldots, m\} - S'$ satisfy $a > b$ and $a + b > m + 1$. Now biject $\{1, 2, \ldots, m\}$ with

$$[\pm l] := \{-1, -2, \ldots, -(l-1), -l, l, l-1, \ldots, 2, 1\}$$

by matching up the corresponding entries of these sets in the order that they are listed. Under this correspondence, one can check that the non-Andrews partitions are exactly the subsets $S \subseteq [\pm l]$ for which there is some i so that $|S \cap [\pm i]| > i$. In the terminology of Sheats [24], a subset of $S \subseteq [\pm l]$ corresponds to a *circle diagram*, and the non-Andrews condition is the same as the circle diagram being *non-admissible*. Sheats explains how the admissible circle diagrams are the same as the *symplectic tableaux* of King [14] and DeConcini [7] indexing the weights of the irreducible representations of sp_m, in the case where the representations are fundamental. Furthermore, Berele [3] showed how to construct the irreducible representations of sp_m in such a way that King's symplectic tableaux index a basis, and hence in the case of the fundamental representations, the basis is indexed by admissible circle diagrams or Andrews partitions. It is easy to check that in the case of the fundamental representations, Berele's construction is exactly the same as our construction following Bourbaki, i.e., $\wedge^k V/\phi(\wedge^{k-2} V)$. □

We also wish to discuss a natural poset structure on the Andrews partitions. Recall that the *Gaussian poset* $L(k, m-k)$ is the distributive lattice formed by all partitions inside an $k \times (m-k)$ box, ordered by inclusion of their Ferrers diagrams. It has rank generating function $\left[{m \atop k}\right]_q$, and the proof of its rank-unimodality using the action of a principal TDS inside of gl_m also proves that this poset is *Peck* by showing that the action of the element e in sl_2 gives rise to an *order-raising operator* on the poset (see [20] for definitions of Peck and order-raising operator).

Similarly, one can easily check that the subset of Andrews partitions inside $L(k, m-k)$ form a distributive sublattice which we will call *Andrews*$(k, m-k)$. One can also easily check that the self-duality on $L(k, m-k)$ given by complementing a partition within the $k \times (m-k)$ box restricts to *Andrews*$(k, m-k)$, so it is also self-dual. A picture of *Andrews*$(3, 3)$ inside of $L(3, 3)$ is shown in Figure 1(a).

Theorem 4 shows that not only is *Andrews*$(k, m-k)$ rank-symmetric and rank-unimodal for m even with $2k \leq m + 2$, but that its elements naturally index the basis for the sp_m-module (and hence sl_2-module) $\wedge^k V/\phi(\wedge^{k-2} V)$. Thus one would hope that the element e in the principal TDS would give rise to an *order-raising operator* on $\wedge^k V/\phi(\wedge^{k-2} V)$ with respect to the order *Andrews*$(k, m-k)$, and hence prove this poset is Peck. This is false, however, already for $k = 2$ and $m \geq 6$. Nevertheless, we had conjectured that the poset *Andrews*$(k, m-k)$ is Peck if m is even and $2k \leq m+2$, and this has been proven very recently by Donnelly [8]. Donnelly constructs the fundamental irreducible representations

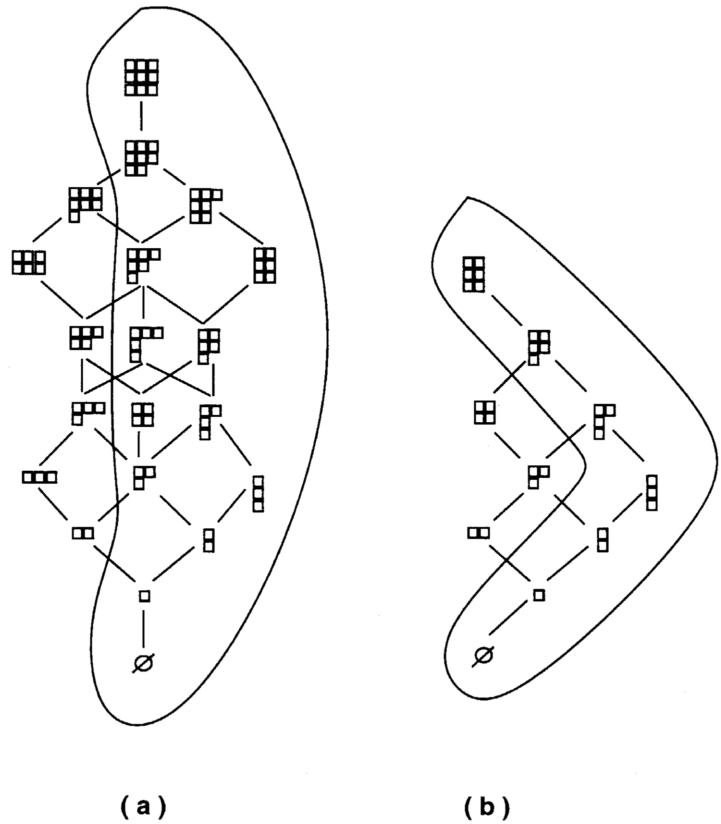

Figure 1. (a) The Gaussian poset $L(3, 3)$ with the elements of *Andrews*$(3, 3)$ shown circled, (b) The Gaussian poset $L(3, 2)$ with the elements of *Good*$(3, 2)$ shown circled.

of sp_m with a basis indexed by admissible circle diagrams, in such a way that the principal TDS has its raising operator acting as an order-raising operator with respect to the partial order on the circle diagrams isomorphic to *Andrews*$(k, m-k)$.

There is also a well-known open problem to determine whether the Gaussian poset $L(k, m-k)$ has a *symmetric chain decomposition* (see [21] for definition and some discussion of this problem). Theorem 4 suggest a natural extension of this problem:

Question *Does there exist a symmetric chain decomposition for $L(k, m-k)$ which restricts to Andrews$(k, m-k)$ for m even*?

If such a symmetric chain decomposition exists, it would by necessity also give a symmetric chain decomposition for the subposet *NonAndrews*$(k, m-k)$ of non-Andrews partitions inside $L(k, m-k)$. Strangely, this poset *NonAndrews*$(k, m-k)$ is *not* isomorphic to the smaller Gaussian poset $L(k-2, m+2-k)$, even though they share the same rank-generating function. In fact *NonAndrews*$(k, m-k)$ is not even a distributive lattice for $k \geq 4$!

3. Another special case

In this section we prove analogous results involving $\mathrm{Sym}^* V$ and the orthogonal Lie algebra, rather than $\wedge^* V$ and the symplectic Lie algebra. This case also leads to a conjecture that another self-dual, rank-unimodal subposet of the Gaussian poset is Peck.

Theorem 5 *If n is an odd positive integer, then*

$$\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q - q^{n-1} \begin{bmatrix} n+k-3 \\ k-2 \end{bmatrix}_q$$

is a symmetric, unimodal polynomial with non-negative coefficients.

Proof: In fact, we will show that this difference is, up to rescaling, the sl_2-character for the principal TDS inside of the orthogonal Lie algebra so_n acting in a certain representation.

Let $n = 2l + 1$. Choose $\langle \cdot, \cdot \rangle$ a non-degenerate symmetric form on an n-dimensional $\mathbb{C}$-vector space, and let the special orthogonal group $SO(V)$ be the subgroup of $GL(V)$ consisting of transformations which have determinant 1 and preserve $\langle \cdot, \cdot \rangle$. Then $SO(V)$ and its Lie algebra so_n act on V, and hence on $\mathrm{Sym}^k V$ with character

$$\mathrm{char}_{so_n}(\mathrm{Sym}^k V) = h_k\big(x_l, x_{l-1}, \ldots, x_2, x_1, 1, x_1^{-1}, x_2^{-1}, \ldots, x_{l-1}^{-1}, x_l^{-1}\big)$$

where h_k is the kth (complete) homogeneous symmetric function. The principal TDS inside of so_n therefore acts on $\mathrm{Sym}^k V$ with character

$$\begin{aligned} \mathrm{char}_{sl_2}(\mathrm{Sym}^k V) &= h_k(q^{n-1}, q^{n-3}, \ldots, q^4, q^2, 1, q^{-2}, q^{-4}, \ldots, q^{3-n}, q^{1-n}) \\ &= q^{k(1-n)} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{q^2} \end{aligned}$$

and hence

$$\begin{aligned} &\mathrm{char}_{sl_2}(\mathrm{Sym}^k V) - \mathrm{char}_{sl_2}(\mathrm{Sym}^{k-2} V) \\ &\quad = q^{k(1-n)} \left(\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{q^2} - (q^2)^{n-1} \begin{bmatrix} n+k-3 \\ k-2 \end{bmatrix}_{q^2} \right). \end{aligned} \tag{3.1}$$

Define $\phi : \mathrm{Sym}^{k-2} V \to \mathrm{Sym}^k V$ to be multiplication by the symmetric 2-tensor $w \in \mathrm{Sym}^2 V$ corresponding to $\langle \cdot, \cdot \rangle$. "Multiplication" by w means the following composite

$$\begin{array}{ccccc} \mathrm{Sym}^{k-2} V & \hookrightarrow & \mathrm{Sym}^{k-2} V \otimes \mathrm{Sym}^2 V & \to & \mathrm{Sym}^k V \\ v & \mapsto & v \otimes w & \mapsto & v \cdot w \end{array}$$

where the second map in the sequence is the shuffle (symmetrization) product.

The map ϕ is so_n-equivariant as before since $\langle \cdot, \cdot \rangle$ is preserved by $SO(V)$, so w is annihilated by sl_2. The map ϕ is an injection, since under the isomorphism of the symmetric algebra $\mathrm{Sym}^* V$ and the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$, the map ϕ corresponds to multiplying the polynomials of degree $k-2$ by a non-zero polynomial of degree 2.

Therefore, the expression on the right-hand side of Eq. (3.1) is the sl_2-character of the quotient module $\mathrm{Sym}^k V/\phi(\mathrm{Sym}^{k-2} V)$, and hence is unimodal. □

Remark Empirically it appears that the assertion of Theorem 5 is also true if k is odd and $n > 2$ is arbitrary, but we have no proof of this for even n.

It can be shown that the so_n representation $\mathrm{Sym}^k V/\phi(\mathrm{Sym}^{k-2} V)$ appearing in the proof of Theorem 5 is irreducible, although it does not correspond to a fundamental representation of so_n (see the discussion after the proof of Theorem 6). We now prove that there is again a natural set of partitions which index a basis for this quotient, deferring a discussion of their relation to known *orthogonal tableaux* until after the proof.

Firstly, note that a partition λ inside a $k \times (n-1)$ rectangle can be thought of as the k-multiset S of its parts in $\{0, 1, \ldots, n-1\}$, and also can be identified with a product of the basis vectors $\{v_0, \ldots, v_{n-1}\}$ of V,

$$v_\lambda = \prod_{i \in S} v_i$$

where here again the product is the commutative symmetrization product in $\mathrm{Sym}^* V$. Thus the set of all such partitions naturally indexes the monomial basis of $\mathrm{Sym}^k V$.

We wish to identify an appropriate subset of these partitions which will index a basis for our quotient module. Again using one of the q-Pascal's triangle recursions, we have

$$\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q - q^{n-1} \begin{bmatrix} n+k-3 \\ k-2 \end{bmatrix}_q = \begin{bmatrix} n+k-3 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n+k-2 \\ k \end{bmatrix}_q,$$

whose right-hand side suggests the set of partitions λ inside in a $k \times (n-1)$ rectangle which satisfy one of these two mutually exclusive conditions: either $\lambda_i \leq n-2$ for all i and $\lambda_k = 0$ (i.e., λ fits inside a $(k-1) \times (n-2)$ "corner" of the box) or $\lambda_k > 0$ (so removing the full first column of λ gives a partition inside a $k \times (n-2)$ box). Say that a partition inside a $k \times (n-1)$ box (or its corresponding multiset or its corresponding basis vector in $\mathrm{Sym}^k V$) satisfying either of these two conditions is *good*, else it is *bad*.

Theorem 6 *For n odd, the images of the good basis vectors in* $\mathrm{Sym}^k V$ *form a basis for* $\mathrm{Sym}^k V/\phi(\mathrm{Sym}^{k-2} V)$.

Proof: Identifying ϕ with its $\binom{n+k-3}{k-2} \times \binom{n+k-1}{k}$ matrix relative to the multiset bases, we will show that the $\binom{n+k-3}{k-2} \times \binom{n+k-3}{k-2}$ square submatrix ϕ' of ϕ obtained by restricting to the bad columns is non-singular, and hence that the images of the good basis vectors form a basis for the quotient $\mathrm{Sym}^k V/\phi(\mathrm{Sym}^{k-2} V)$.

Directly translating the condition for a partition λ to be bad via the correspondence with multisets, one can check that a k-multiset S' is bad if and only if it contains a copy of the pair $\{0, n-1\}$.

By the canonical forms for symmetric non-degenerate bilinear forms over $\mathbb{C}$, we can assume w is given by

$$w = \sum_{i=0}^{l} v_i \cdot v_{n-1-i}.$$

After identifying basis elements of $\mathrm{Sym}^{k-2}V$, $\mathrm{Sym}^k V$ with $(k-2)$-multisets, k-multisets, respectively, ϕ sends a basis $(k-2)$-multiset S to the sum of all k-multisets S' obtained by adjoining a *new copy* of the pair $\{i, n-1-i\}$ to S.

We now decompose the matrix ϕ into a certain block form. Call the subsets of the form $\{i, n-1-i\}$ *pairs*, and note that any subset S can be decomposed uniquely $S = P \cup U$ where P is a union of some pairs, and U consists of the unpaired elements (either i or $n-1-i$) within S. For example, if $n = 7$ and $S = \{0, 0, 1, 2, 2, 3, 5, 6, 6, 6\}$, then $P = \{0, 0, 1, 5, 6, 6\}$, $U = \{2, 2, 3, 6\}$. Note that if $\phi(S)$ contains some multiset S' with non-zero coefficient, then S' must have the same multiset of unpaired elements U as S, and it must contain exactly one more pair $\{i, n-1-i\}$ than S did. Therefore, if we fix a possible multiset of unpaired elements U (that is, any multiset on $\{0, 1, \ldots, n-1\}$ which contains at most one element from $\{i, n-1-i\}$ for all i), and let $\mathcal{S}_U, \mathcal{S}'_U$ be the collection of $(k-2)$-multisets, k-multisets on $\{0, 1, \ldots, n-1\}$ whose unpaired elements are exactly U, then ϕ will be block diagonal, with each non-zero block representing the map from subspace spanned by $\mathcal{S}_U$ into that spanned by $\mathcal{S}'_U$. Let ϕ_U be the restriction of ϕ to the spaces spanned by $\mathcal{S}_U, \mathcal{S}'_U$, and ϕ'_U the restriction of ϕ_U to its bad columns. It remains to show that each ϕ'_U is square and non-singular.

Trivially, we can reduce to the case where U is empty, since removing the unpaired elements from S, S' does not affect the matrix entry $\phi_U(S, S')$, and does not affect whether S' is good or bad. When U is empty, since S, S' are unions of pairs $\{i, n-1-i\}$, we lose no information if we replace S, S' by the multisets T, T' of $\{0, 1, \ldots, (n-1)/2\}$ obtained by replacing each pair $\{i, n-1-i\}$ with the smaller of the two numbers in the pair. We will have

$$\phi_U(S, S') = \begin{cases} 1 & \text{if } T \subset T' \\ 0 & \text{else} \end{cases}$$

and S' is bad if and only if $0 \in T'$.

Thus it only remains to observe the following: For any positive integers m, r, let M be the inclusion incidence matrix with rows, columns indexed by $(r-1)$ and r-multisets on $\{0, 1, \ldots, m\}$ respectively. Let M' be its restriction to the columns indexed by multisets containing 0. Then M' is square and invertible. To see this, note that if we order the rows and columns by lexicographic order on multisets with 0 coming first, 1 next, etc., then this matrix is upper unitriangular. □

It follows from Littlewood's branching rules for restricting irreducible gl_n-characters to so_n-characters [17], that the representation $\mathrm{Sym}^k V/\phi(\mathrm{Sym}^{k-2} V)$ is an irreducible so_n-representation and corresponds to the partition (k) having a single part of size k. *Orthogonal tableaux* indexing the weights of these irreducible representations have been given by King [14], Koike and Terada [16], Proctor [23], and Sundaram [26]. It is easy to check in the case of the irreducible corresponding to the partition (k) that each of these sets of orthogonal tableaux reduces to the disjoint union of two sets, consisting of all k-multisets and all $(k-1)$-multisets on an $(n-1)$-set, respectively. This is easily seen to correspond bijectively with the two kinds of good partitions in Theorem 6. We are not aware, however, of any explicit construction of the irreducible representations of so_m which coincides with our construction $\mathrm{Sym}^k V/\phi(\mathrm{Sym}^{k-2} V)$ in this special case, and hence which would imply Theorem 6 in the way that Berele's construction implied Theorem 4.

As in the previous section, one can consider a natural partial order on the good partitions. The q-binomial coefficient $\left[{n+k-1 \atop k}\right]_q$ is the rank generating function for the Gaussian poset $L(k, n-1)$, and one can easily check that the subset of good partitions inside $L(k, n-1)$ form a distributive sublattice which we will call $Good(k, n-1)$. One can also check that the self-duality on $L(k, n-1)$ given by complementing a partition within the $k \times (n-1)$ box restricts to $Good(k, n-1)$, so it is also self-dual. A picture of $Good(3, 2)$ is shown in Figure 1(b). Theorem 6 shows that not only is $Good(k, n-1)$ rank-symmetric and rank-unimodal for n odd, but that its elements naturally index the basis for an sl_2-module. Thus one would again hope that the element e in sl_2 would give rise to an *order-raising operator* on $Good(k, n-1)$, and hence prove that it is Peck. This hope is again false, already for $k = 2$ and $n \geq 5$. Nevertheless, we still make the following conjecture.

Conjecture 7 *The poset $Good(k, n-1)$ is Peck for n odd.*

It would be interesting to see if the methods of Donnelly mentioned in the previous section can be modified to prove this.

As in the discussion at the end of the previous section, it is natural to extend the question of whether there is a symmetric chain decomposition of the Gaussian poset $L(k, n-1)$ to ask whether there is one which restricts to $Good(k, n-1)$. In contrast to the case of $Andrews(k, m-k)$, the question seems more hopeful in this case because the subposet $Bad(k, n-1)$ consisting of the bad partitions in $L(k, n-1)$ is easily seen to be isomorphic to the smaller Gaussian poset $L(k-2, n-1)$. Therefore, one could hope for existence of a symmetric chain decomposition defined recursively on $L(k-2, n-1)$, which then extends over $Good(k, n-1)$ to the rest of $L(k, n-1)$.

4. Schur functions

We now generalize Theorems 1 and 5 by proving a unimodality result for certain differences of principally specialized Schur functions. For a partition λ, we let $s_\lambda(x_1, \ldots, x_n)$ denote the *Schur function* in the variables $x_1, \ldots, x_n$ associated to λ [18]. The *principal specialization* of s_λ is $s_\lambda(1, q, q^2, \ldots, q^{n-1})$, and is known to be a symmetric, unimodal polynomial in q with non-negative coefficients [25, Theorem 13].

The main result of this section is the following theorem.

Theorem 8 *Under the following conditions on $\lambda, \tilde{\lambda}$ and the parity of n, the centered difference of principal specializations*

$$s_{\tilde{\lambda}}(1, q, q^2, \ldots, q^{n-1}) - q^{n-1} s_\lambda(1, q, q^2, \ldots, q^{n-1})$$

is the principal specialization of an sp_n-character $(n$ even$)$ or so_n-character $(n$ odd$)$, and hence a symmetric, unimodal polynomial in q with non-negative coefficients:

(1) $\tilde{\lambda}$ *is obtained from* λ *by adding two cells to the first row, and n is odd.*
(2) $\tilde{\lambda}$ *is obtained from* λ *by adding a new part of size* 2, *n is odd, and $n > a + b$ where a, b are the lengths of the first two columns of* λ.
(3) $\tilde{\lambda}$ *is obtained from* λ *by adding two cells to the first column, n is even, and* $n \geq 2(l(\lambda) + 1)$ *where* $l(\lambda)$ *is the number of parts of* λ.
(4) $\tilde{\lambda}$ *is obtained from* λ *by adding a new column of size* 2, *n is even, and* $n \geq 2$.

Before giving the proof of each case of the theorem, we give a sketch of the basic idea underlying all four cases. Our first step is to interpret $s_\lambda(x_1, \ldots, x_n)$ as the formal character of an explicitly constructed irreducible representation of $GL(V)$, where V is an n-dimensional $\mathbb{C}$-vector space as usual. To this end, recall that for an n-dimensional vector space V over $\mathbb{C}$, the *Schur module* (or *co-Schur module*) $S_\lambda V$ constructs the irreducible representation of $GL(V)$ corresponding to λ, and the formal character is $s_\lambda(x_1, \ldots, x_n)$ (see [2] for definitions and details about (co-)Schur modules). Because we are working over $\mathbb{C}$, the Schur module and co-Schur module are isomorphic as $GL(V)$-representations, so we will abuse notation and use $S_\lambda V$ for both. By choosing a non-degenerate symmetric (resp. skew-symmetric) form $\langle \cdot, \cdot \rangle$ on V when n is odd (resp. even), and letting $SO(V)$, so_n (resp. $Sp(V)$, sp_n) be the classical simple Lie group and Lie algebra associated to the form $\langle \cdot, \cdot \rangle$, the Schur module $S_\lambda V$ is also a representation for so_n (resp. sp_n) whose formal character is

$$\begin{aligned} \mathrm{char}_{so_n} S_\lambda V &= s_\lambda\big(x_l, x_{l-1}, \ldots, x_1, 1, x_1^{-1}, \ldots, x_{l-1}^{-1}, x_l^{-1}\big) \\ \mathrm{char}_{sp_n} S_\lambda V &= s_\lambda\big(x_l, x_{l-1}, \ldots, x_1, x_1^{-1}, \ldots, x_{l-1}^{-1}, x_l^{-1}\big) \end{aligned}$$

and whose sl_2-character when restricted to the principal TDS inside of so_n or sp_n is

$$\begin{aligned} \mathrm{char}_{sl_2} S_\lambda V &= s_\lambda(q^{1-n}, q^{3-n}, \ldots, q^{n-3}, q^{n-1}) \\ &= q^{(1-n)|\lambda|} s_\lambda\big(1, q^2, q^4, \ldots, q^{2(n-1)}\big) \end{aligned}$$

where $|\lambda|$ is the sum of the parts of λ. It therefore suffices to prove (for each case asserted in the theorem) that there exists an so_n or sp_n-equivariant injection

$$\phi : S_\lambda V \hookrightarrow S_{\tilde{\lambda}} V.$$

This implies that the difference of principal specializations will be (up to a shift by a power of q, and the substitution $q \mapsto q^2$) the sl_2-character for the quotient $S_{\tilde{\lambda}} V / \phi(S_\lambda V)$, and then symmetry, unimodality and non-negativity of the coefficients follow as before from [25, Theorem 15].

The injection ϕ may be uniformly described in each case as the following composite of three maps:

$$S_\lambda V \to S_\lambda V \otimes (V \otimes V) \to \bigoplus_{\substack{\lambda \subset \tilde{\lambda} \\ |\tilde{\lambda}| = |\lambda| + 2}} S_{\tilde{\lambda}} V \to S_{\tilde{\lambda}} V$$

Here the first map is simply tensoring with w, the symmetric or skew-symmetric 2-tensor in $V \otimes V$ which corresponds to the form $\langle \cdot, \cdot \rangle$, and which is annihilated by so_n or sp_n. The second map comes from the Pieri formula for Schur modules or Schur functions ([19], [18], p. 42), and the third map is just the canonical projection onto a summand in the direct sum.

It is clear that ϕ is so_n or sp_n-equivariant as before since w is annihilated by so_n or sp_n. It only remains to check that in each case asserted by the theorem, ϕ is injective. While the composite "Pieri map"

$$S_\lambda V \otimes (V \otimes V) \to S_{\tilde{\lambda}} V$$

is somewhat complicated to describe explicitly for general $\tilde{\lambda}, \lambda$ (see [19]), in each of the cases asserted in the theorem we have a simple description, which allows us to conclude that ϕ is injective.

Proof of Theorem 8: From the previous discussion, we only need to show in each case of the theorem how to describe the map ϕ explicitly, and check that it is injective. We will use the fact that the (co-)Schur functor construction may be applied for any skew Ferrers diagram D, i.e., D need not necessarily correspond to a partition. We introduce the following terminology: for a non-negative integer m, $\mathrm{Row}(m)$ denotes a Ferrers diagram consisting of a single row with m cells, and $\mathrm{Col}(m)$ is a single column with m cells. Given two skew diagrams D and D', let $D * D'$ denote the skew diagram obtained by placing D' strictly north and east of D so that they have no cells in the same row or column. We will use without further mention the facts that

$$\begin{aligned} S_{\mathrm{Row}(m)} V &\cong \mathrm{Sym}^m V \\ S_{\mathrm{Col}(m)} V &\cong \wedge^m V. \end{aligned}$$

For case 1, consider the following commutative diagram of maps

$$\begin{array}{ccc} S_\lambda & \stackrel{i_1}{\to} & S_{\mathrm{Row}(\lambda_k)*\mathrm{Row}(\lambda_{k-1})*\cdots*\mathrm{Row}(\lambda_2)*\mathrm{Row}(\lambda_1)} \\ \otimes w \downarrow & & \downarrow \otimes w \\ S_{\lambda*\mathrm{Row}(2)} & \stackrel{i_2}{\to} & S_{\mathrm{Row}(\lambda_k)*\mathrm{Row}(\lambda_{k-1})*\cdots*\mathrm{Row}(\lambda_2)*\mathrm{Row}(\lambda_1)*\mathrm{Row}(2)} \\ \hat{\pi} \downarrow & & \downarrow \pi \\ S_{\tilde{\lambda}} & \stackrel{i_3}{\to} & S_{\mathrm{Row}(\lambda_k)*\mathrm{Row}(\lambda_{k-1})*\cdots*\mathrm{Row}(\lambda_2)*\mathrm{Row}(\lambda_1+2)} \end{array} \tag{4.1}$$

where here S_D denotes the Schur module construction (as opposed to the co-Schur module) applied to V. The horizontal maps i_1, i_2, i_3 are inclusions which come from the definition of a Schur module $S_D V$ as the image of a certain map into $\mathrm{Sym}^{\mu_1} V \otimes \mathrm{Sym}^{\mu_2} V \cdots \otimes \mathrm{Sym}^{\mu_l} V$, where μ_i is the size of the ith row of the skew diagram D. Also the maps $\otimes w$ from the first row to the second row are defined because w is a symmetric 2-tensor, i.e., it is in $S_{\mathrm{Row}(2)} V$, because n is odd. The map π is defined by $\pi = id \otimes \cdots \otimes id \otimes g$, where g is the symmetrization map $g : \mathrm{Sym}^{\lambda_1} V \otimes \mathrm{Sym}^2 V \to \mathrm{Sym}^{\lambda_1+2} V$. The map $\hat{\pi}$ is defined because the composite $\pi \circ i_2$ happens to factor through $S_{\tilde{\lambda}}$, as is easy to check from the definition of the Schur module.

The composite $\pi \circ (\otimes w)$ of the two maps in the right column is injective, because it is $id \otimes \cdots \otimes id \otimes h$, where h is the same map which was shown to be injective in the proof of Theorem 5. Since i_1 is an injection, this implies that our map $\phi = \hat{\pi} \circ (\otimes w)$ is an injection, as desired.

For case 3, one does the "transpose" of the argument just given, replacing Schur modules with co-Schur modules, and rows by columns. In the second-to-last sentence of the argument, one uses Proposition 2.

For case 4, consider the following commutative diagram

$$\begin{array}{ccc} S_\lambda & \stackrel{i_1}{\to} & S_{\mathrm{Row}(\lambda_k)*\mathrm{Row}(\lambda_{k-1})*\cdots*\mathrm{Row}(\lambda_2)*\mathrm{Row}(\lambda_1)} \\ \otimes w \downarrow & & \downarrow \otimes w \\ S_{\lambda*\mathrm{Row}(1)*\mathrm{Row}(1)} & \stackrel{i_2}{\to} & S_{\mathrm{Row}(\lambda_k)*\mathrm{Row}(\lambda_{k-1})*\cdots*\mathrm{Row}(\lambda_2)*\mathrm{Row}(\lambda_1)*\mathrm{Row}(1)*\mathrm{Row}(1)} \\ \hat{\pi} \downarrow & & \downarrow \pi \\ S_{\tilde{\lambda}} & \stackrel{i_3}{\to} & S_{\mathrm{Row}(\lambda_k)*\mathrm{Row}(\lambda_{k-1})*\cdots*\mathrm{Row}(\lambda_2+1)*\mathrm{Row}(\lambda_1+1)} \end{array} \tag{4.2}$$

where here S_D denotes the Schur module construction applied to V. The horizontal maps i_1, i_2, i_3 are the Schur modules' defining inclusions as before. The map π is defined by $\pi = id \otimes \cdots \otimes id \otimes g_1 \otimes g_2$, where g_1, g_2 are the symmetrization maps

$$g_1 : \mathrm{Sym}^{\lambda_1} V \otimes V \to \mathrm{Sym}^{\lambda_1+1} V$$
$$g_2 : \mathrm{Sym}^{\lambda_2} V \otimes V \to \mathrm{Sym}^{\lambda_2+1} V$$

The map $\hat{\pi}$ is defined because the composite $\pi \circ i_2$ happens to factor through $S_{\tilde{\lambda}}$, as is easy to check from the definition of the co-Schur module, using the fact that w is already a skew-symmetric 2-tensor (since n is even).

Since i_1 is an injection, our composite map $\phi = \hat{\pi} \circ (\otimes w)$ in the first column will be an injection, as long as we can show that the composite $\pi \circ (\otimes w)$ in the right column is injective. But this map is $id \otimes \cdots \otimes id \otimes h$, where h is the same map in the case where there are only 2 rows in λ. Thus we need a lemma which says that if w is a non-degenerate skew-symmetric 2-tensor, $n \geq 2$, and λ has only 2 rows, then the composite map $(g_1 \otimes g_2) \circ (\otimes w)$ is injective. This is easy to prove using the same sort of block-diagonal decomposition technique used to prove injectivity of the map in Theorem 6, so we will omit the details.

For case 2, one does the "transpose" of the argument just given, replacing Schur modules with co-Schur modules, and rows by columns. In the second-to-last sentence of the argument, one needs to show that if $a \geq b$ with $n > a + b$, and w is a non-degenerate symmetric 2-tensor, then the following composite map is injective:

$$S_{\mathrm{Col}(a)*\mathrm{Col}(b)} \underset{\otimes w}{\to} S_{\mathrm{Col}(a)*\mathrm{Col}(b)*\mathrm{Col}(1)*\mathrm{Col}(1)} \underset{g_1 \otimes g_2}{\to} S_{\mathrm{Col}(a+1)*\mathrm{Col}(b+1)}$$

where g_1, g_2 are the antisymmetrization maps

$$g_1 : \wedge^a V \otimes V \to \wedge^{a+1} V$$
$$g_2 : \wedge^b V \otimes V \to \wedge^{b+1} V$$

Again this is easy to prove using the same block-diagonal decomposition technique used to prove injectivity of the map in Theorem 6, and we omit the details. $\square$

Remarks

1. It is not hard to give combinatorial injections proving that in all of the cases (1)–(4), the appropriate differences

$$s_{\tilde{\lambda}}\left(x_l, x_{l-1}, \ldots, x_1, 1, x_1^{-1}, \ldots, x_{l-1}^{-1}, x_l^{-1}\right) \\ - s_{\lambda}\left(x_l, x_{l-1}, \ldots, x_1, 1, x_1^{-1}, \ldots, x_{l-1}^{-1}, x_l^{-1}\right)$$

or

$$s_{\tilde{\lambda}}\left(x_l, x_{l-1}, \ldots, x_1, x_1^{-1}, \ldots, x_{l-1}^{-1}, x_l^{-1}\right) \\ - s_{\lambda}\left(x_l, x_{l-1}, \ldots, x_1, x_1^{-1}, \ldots, x_{l-1}^{-1}, x_l^{-1}\right)$$

where $l = \lfloor \frac{n}{2} \rfloor$ have non-negative coefficients as a Laurent polynomial in $x_1, \ldots, x_l$, *regardless of the parity of* n. However, these differences will not always have meaning as so_n or sp_n-characters, and unimodality of their principal specializations requires the parity conditions stated in each case.
2. There is an alternative proof of Theorem 8 relying on Littlewood's identities [17] giving the branching rules for decomposing into irreducibles the restriction of an irreducible gl_n-representation $S_\lambda V$ to so_n or sp_n. In the alternative proof, one shows that when $\lambda, \tilde{\lambda}, n$ satisfy the hypotheses of the theorem, the decomposition coefficients for $\tilde{\lambda}$ always dominate those of λ, so that there must be an injection of representations. Such a program would not be hard to carry out, but we feel that such a proof is somewhat less illuminating than actually constructing the injections as above.
3. One might hope that for any $\tilde{\lambda}$ obtained from λ by adding two cells, the centered difference considered in Theorem 8 is unimodal (it will trivially be symmetric) under some parity conditions on n. However, this is false in general. For example, if $\lambda, \tilde{\lambda} = (3, 1), (3, 3)$ then the difference is not unimodal for $n = 4, 5, 6, 8$, if $\lambda, \tilde{\lambda} = (2, 1, 1), (2, 2, 2)$ then the difference is not unimodal for $n = 5, 6, 7$, and if $\lambda, \tilde{\lambda} = (3, 2, 1), (3, 3, 2)$ then the difference is not unimodal for $n = 5, 6$. Interestingly, in each of these examples, the difference does appear to be unimodal for n sufficiently large, regardless of its parity!
4. One might also ask whether there is a generalization of Theorems 4 and 6, and Conjecture 7 about posets. There is a good candidate to replace the Gaussian poset $L(k, n)$, namely the poset $L(\lambda, n)$ consisting of all column-strict tableaux of shape λ ordered entry-wise, which was conjectured to be Peck by Stanley, and proven using sl_2-representations in [21]. Unfortunately, we do not know of good candidates for the analogues of the subposets of Andrews and good partitions in Sections 2 and 3, which would index basis elements in the quotient $S_{\tilde{\lambda}} V / \phi(S_\lambda V)$.

5. A strange conjecture

The KOH identity [28] writes a q-binomial coefficient as

$$\begin{bmatrix} m \\ k \end{bmatrix}_q = \sum_{\nu} G_\nu(q),$$

where ν ranges over all partitions of k, and $G_\nu(q)$ is a certain shifted product of q-binomial coefficients, which are all symmetric and centered at the same power of q. Similarly the generalization of KOH to Schur functions of Kirillov [15] is

$$s_\lambda(1, q, \ldots, q^{m-1}) = \sum_{\vec{\nu}} G_{\vec{\nu}}(q),$$

where here $\vec{\nu}$ ranges over certain sequences of partitions, called *configurations*, and $G_{\vec{\nu}}(q)$ is another shifted product of q-binomial coefficients. Therefore, one might try to prove a refinement of Theorem 8, namely, that the centered difference

$$G_{\tilde{\vec{\nu}}}(q) - q^{m-1} G_{\vec{\nu}}(q)$$

is symmetric, unimodal with non-negative coefficients, under some natural conditions on $m, \tilde{\vec{\nu}}, \vec{\nu}$.

In case (3) of Theorem 8 we have such a conjecture. The new configuration $\tilde{\vec{\nu}}$ is obtained by adding two cells to the first column of each partition of $\vec{\nu}$, and appending 11 and 1 as new partitions to $\vec{\nu}$. Here, m is even, $m \geq 2(l(\vec{\nu}_1) + 1)$. We cannot even verify that for $q = 1$ the integer representing this difference is non-negative.

By considering an iterate of the $\vec{\nu} = (1^{k-2}, 0, 0, \ldots)$ term of the above conjecture, we conjecture the following generalization of Theorem 1.

Conjecture 9 *If n is odd, and r and k are non-negative integers with $n \geq 2rk - 4r + 3$, then*

$$\begin{bmatrix} n-1 \\ k \end{bmatrix}_q - q^{n-2rk+1+4(r-1)} \begin{bmatrix} n-1+4(r-1) \\ k-2 \end{bmatrix}_q$$

is a symmetric, unimodal polynomial in q with non-negative coefficients.

Appendix: A lemma on the canonical matching

There is a well-known matching in the incidence graph for the inclusion relation between the $(r-1)$ and r-subsets of an n-element set, which has been discovered and rediscovered by many authors in various guises [1, 11, 27]. For this reason we call it *the canonical matching*. Our original proof of Theorem 4 (before we were aware of Berele's work [3]) relied on a decomposition of the matrix for the map $\phi\colon \wedge^{k-2} V \to \wedge^k V$ into rectangular blocks, very similar to the proof of Theorem 6. It was shown that in each rectangular block the non-Andrews partitions naturally indexed a set of columns which selected out an invertible square submatrix, and hence that the Andrews partitions formed a basis for the quotient $\wedge^k V/\phi(\wedge^{k-2} V)$. The crucial lemma in this proof was the following statement about the canonical matching, which we think is of independent interest.

Lemma 10 *Assume $2r \leq n+1$ and let $M(n, r)$ be the $\binom{n}{r-1} \times \binom{n}{r-1}$ incidence matrix obtained by restricting the inclusion incidence matrix between $(r-1)$ and r-subsets of* *an n-set to the columns indexed by those r-subsets which are matched in the canonical matching. Then $M(n, r)$ is square and invertible.*

Proof: Let $\theta_{n,r} : T \mapsto T'$ be the canonical matching. From any of the descriptions of θ [1, 11, 27] the following two properties of θ are easy to check, *assuming* $r \leq \lceil \frac{n}{2} \rceil$:

(1) If $n \notin T$ then $n \notin \theta_{n,r}(T)$.
(2) If $n \in T$ then $n \in \theta_{n,r}(T)$ and

$$\theta_{n,r}(T) = \theta_{n-1,r-1}(T - \{n\}) \cup \{n\}$$

From this it follows that reordering both the rows and columns of $M(n, r)$ so that the subsets not containing n come first, produces a block upper-triangular form for $M(n, r)$:

$$M(n, r) = \begin{pmatrix} M(n-1, r) & * \\ 0 & M(n-1, r-1) \end{pmatrix}.$$

Thus by induction on $r + n$ it only remains to show that $M(2r - 1, r)$ is invertible. But $M(2r - 1, r)$ is the *entire* inclusion incidence matrix between the middle ranks in a Boolean algebra of odd rank, which is known to be invertible [13]. □

Acknowledgments

The authors would like to thank Georgia Benkart, Andrzej Daszkiewicz, Kathy O'Hara, Peter Olver, Bob Proctor, and Andrei Zelevinsky for helpful comments, suggestions, and references. We also thank Rob Donnelly for advance information of his work.

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