Journal of Algebraic Combinatorics **8** (1998), 115–126

On Elliptic Product Formulas for Jackson Integrals Associated with Reduced Root Systems

KAZUHIKO AOMOTO
Graduate School of Mathematics, Nagoya University, Furo-cho 1, Chikusa-ku, Nagoya 464-8602, Japan

Received December 13, 1995; Revised May 5, 1997

Abstract. In this note, we state certain product formulae for Jackson integrals associated with any root systems, involved in elliptic theta functions which appear as connection coefficients. The fomulae arise naturally in case of arbitrary root systems by extending the connection problem which has been investigated in [1, 4] in case of A type root system. This is also connected with the Macdonald-Morris constant term identity investigated by I. Cherednik [6], and K. Kadell [15] on the one hand, and of the Askey-Habsieger-Kadell's q-Selberg integral formula and its extensions [4, 8, 12, 14, 15] on the other. This is also related with some of the results due to R.A. Gustafson [10, 11], although our integrands are different from his.

Keywords: elliptic theta function, Jackson integral, reduced root system, product formula, q-difference

1. The product formula

Let q be the elliptic modulus satisfying $|q| < 1$. Let $\mathbf{h}$ and H be the n-dimensional Cartan subalgebra and its Cartan subgroup, associated with a simple Lie algebra $\mathcal{G}$ of rank n over $\mathbf{C}$. Let $\mathbf{h}^*$ be the dual of $\mathbf{h}$. Let $R_+ \subset \mathbf{h}^*$ be the positive root system on $\mathbf{h}$. For $\alpha, \beta \in R_+$, we denote by (α, β) the inner product induced by the Killing form on $\mathcal{G}$. We may identify the vector space $\mathbf{h}^*$ with its dual $\mathbf{h}$ through the inner product as follows:

$$\alpha \subset \mathbf{h}^* \to h_\alpha \in \mathbf{h},$$

such that $(\alpha, \mu) = \mu(h_\alpha)$ for any $\mu \in \mathbf{h}^*$.

Let X be the n-dimensional coweight lattice $\cong \mathbf{Z}^n$ in $\mathbf{h}$ consisting of $h \in \mathbf{h}$ such that $\alpha(h) \in \mathbf{Z}$. We take a suitable basis $\chi_1, \ldots, \chi_n$, so that an arbitrary element χ of X can be described as a linear combination

$$\chi = \sum_{j=1}^{n} \nu_j \chi_j$$

for $(\nu_1, \ldots, \nu_n) \in \mathbf{Z}^n$. $\mathbf{h}$ and H are isomorphic to the tensor products $X \otimes \mathbf{C}$, and $X \otimes \mathbf{C}^*$, respectively. X can also be embedded in the n-dimensional algebraic torus H (Cartan subgroup) isomorphic to $(\mathbf{C}^*)^n$. We denote this identification by

$$\chi \in X \to q^\chi = (q^{\nu_1}, \ldots, q^{\nu_n}) \in (\mathbf{C}^*)^n.$$

Let $\gamma_1, \gamma_2, \gamma_3, \ldots$ be arbitrary complex numbers. Each $\mu \in \mathbf{h}^*$ defines a monomial $t^\mu = t_1^{\mu(\chi_1)} \cdots t_n^{\mu(\chi_n)}$ for $t = (t_1, \ldots, t_n) \in H$. We denote the two linear functions λ and λ' as $\lambda = \frac{1}{2}\sum_{\alpha\in R_+}(1-2\gamma_{(\alpha,\alpha)})\alpha$ and $\lambda' = \lambda - \frac{1}{2}\sum_{\alpha\in R_+}\alpha$, respectively.

We denote by $Q^\chi f(t) = f(q^\chi t)$ the q-shift by $\chi \in X$ for a function $f(t)$ on H. We consider the q-multiplicative function $\Phi(t)$ and $\Phi_0(t)$ on H as

$$\Phi(t) = t^\lambda \prod_{\alpha\in R_+} \frac{(q^{1-\gamma_{(\alpha,\alpha)}} \cdot t^\alpha)_\infty}{(q^{\gamma_{(\alpha,\alpha)}} \cdot t^\alpha)_\infty}, \tag{1.1}$$

$$\Phi_0(t) = \Phi(t) \prod_{\alpha\in R_+} (t^{\alpha/2} - t^{-\alpha/2}). \tag{1.2}$$

Here $(x)_\infty = (x;q)_\infty$ denotes the infinite product $\prod_{\nu=0}^\infty (1 - xq^\nu)$. It is known that there are at most 2 different $\gamma_{(\alpha,\alpha)}$ which appear in the RHS of (1.1).

Then $\Phi(t)$, $\Phi_0(t)$ have the quasi-symmetry with respect to the Weyl group W associated with the root system R_+.

$$\sigma\Phi(t) = \Phi(\sigma^{-1}(t)) = U_\sigma(t)\Phi(t), \tag{1.3}$$

$$\sigma\Phi_0(t) = \Phi_0(\sigma^{-1}(t)) = \mathrm{sgn}(\sigma)U_\sigma(t)\Phi(t), \tag{1.4}$$

where $\{U_\sigma(t)\}_{\sigma\in W}$ defines one cocycle on W with values in pseudo-constants i.e., q-periodic functions with respect to t.

Indeed $U_\sigma(t)$ satisfies the properties

$$U_{\sigma\tau}(t) = U_\sigma(t) \cdot \sigma U_\tau(t) \quad \text{for } \sigma, \tau \in W,\, U_e(t) = 1 \quad (e = \text{the identity})$$

and $Q^\chi U_\sigma(t) = U_\sigma(t)$ for any $\chi \in X$.

It is given in an explicit form as

$$U_\sigma(t) = \prod_{\alpha\in R_+, \sigma(\alpha)<0} \left\{ t^{(2\gamma_{(\alpha,\alpha)}-1)\alpha} \frac{\theta(q^{\gamma_{(\alpha,\alpha)}} t^\alpha)}{\theta(q^{1-\gamma_{(\alpha,\alpha)}} t^\alpha)} \right\}, \tag{1.5}$$

where $\theta(x)$ denotes the Jacobi elliptic theta function $\theta(x) = (x)_\infty \cdot (q/x)_\infty \cdot (q)_\infty$.

We now consider the following Jackson integral

$$\begin{aligned} J(\xi) &= \int_{[0,\xi\infty]_q} \Phi_0(t)\varpi, \quad \left(\varpi = \frac{d_qt_1}{t_1} \wedge \cdots \wedge \frac{d_qt_n}{t_n}\right) \\ &= (1-q)^n \sum_{\chi\in X} \Phi_0(q^\chi \xi), \end{aligned} \tag{1.6}$$

where ξ denotes an arbitrary point of H.

Assume now that $\lambda(\chi) + \frac{1}{2}\sum_{\alpha\in R_+}\alpha(\chi) > 0$ for all $\chi \neq 0$ of the fundamental domain Δ_+ in $\mathbf{h}$ defined by the inequalities $\alpha(\chi) \geq 0$ for $\alpha \in R_+$.

Then (1.6) is summable on $X \cap \Delta_+$ because $\prod_{\alpha \in R_+} \{(q^{1-\gamma_{(\alpha,\alpha)}} t^\alpha)_\infty / (q^{\gamma_{(\alpha,\alpha)}} t^\alpha)_\infty\}$ is bounded on Δ_+. Hence, (1.6) is also summable, in view of the quasi-symmetry (1.4) of $\Phi_0(t)$. It is important to note that this property does not depend on the choice of ξ, whence $\xi^{-\lambda} J(\xi)$ becomes a meromorphic function of $\xi \in (\mathbf{C}^*)^n$.

Then by definition $J(\xi)$ is a q-periodic function of ξ on H:

$$Q^\chi J(\xi) = J(\xi) \quad \text{for } \chi \in X.$$

The formula which we want to propose is the following.

Proposition 1 *$J(\xi)$ can be described as*

$$J(\xi) = C_1 \xi^{\lambda'} \prod_{\alpha \in R_+} \frac{\theta(q\xi^\alpha)}{\theta(q^{1+\gamma_{(\alpha,\alpha)}} \xi^\alpha)}, \tag{1.7}$$

(*we shall denote by $\tilde{\psi}_*(\xi)$ the term in the RHS divided by C_1 in the sequel*) *or equivalently*

$$J(\xi) = C_2 \sum_{\sigma \in W} \operatorname{sgn} \sigma \cdot U_\sigma(\xi)^{-1} \cdot \sigma \psi(\xi), \tag{1.8}$$

where C_1 and C_2 are constants with respect to ξ. Let $\psi(\xi)$ be a pseudo-constant defined as

$$\psi(\xi) = \xi^{\lambda'} \prod_{j=1}^{n} \frac{\theta(q^{c_j + \gamma_{(\omega_j,\omega_j)}} \xi^{\omega_j})}{\theta(q^{\gamma_{(\omega_j,\omega_j)}} \xi^{\omega_j})}, \tag{1.9}$$

where $\{c_j\}_{j=1}^n$ are uniquely determined by the expression $\omega = \sum_{j=1}^n c_j \omega_j$ with respect to the positive simple roots $\{\omega_j\}_{j=1}^n$ such that the corresponding positive roots are exactly R_+.

Remark 1 Since the symmetric n-dimensional cohomology associated with the Jackson integrals (1.6) is one dimensional, it may be conjectured that C_1 and C_2 can be written in product form by using q-gamma functions of $\gamma_{(\alpha,\alpha)}$. The explicit forms are presented in [13] but not yet proved except in the two-dimensional cases. In [17], Macdonald has given a formal proof of Ito's formula in an entirely different way. In [10, 11] Gustafson obtains various product formulas under a slightly different situation from ours. In [7], van Diejen obtains a similar product formula for BC_n type, by using Gustafson's result.

Proof 1: $J(\xi)$ satisfies the same quasi-symmetry as $\Phi_0(t)$:

$$\sigma J(\xi) = \operatorname{sgn} \sigma \cdot U_\sigma(\xi) \cdot J(\xi), \quad \sigma \in W \tag{1.10}$$

Since $J(\xi)\xi^{-\lambda}$ is a meromorphic function on H with poles lying in the set: $\prod_{\alpha \in R_+} \theta(q^{1+\gamma_{(\alpha,\alpha)}} \xi^\alpha) = 0$, $J(\xi)$ can be written as

$$J(\xi) = \xi^\lambda \frac{f(\xi)}{\prod_{\alpha \in R_+} \theta(q^{1+\gamma_{(\alpha,\alpha)}} \xi^\alpha)}, \tag{1.11}$$

where $f(\xi)$ denotes a quasi-periodic function of ξ in the sense that $Q^\chi f(\xi)/f(\xi)$ is a monomial in ξ for any $\chi \in X$, and satisfies the skew-symmetry: $\sigma f(\xi) = \operatorname{sgn}\sigma f(\xi)$ for $\sigma \in W$. This implies $f(\xi)$ vanishes if $\xi^\alpha = 1, q^{\pm 1}, q^{\pm 2}, \ldots$ Hence, $f(\xi)$ is divided out by the product $\prod_{\alpha\in R_+}\theta(q\xi^\alpha)$ and also by the product $\xi^{\lambda'}\prod_{\alpha\in R_+}\theta(q\xi^\alpha)$:

$$J(\xi) = \xi^{\lambda'} g(\xi) \prod_{\alpha\in R_+} \frac{\theta(q\xi^\alpha)}{\theta(q^{1+\gamma_{(\alpha,\alpha)}}\xi^\alpha)}, \tag{1.12}$$

where $g(\xi)$ denotes a holomorphic function on H. Since $g(\xi)$ is q-periodic, it must be constant. And the formula (1.7) follows.

To prove (1.8), we denote by $\tilde\psi(\xi)$ the sum $\sum_{\sigma\in W}\operatorname{sgn}\sigma U_\sigma^{-1}(\xi)\cdot\sigma\psi(\xi)$. Since $U_\sigma(t)$ has the expression (1.5), the poles of $\tilde\psi(\xi)$ lie in the set $\{\xi\in H;\ \prod_{\alpha\in R_+}\theta(q^{1+\gamma_{(\alpha,\alpha)}}\xi^\alpha)=0\}$. Moreover, it satisfies the same quasi-symmetry as in (1.10). Hence, it can be expressed as in the RHS of (1.8). It is sufficient to prove that it does not vanish identically. We evaluate the residues of $\tilde\psi(\xi)$ at the points of the equations $\theta(q^{\gamma_{(\omega_j,\omega_j)}}\xi^{\omega_j}) = 0$, say, of the equations

$$q^{\gamma_{(\omega_j,\omega_j)}}\xi^{\omega_j} = 1 \quad \text{for } 1\le j\le n. \tag{1.13}$$

These points appear only for the residues of the summand $\psi(\xi)$ itself in the sum $\tilde\psi(\xi)$. In fact, $\sigma(\omega_j) = \omega_j$, $1\le j\le n$ if and only if $\sigma =$ the identity. Since they do not vanish identically, $\tilde\psi(\xi)$ does not vanish identically. Let $\xi = \zeta$ be a point satisfying this system of equations. Then

$$\operatorname{Res}_{\xi=\zeta}\tilde\psi(\xi) = \operatorname{Res}_{\xi=\zeta}\psi(\xi) = \zeta^{\lambda'}\frac{\prod_{j=1}^n\theta(q^{c_j})}{\theta'(1)^n}.$$

Equation (1.8) is thus proved. □

2. A-type root system

In the sequel $\nu_1, \nu_2, \nu_3, \ldots$ will denote integers.

At first we can take $\Phi(t)$ in a slightly more general way than (1.1), namely we take $\Phi(t)$ as

$$\Phi(t) = t_1^{\alpha_1}\cdots t_n^{\alpha_n}\prod_{j=1}^n\frac{(t_j)_\infty}{(q^\beta t_j)_\infty}\prod_{1\le i<j\le n}\frac{(q^{1-\gamma}t_j/t_i)_\infty}{(q^\gamma t_j/t_i)_\infty} \tag{2.1}$$

for $\alpha_j, \beta, \gamma \in \mathbf{C}$ such that $\alpha_j = \alpha_1 + (j-1)(1-2\gamma)$, and

$$\Phi_0(t) = \Phi(t)D(t), \quad \text{for } D(t) = \prod_{1\le i<j\le n}(t_i - t_j). \tag{2.2}$$

X consists of the lattice points $x=(\nu_1,\ldots,\nu_n)\in\mathbf{Z}^n$. In this case, by the same argument as in the preceding section, the sum (1.6) is summable provided

$$-\beta>\alpha_1+n-1>0,\quad -\beta>\alpha_1+(n-1)(2-2\gamma)>0.$$

The Weyl group W is isomorphic to the symmetric group $\mathcal{S}_n$ of nth degree and the one-cocycle $\{U_\sigma(t)\}$ is given as

$$U_\sigma(t)=\prod_{i<j,\sigma^{-1}(i)>\sigma^{-1}(j)}\left(\frac{t_j}{t_i}\right)^{2\gamma-1}\frac{\theta(q^\gamma t_j/t_i)}{\theta(q^{1-\gamma}t_j/t_i)} \tag{2.3}$$

for any permutation σ among the n letters $\{1,2,\ldots,n\}$.

As a special case of Jackson integrals, we take $\xi=\xi_F$ for

$$\xi_1=q,\quad \xi_2=q^{1+\gamma},\ldots,\xi_n=q^{1+(n-1)\gamma}.$$

The integral J over $[0,\xi_F\infty]_q$ is done only over the set $\langle\xi_F\rangle$ consisting of the points t such that $t_1=q^{1+\nu_1}, t_2/t_1=q^{\gamma+\nu_2},\ldots,t_n/t_{n-1}=q^{\gamma+\nu_n}$ for each $\nu_j\in\mathbf{Z}_{\geq 0}$. $\langle\xi_F\rangle$ is clearly a subset of $[0,\xi_F\infty]_q$. We call it "α-stable cycle", since the absolute value $|t^\alpha|$ is maximal at ξ_F and smaller elsewhere in $\langle\xi_F\rangle$ than the value at ξ_F. (Here the terminology "stable" is used as a discrete analog of the one in case of ordinary integrals.)

Hence, the substitution of the point ξ_F into $\Phi_0(t)$ gives the asymptotic behavior of $J=J(\alpha_1)$ for $\alpha_1\to+\infty$ as

$$J=q^{A_n}\prod_{j=1}^n\frac{\Gamma_q(\beta+1+(j-1)\gamma)\Gamma_q(j\gamma)}{\Gamma_q(\gamma)}(1+O(q^{\alpha_1})) \tag{2.4}$$

for $A_n=\sum_{j=1}^n(\alpha_j+n-j)[1+(j-1)\gamma]$, where $\Gamma_q(u)$ denotes the q-gamma function $(1-q)^{1-u}(q)_\infty/(q^u)_\infty$.

On the other hand, we can take $\xi=\eta_F$ for

$$\xi_1=q^{-\beta},\quad \xi_2=q^{-\beta-\gamma},\ldots,\xi_n=q^{-\beta-(n-1)\gamma}.$$

The Jackson integral J over $[0,\eta_F\infty]_q$ is meaningless, since $\Phi_0(t)$ has poles there. We denote by $\langle\eta_F\rangle$ the set of points t such that

$$t_1=q^{-\beta-\nu_1},\quad t_2/t_1=q^{-\gamma-\nu_2},\ldots,t_n/t_{n-1}=q^{-\gamma-\nu_n}$$

for $\nu_j\in\mathbf{Z}_{\geq\mathbf{0}}$. $\langle\eta_F\rangle$ is a subset of $[0,\eta_F\infty]_q$. Then we can replace the sum (1.6) by the following regularization (see [1] or [4] for details):

$$\begin{aligned}&\operatorname{reg}\int_{[0,\eta_F\infty]_q}\Phi_0(t)\cdot\varpi\quad\left(\text{also denoted by }\int_{\mathrm{reg}\langle\eta_\mathrm{F}\rangle}\Phi_0(t)\cdot\varpi\right)\\&\quad=\sum_{\nu_j\geq 0}\prod_{j=1}^n\operatorname{Res}_{t_j/t_{j-1}=q^{-\gamma-\nu_j}}\left[\Phi_0(t)\frac{dt_1}{t_1}\wedge\cdots\wedge\frac{dt_n}{t_n}\right],\end{aligned} \tag{2.5}$$

where $\prod_{j=1}^{n}$ Res denotes the residue at each point of $\langle \eta_F \rangle$. Here we have put $t_0 = q^{-\beta+\gamma}$. Equation (2.5) is well defined because $\Phi_0(t)$ has simple poles at each point of $\langle \eta_F \rangle$.

We define the quotient of theta functions

$$\psi_n(\xi, \eta_F) = (1-q)^n \prod_{j=1}^{n} \left(\xi_j q^{\beta+(j-1)\gamma}\right)^{\alpha_j} \frac{(q)_\infty^3 \theta(q^{\alpha_j+\cdots+\alpha_n+\gamma+1}\xi_j/\xi_{j-1})}{\theta(q^{\alpha_j+\cdots+\alpha_n+1})\theta(q^{\gamma+1}\xi_j/\xi_{j-1})} \tag{2.6}$$

for $\xi_0 = q^{-\beta+\gamma}$. It is a pseudo constant. The connection coefficient (or ratio) between the Jackson integrals over the cycles $[0, \xi\infty]_q$ and $\mathrm{reg}\langle \eta_F \rangle$ is denoted by $([0, \xi\infty]_q : \mathrm{reg}\langle \eta_F \rangle)_{\Phi_0}$.

Then we can give the following formula (see [4]):

$$([0, \xi\infty]_q : \mathrm{reg}\langle \eta_F \rangle)_{\Phi_0} = \sum_{\sigma \in \mathcal{S}_n} \sigma \psi_n(\xi, \eta_F) \cdot \mathrm{sgn}\, \sigma \cdot U_\sigma(\xi)^{-1}, \tag{2.7}$$

where $\sigma\varphi(\xi)$ denotes $\varphi(\sigma^{-1}(\xi))$.

We evaluate the RHS of (2.7) in case where β vanishes. We denote by $F_n(\xi) = F_n(\xi, \alpha_1, \gamma)$ the sum

$$\sum_{\sigma \in \mathcal{S}_n} \mathrm{sgn}\, \sigma \cdot U_\sigma(\xi)^{-1} \cdot \sigma \left\{ \prod_{j=1}^{n} \xi_j^{\alpha_j} \frac{\theta(q^{\alpha_j+\cdots+\alpha_n+\gamma+1}\xi_j/\xi_{j-1})}{\theta(q^{\gamma+1}\xi_j/\xi_{j-1})} \right\} \tag{2.8}$$

for $\xi_0 = q^\gamma$. Then we have the product formula (see [2]):

$$F_n(\xi, \alpha_1, \gamma) = f_n(\alpha_1, \gamma) \cdot g_n(\xi, \alpha_1, \gamma), \tag{2.9}$$

where

$$f_n(\alpha_1, \gamma) = (-1)^{n(n-1)/2} \prod_{j=1}^{n} q^{-(j-1)^2\gamma} \frac{\theta(q^{\alpha_j+\cdots+\alpha_n+1})}{\theta(q^{1+\alpha_1-(n+j-2)\gamma})}, \tag{2.10}$$

$$g_n(\xi, \alpha_1, \gamma) = \prod_{j=1}^{n} \xi_j^{\alpha_1-2(j-1)\gamma} \frac{\theta(q^{\alpha_1+1-(n-1)\gamma}\xi_j)}{\theta(q\xi_j)} \cdot \prod_{1\le i<j\le n} \frac{\xi_i \theta(\xi_j/\xi_i)}{\theta(q^{\gamma+1}\xi_j/\xi_i)}. \tag{2.11}$$

Hence, $\xi_j\,(j \geq 1)$ being replaced by $\xi_j q^\beta$ in (2.9),

$$\begin{aligned} &([0, \xi\infty]_q : \mathrm{reg}\langle \eta_F \rangle)_{\Phi_0} \\ &\quad = (1-q)^n \prod_{j=1}^{n} \frac{(q)_\infty^3 q^{\alpha_j\gamma(j-1)}}{\theta(q^{\alpha_j+\cdots+\alpha_n+1})} \cdot f_n(\alpha_1, \gamma) \cdot g_n(q^\beta\xi, \alpha_1, \gamma). \end{aligned} \tag{2.12}$$

It is also possible to evaluate explicitly $\int_{\langle \xi_F \rangle} \Phi_0(t) \cdot \varpi$ itself. In fact, the following formula is known (see [16]).

Proposition 2

$$
\begin{aligned}
&\int_{\langle\xi_F\rangle} \Phi_0(t)\cdot\varpi \\
&\quad = q^{A_n}\prod_{j=1}^{n}\frac{\Gamma_q(\beta+1+(j-1)\gamma)\Gamma_q(\alpha_1+n-1-(n+j-2)\gamma)\Gamma_q(j\gamma)}{\Gamma_q(\gamma)\Gamma_q(\alpha_1+\beta+n-(n-j)\gamma)}.
\end{aligned}
$$

In particular, we have

$$
\begin{aligned}
&(\langle\xi_F\rangle : \mathrm{reg}\,\langle\eta_F\rangle)_{\Phi_0} \\
&\quad = (1-q)^n(q)_\infty^{3n}\cdot q^{K_n}\cdot\prod_{j=1}^{n}\frac{\theta(q^{\alpha_1+\beta+2-(n-j)\gamma})\theta(q^\gamma)}{\theta(q^{\alpha_1+1-(n+j-2)\gamma})\theta(q^{\beta+2+(j-1)\gamma})\theta(q^{j\gamma})},
\end{aligned}
\tag{2.13}
$$

where K_n denotes

$$
\begin{aligned}
K_n = &-\frac{2}{3}n(n-1)(2n-1)\gamma^2-\frac{n(n-1)}{2}\gamma+n\beta\{\alpha_1-(n-1)\gamma\} \\
&+n\alpha_1\{1+(n-1)\gamma\}.
\end{aligned}
\tag{2.14}
$$

Next, we take as $\Phi(t)$ the q-multiplicative function

$$
\Phi(t) = t_1^{\alpha_1}\cdots t_n^{\alpha_n}\frac{(t_1\cdots t_n)_\infty}{(q^\beta t_1\cdots t_n)_\infty}\prod_{1\le i<j\le n}\frac{(q^{1-\gamma}t_j/t_i)_\infty}{(q^\gamma t_j/t_i)_\infty}
\tag{2.15}
$$

for $\alpha_j = \alpha_1+(j-1)(1-2\gamma)$. Let $\Phi_0(t)$ be as in (2.2). As in case of (2.1), $J(\xi)$ is summable, provided

$$
-\beta > \alpha_1+n-1 > 0, \quad -\beta > \alpha_1+(n-1)(2-2\gamma) > 0.
$$

X consists of the points $x = (\nu_1,\ldots,\nu_n)\in\mathbf{Z}^n$. Then by the argument in Section 1, we have the formula (1.7) for

$$
J(\xi) = C_1\prod_{j=1}^{n}\xi_j^{\alpha_1-2(j-1)\gamma}\frac{\theta(q\xi_1\cdots\xi_n)}{\theta(q^{1+\beta}\xi_1\cdots\xi_n)}\prod_{1\le i<j\le n}\frac{\theta(q\xi_j/\xi_i)}{\theta(q^{1+\gamma}\xi_j/\xi_i)}.
\tag{2.16}
$$

3. B_n, C_n, D_n, G_2, F_4 and $E_6 \sim E_8$ type root systems

For B_n-type, we take as $\Phi(t)$ the multiplicative function

$$
\Phi(t) = t_1^{\alpha_1}\cdots t_n^{\alpha_n}\prod_{j=1}^{n}\frac{(t_jq^{1-\beta})_\infty}{(t_jq^\beta)_\infty}\prod_{1\le i<j\le n}\frac{(q^{1-\gamma}t_j/t_i)_\infty\cdot(q^{1-\gamma}t_it_j)_\infty}{(q^\gamma t_j/t_i)_\infty\cdot(q^\gamma t_it_j)_\infty}
\tag{3.1}
$$

for $\alpha_j = \alpha_1 + (j-1)(1-2\gamma), \alpha_1 = \frac{1}{2} - \beta$. This is exactly the one obtained from (1.1) for the root system B_n when an explicit realization of positive roots is properly chosen for $\beta = \gamma_1, \gamma = \gamma_2$. The Weyl group W is isomorphic to the semi-direct product $\mathbf{Z}_2^n \times \mathcal{S}_n$. $\Phi_0(t)$ is given by

$$\begin{aligned}\Phi_0(t) &= \Phi(t)\mathcal{A}\left(t_1^{-(1/2)+n} t_2^{-(3/2)+n} \cdots t_n^{(1/2)}\right)\\ &= (-1)^{n(n+1)/2}\Phi(t)(t_1\cdots t_n)^{-n+(1/2)}\prod_{j=1}^{n}(1-t_j)\\ &\quad\cdot\prod_{1\le i<j\le n}(t_i - t_j)(1-t_it_j),\end{aligned} \tag{3.2}$$

where $\mathcal{A}$ denotes the alternating sum with respect to the Weyl group W.

X consists of the points $x = (\nu_1, \dots, \nu_n) \in \mathbf{Z}^n$.

Let $\tilde{\psi}_*(\xi)$ denote the product $\xi^{\lambda'}\prod_{\alpha\in R_+}\frac{\theta(q\xi^\alpha)}{\theta(q^{1+\gamma(\alpha,\alpha)}\xi^\alpha)}$ in the RHS of (1.7). Then $\tilde{\psi}_*(\xi)$ is represented more concretely as

$$\tilde{\psi}_*(\xi) = \prod_{j=1}^{n}\xi_j^{\alpha_j-(j-\frac{1}{2})}\frac{\theta(q\xi_j)}{\theta(q^{\beta+1}\xi_j)}\prod_{1\le i<j\le n}\frac{\theta(q\xi_j/\xi_i)\theta(q\xi_i\xi_j)}{\theta(q^{1+\gamma}\xi_j/\xi_i)\theta(q^{1+\gamma}\xi_i\xi_j)}. \tag{3.3}$$

The explicit formula for C_1 has given by Ito [13] in case $n = 2$.

Similarly for C_n-type, we take as $\Phi(t)$ the function

$$\Phi(t) = t_1^{\alpha_1}\cdots t_n^{\alpha_n}\prod_{j=1}^{n}\frac{\left(q^{1-\beta}t_j^2\right)_\infty}{\left(q^{\beta}t_j^2\right)_\infty}\prod_{1\le i<j\le n}\frac{(q^{1-\gamma}t_j/t_i)_\infty\cdot(q^{1-\gamma}t_it_j)_\infty}{(q^{\gamma}t_j/t_i)_\infty\cdot(q^{\gamma}t_it_j)_\infty} \tag{3.4}$$

for $\alpha_1 = 1-2\beta, \alpha_j = \alpha_1 + (j-1)(1-2\gamma)$. This corresponds to (1.1) for $\gamma = \gamma_2$, $\beta = \gamma_4$. The Weyl group W is isomorphic to the one of B_n-type. $\Phi_0(t)$ is equal to $\Phi(t)\mathcal{A}(t_1^n t_2^{n-1}\cdots t_n)$.

X consists of the points $x = (\nu_1, \dots, \nu_n) \in \mathbf{Z}^n$ or $(\nu_1 + \frac{1}{2}, \dots, \nu_n + \frac{1}{2}) \in \mathbf{Z}^n + (\frac{1}{2}, \dots, \frac{1}{2})$. $\tilde{\psi}_*(\xi)$ is then given as

$$\tilde{\psi}_*(\xi) = \prod_{j=1}^{n}\xi_j^{\alpha_j-j}\frac{\theta\left(q\xi_j^2\right)}{\theta\left(q^{\beta+1}\xi_j^2\right)}\prod_{1\le i<j\le n}\frac{\theta(q\xi_j/\xi_i)\theta(q\xi_i\xi_j)}{\theta(q^{1+\gamma}\xi_j/\xi_i)\theta(q^{1+\gamma}\xi_i\xi_j)}. \tag{3.5}$$

For D_n-type we take as $\Phi(t)$ the function

$$\Phi(t) = \prod_{j=1}^{n}t_j^{\alpha_j-j+1}\prod_{1\le i<j\le n}\frac{(q^{1-\gamma}t_j/t_i)_\infty\cdot(q^{1-\gamma}t_it_j)_\infty}{(q^{\gamma}t_j/t_i)_\infty\cdot(q^{\gamma}t_it_j)_\infty} \tag{3.6}$$

for $\alpha_1 = 0$ and $\alpha_j = \alpha_1 + (j-1)(1-2\gamma)$. This corresponds to (1.1) for $\gamma = \gamma_2$. $\Phi_0(t)$ is equal to $\Phi(t)\mathcal{A}(t_1^n t_2^{n-1}\cdots t_n)$. The Weyl group W is isomorphic to the semi-direct product of $\mathbf{Z}_2^n$ and $\mathcal{S}_n$. X is the same as in case of C_n type. $\tilde{\psi}_*(\xi)$ is then equal to

$$\tilde{\psi}_*(\xi) = \prod_{j=1}^{n}\xi_j^{\alpha_j-(j-1)}\prod_{1\le i<j\le n}\frac{\theta(q\xi_j/\xi_i)\theta(q\xi_i\xi_j)}{\theta(q^{1+\gamma}\xi_j/\xi_i)\theta(q^{1+\gamma}\xi_i\xi_j)}. \tag{3.7}$$

For E_8-type we can choose an orthonormal basis $\{\epsilon_j\}_{j=1}^8$ in $\mathbf{h}^* \cong \mathbf{R}^8$ such that the positive simple roots are

$$\omega_1 = \frac{1}{2}(\epsilon_1 + \epsilon_8 - \epsilon_2 - \epsilon_3 - \epsilon_4 - \epsilon_5 - \epsilon_6 - \epsilon_7),$$
$$\omega_2 = \epsilon_1 + \epsilon_2, \quad \omega_3 = \epsilon_2 - \epsilon_1, \quad \omega_4 = \epsilon_3 - \epsilon_2, \quad \omega_5 = \epsilon_4 - \epsilon_3, \quad \omega_6 = \epsilon_5 - \epsilon_4,$$
$$\omega_7 = \epsilon_6 - \epsilon_5, \quad \omega_8 = \epsilon_7 - \epsilon_6.$$

The positive roots are $\pm\epsilon_i + \epsilon_j (1 \le i < j \le 8)$ and $\frac{1}{2}(\epsilon_8 + \sum_{i=1}^7 \nu(i)\epsilon_i)$ where $\{\nu(k)\}_{k=1}^7$ moves over the set ± 1 such that the number of $\nu(k)$ which are equal to -1 is even. Then $\Phi(t)$ has the quasi-symmetry when and only when

$$\alpha_j = (j-1)(1-2\gamma)(1 \le j \le 7), \quad \alpha_8 = 23(1-2\gamma).$$
$$\Phi(t) = \prod_{j=1}^{8} t_j^{\alpha_j} \prod_{1\le i<j\le 8} \frac{(q^{1-\gamma}t_j/t_i)_\infty (q^{1-\gamma}t_i t_j)_\infty}{(q^{\gamma}t_j/t_i)_\infty (q^{\gamma}t_i t_j)_\infty}$$
$$\cdot \prod_{\nu(k)=\pm 1} \frac{\left(q^{1-\gamma}t_1^{\nu(1)/2}\cdots t_7^{\nu(7)/2}t_8^{1/2}\right)_\infty}{\left(q^{\gamma}t_1^{\nu(1)/2}\cdots t_7^{\nu(7)/2}t_8^{1/2}\right)_\infty}. \tag{3.8}$$

The pairing between $\mu = \sum_{j=1}^8 \mu_j \epsilon_j \in \mathbf{h}^*$ and $x = (x_1, \ldots, x_8) \in \mathbf{h}(\cong \mathbf{R}^8)$ is given by

$$\mu, x \to \sum_{j=1}^{8} \mu_j x_j.$$

X is the eight-dimensional lattice in $\mathbf{h}$ consisiting of the points $x = (x_1, \ldots, x_6, x_7, x_8) \in \mathbf{h}$ such that $\alpha(x) \equiv 0(\mathbf{Z})$ for $\alpha \in R_+$. In other words, $x \in X$ consists of the points x such that $x_j = \nu_j, (1 \le j \le 8)$ or $x_j = \nu_j + \frac{1}{2}, (1 \le j \le 8)$ satisfying the equality

$$\nu_1 + \cdots + \nu_8 \equiv 0(2).$$

$\tilde{\psi}_*(\xi)$ is then equal to

$$\tilde{\psi}_*(\xi) = \left\{\prod_{j=1}^{7} \xi_j^{-2(j-1)\gamma}\right\} \xi_8^{-46\gamma} \prod_{1\le i<j\le 8} \frac{\theta(q\xi_j/\xi_i)\theta(q\xi_i\xi_j)}{\theta(q^{1+\gamma}\xi_j/\xi_i)\theta(q^{1+\gamma}\xi_i\xi_j)}$$
$$\cdot \prod_{\nu(k)=\pm 1} \frac{\theta\left(q\xi_1^{\nu(1)/2}\cdots \xi_7^{\nu(7)/2}\xi_8^{1/2}\right)}{\theta\left(q^{\gamma+1}\xi_1^{\nu(1)/2}\cdots \xi_7^{\nu(7)/2}\xi_8^{1/2}\right)}. \tag{3.9}$$

For E_7-type, in terms of the above basis $\{\epsilon_j\}_{j=1}^8$, the positive simple roots are

$$\omega_1 = \frac{1}{2}(\epsilon_1 + \epsilon_8 - \epsilon_2 - \epsilon_3 - \epsilon_4 - \epsilon_5 - \epsilon_6 - \epsilon_7),$$
$$\omega_2 = \epsilon_1 + \epsilon_2, \quad \omega_3 = \epsilon_2 - \epsilon_1, \quad \omega_4 = \epsilon_3 - \epsilon_2, \quad \omega_5 = \epsilon_4 - \epsilon_3, \quad \omega_6 = \epsilon_5 - \epsilon_4,$$
$$\omega_7 = \epsilon_6 - \epsilon_5.$$

The positive roots are $\pm\epsilon_i+\epsilon_j (1 \le i < j \le 6)$, $-\epsilon_7+\epsilon_8$ and $\frac{1}{2}(-\epsilon_7+\epsilon_8+\sum_{i=1}^{6}\nu(i)\epsilon_i)$ where $\{\nu(k)\}_{k=1}^{6}$ moves over the set ± 1 such that the number of $\nu(k)$ which are equal to -1 is odd. Then $\Phi(t)$ has the quasi-symmetry when and only when

$$\alpha_j = (j-1)(1-2\gamma)(1 \le j \le 6), \quad -\alpha_7 = \alpha_8 = \frac{17}{2}(1-2\gamma).$$

$$\Phi(t) = \left\{\prod_{j=1}^{6} t_j^{\alpha_j}\right\} t_8^{\alpha_8} \cdot \left\{\prod_{1\le i<j\le 6} \frac{(q^{1-\gamma}t_j/t_i)_\infty (q^{1-\gamma}t_i t_j)_\infty}{(q^\gamma t_j/t_i)_\infty (q^\gamma t_i t_j)_\infty}\right\} \cdot \frac{(q^{1-\gamma}t_8)_\infty}{(q^\gamma t_8)_\infty}$$
$$\cdot \prod_{\nu(k)=\pm 1} \frac{\left(q^{1-\gamma} t_1^{\nu(1)/2}\cdots t_6^{\nu(6)/2} t_8^{1/2}\right)_\infty}{\left(q^{\gamma} t_1^{\nu(1)/2}\cdots t_6^{\nu(6)/2} t_8^{1/2}\right)_\infty}. \tag{3.10}$$

X consists of the points x such that either $x_j = \nu_j (1 \le j \le 8)$, $x_7 = 0$ where $\sum_{j=1}^{6}\nu_j + \nu_8 \equiv 0(2)$, or $x_j = \nu_j + \frac{1}{2}(1 \le j \le 6)$, $x_7 = 0$, $x_8 = \nu_8$ where $\sum_{j=1}^{6}\nu_j + \nu_8 \equiv 1(2)$.

We normalize ξ as $\xi_7 = 1$. $\tilde{\psi}_*(\xi)$ is then equal to

$$\tilde{\psi}_*(\xi) = \left\{\prod_{j=1}^{6} \xi_j^{-2(j-1)\gamma}\right\} \cdot \xi_8^{-17\gamma} \prod_{1\le i<j\le 6} \frac{\theta(q\xi_j/\xi_i)\theta(q\xi_i\xi_j)}{\theta(q^{1+\gamma}\xi_j/\xi_i)\theta(q^{1+\gamma}\xi_i\xi_j)} \cdot \frac{\theta(q\xi_8)}{\theta(q^{\gamma+1}\xi_8)}$$
$$\cdot \prod_{\nu(k)=\pm 1} \frac{\theta\left(q\xi_1^{\nu(1)/2}\cdots \xi_6^{\nu(6)/2}\xi_8^{1/2}\right)}{\theta\left(q^{\gamma+1}\xi_1^{\nu(1)/2}\cdots \xi_6^{\nu(6)/2}\xi_8^{1/2}\right)}. \tag{3.11}$$

For E_6-type, in $\mathbf{R}^8$ the positive simple roots are

$$\omega_1 = \frac{1}{2}(\epsilon_1+\epsilon_8-\epsilon_2-\epsilon_3-\epsilon_4-\epsilon_5-\epsilon_6-\epsilon_7),$$
$$\omega_2 = \epsilon_1+\epsilon_2, \quad \omega_3 = \epsilon_2-\epsilon_1, \quad \omega_4 = \epsilon_3-\epsilon_2, \quad \omega_5 = \epsilon_4-\epsilon_3, \quad \omega_6 = \epsilon_5-\epsilon_4.$$

The positive roots are $\pm\epsilon_i+\epsilon_j (1 \le i < j \le 5)$ and $\frac{1}{2}(\epsilon_8-\epsilon_6-\epsilon_7+\sum_{i=1}^{5}\nu(i)\epsilon_i)$ where $\{\nu(k)\}_{k=1}^{6}$ moves over the set ± 1 such that the number of $\nu(k)$ which are equal to -1 is even. Then $\Phi(t)$ has the quasi-symmetry when and only when

$$\alpha_j = (j-1)(1-2\gamma)(1 \le j \le 5), \quad -\alpha_6 = -\alpha_7 = \alpha_8 = 4(1-2\gamma).$$

$$\Phi(t) = \left\{\prod_{j=1}^{5} t_j^{\alpha_j}\right\} t_8^{\alpha_8} \cdot \left\{\prod_{1\le i<j\le 5} \frac{(q^{1-\gamma}t_j/t_i)_\infty (q^{1-\gamma}t_i t_j)_\infty}{(q^\gamma t_j/t_i)_\infty (q^\gamma t_i t_j)_\infty}\right\}$$
$$\cdot \prod_{\nu(k)=\pm 1} \frac{\left(q^{1-\gamma} t_1^{\nu(1)/2}\cdots t_5^{\nu(5)/2} t_8^{1/2}\right)_\infty}{\left(q^{\gamma} t_1^{\nu(1)/2}\cdots t_5^{\nu(5)/2} t_8^{1/2}\right)_\infty}. \tag{3.12}$$

X consists of the points x such that either $x_j = \nu_j\ (1 \le j \le 8)$, $x_6 = x_7 = 0$ where $\sum_{j=1}^{5}\nu_j + \nu_8 \equiv 0(2)$, or $x_j = \nu_j + \frac{1}{2}\ (1 \le j \le 5)$, $x_6 = x_7 = 0$, $x_8 = \nu_8 - \frac{1}{2}$ where $\sum_{j=1}^{5}\nu_j + \nu_8 \equiv 1(2)$.

We normalize ξ as $\xi_6 = \xi_7 = 1$. $\tilde{\psi}_*(\xi)$ is then equal to

$$\tilde{\psi}_*(\xi) = \left\{ \prod_{j=1}^{5} \xi_j^{-2(j-1)\gamma} \right\} \cdot \xi_8^{-8\gamma} \prod_{1 \le i < j \le 5} \frac{\theta(q\xi_j/\xi_i)\theta(q\xi_i\xi_j)}{\theta(q^{1+\gamma}\xi_j/\xi_i)\theta(q^{1+\gamma}\xi_i\xi_j)}$$
$$\cdot \prod_{\nu(k)=\pm 1} \frac{\theta\big(q\xi_1^{\nu(1)/2} \cdots \xi_5^{\nu(5)/2}\xi_8^{1/2}\big)}{\theta\big(q^{\gamma+1}\xi_1^{\nu(1)/2} \cdots \xi_5^{\nu(5)/2}\xi_8^{1/2}\big)}. \tag{3.13}$$

For F_4-type we can choose an orthonormal basis $\{(\epsilon_j)\}_{j=1}^4$ such that the positive simple roots are $\omega_1 = \epsilon_2 - \epsilon_3, \omega_2 = \epsilon_3 - \epsilon_4, \omega_3 = \epsilon_4, \omega_4 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)$. The positive roots are $\epsilon_j, \epsilon_i \pm \epsilon_j$ $(1 \le i < j \le 4)$ and $\frac{1}{2}$ $(\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)$. We put

$$\Phi(t) \;=\; t_1^{\frac{11}{2}-5\beta-6\beta} t_2^{\frac{5}{2}-\beta-4\gamma} t_3^{\frac{3}{2}-\beta-2\gamma} t_4^{\frac{1}{2}-\beta} \cdot \prod_{j=1}^{4} \frac{(q^{1-\beta}t_j)_\infty}{(q^\beta t_j)_\infty}$$
$$\cdot \prod_{1 \le i < j \le 4} \frac{(q^{1-\gamma}t_i/t_j)_\infty (q^{1-\gamma}t_it_j)_\infty}{(q^\gamma t_i/t_j)_\infty (q^\gamma t_it_j)_\infty} \cdot \prod_{\nu(k)=\pm 1} \frac{\big(q^{1-\beta}t_1^{1/2}t_2^{\nu(2)/2}t_3^{\nu(3)/2}t_4^{\nu(4)/2}\big)_\infty}{\big(q^{\beta}t_1^{1/2}t_2^{\nu(2)/2}t_3^{\nu(3)/2}t_4^{\nu(4)/2}\big)_\infty}, \tag{3.14}$$

where $\nu(k)$ moves over the set $\{\pm 1\}$.

X consists of the points $x = (x_1, x_2, x_3, x_4)$ such that $x_j = \nu_j$ where $\nu_1 + \nu_2 + \nu_3 + \nu_4 \equiv 0(2)$.

$\tilde{\psi}_*(\xi)$ is then equal to

$$\tilde{\psi}_*(\xi) = \xi_1^{-5\beta-6\gamma}\xi_2^{-\beta-4\gamma}\xi_3^{-\beta-2\gamma}\xi_4^{-\beta} \prod_{j=1}^{4} \frac{\theta(q\xi_i)}{\theta(q^{1+\beta}\xi_i)}$$
$$\cdot \prod_{1 \le i < j \le 4} \frac{\theta(q\xi_i/\xi_j)\theta(q\xi_i\xi_j)}{\theta(q^{1+\gamma}\xi_i/\xi_j)\theta(q^{1+\gamma}\xi_i\xi_j)} \cdot \prod_{\nu(k)=\pm 1} \frac{\theta\big(q\xi_1^{1/2}\xi_2^{\nu(2)/2}\xi_3^{\nu(3)/2}\xi_4^{\nu(4)/2}\big)}{\theta\big(q^{1+\beta}\xi_1^{1/2}\xi_2^{\nu(2)/2}\xi_3^{\nu(3)/2}\xi_4^{\nu(4)/2}\big)}. \tag{3.15}$$

For G_2-type we can choose an orthonormal basis $\epsilon_1, \epsilon_2, \epsilon_3$ such that the positive simple roots are $\epsilon_1 - \epsilon_2, -2\epsilon_1 + \epsilon_2 + \epsilon_3$ and that the positive roots are $\epsilon_1 - \epsilon_2, -\epsilon_2 + \epsilon_3, -\epsilon_1 + \epsilon_3, \epsilon_1 + e_3 - 2\epsilon_2, \epsilon_2 + \epsilon_3 - 2\epsilon_1, 2\epsilon_3 - \epsilon_1 - \epsilon_2$. $\Phi(t)$ is given by

$$\Phi(t) = t_1^{2\gamma-1}t_3^{3-2\beta-4\gamma} \frac{(q^{1-\beta}t_1)_\infty (q^{1-\beta}t_3)_\infty (q^{1-\beta}t_3/t_1)_\infty}{(q^\beta t_1)_\infty (q^\beta t_3)_\infty (q^\beta t_3/t_1)_\infty}$$
$$\times \frac{\big(q^{1-\gamma}t_3/t_1^2\big)_\infty \big(q^{1-\gamma}t_3^2/t_1\big)_\infty (q^{1-\gamma}t_1t_3)_\infty}{\big(q^\gamma t_3/t_1^2\big)_\infty \big(q^\gamma t_3^2/t_1\big)_\infty (q^\gamma t_1t_3)_\infty}. \tag{3.16}$$

X consists of the points $x = (x_1, x_2, x_3)$ such that $x_1 = \nu_1, x_3 = \nu_3, x_2 = 0$. We normalize ξ as $\xi_2 = 1$. $\tilde{\psi}_*(\xi)$ is then equal to

$$\tilde{\psi}_*(\xi) = \xi_1^{2\gamma} \xi_3^{-2\beta-4\gamma} \frac{\theta(q\xi_1)\theta(q\xi_3)\theta(q\xi_3/\xi_1)}{\theta(q^{\beta+1}\xi_1)\theta(q^{\beta+1}\xi_3)\theta(q^{\beta+1}\xi_3\xi_1)} \times \frac{\theta\left(q\xi_3/\xi_1^2\right)\theta\left(q\xi_3^2/\xi_1\right)\theta(q\xi_1\xi_3)}{\theta\left(q^{\gamma+1}\xi_3/\xi_1^2\right)\theta\left(q^{\gamma+1}\xi_3^2/\xi_1\right)\theta(q^{\gamma+1}\xi_1\xi_3)}. \tag{3.17}$$

References

1. K. Aomoto, "On connection coefficients for q-difference system of A type Jackson integral," *SIAM J. Math. Anal.* **25** (1994), 256–273.
2. K. Aomoto, "On a theta product formula for the symmetric A type connection function," *Osaka J. Math.* **32** (1995), 35–39.
3. K. Aomoto and Y. Kato, "Gauss decomposition of connection matrices and application to Yang-Baxter equation, I," *Proc. Japan Acad.* **69** (1993), 238–241; II, ibid., 341–344.
4. K. Aomoto and Y. Kato, "Connection coefficients for symmetric A type Jackson integrals," *Duke Math. Jour.* **74** (1994), 129–143.
5. R. Askey, "Some basic hypergeometric extensions of integrals of Selberg and Andrews," *SIAM J. Math. Anal.* **11** (1980), 938–951.
6. I. Cherednik, "The Macdonald constant term conjecture," *Int. Math. Res. Notices* (6) (1993), 165–177.
7. J.F. van Diejen, "On certain multiple Bailey, Rogers and Dougall type summation formulas," Pub. of R.I.M.S., 33(1997), 483–508.
8. R. Evans, "Multidimensional q-Beta integrals," *SIAM J. Math. Anal.* **23** (1992), 758–765.
9. P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John Wiley and Sons, 1978.
10. R.A. Gustafson, "Some q-Beta integrals on $SU(n)$ and $S_p(n)$ that generalize the Askey-Wilson and Nasrallah-Rahman integrals," *SIAM J. Math. Anal.* **25** (1994), 441–449.
11. R.A. Gustafson, "Some q-beta and Mellin-Barnes integrals on compact Lie groups and Lie algebras," *Trans. AMS* **341** (1994), 69–119.
12. L. Habsieger, "Une q-integral de Selberg et Askey," *Trans. AMS* **19** (1988), 1475–1489.
13. Masa Ito, "On theta product formula for Jackson integrals associated with root systems of rank two," preprint, 1995.
14. K. Kadell, "A proof of Askey's conjectured q-analog of Selberg's integral and a conjecture of Morris," *SIAM J. Math. Anal.* **19** (1988), 969–986.
15. K. Kadell, "A proof of the q Macdonald-Morris conjecture for BC_n," *Mem. AMS* **108** (516) (1994), 1–80.
16. J. Kaneko, "q-Selberg integrals and Macdonald polynomials," *Ann. Ecole Norm. Sup.* **29** (1996), 583–637.
17. I. Macdonald, "A formal identity for affine root systems," preprint, 1996.
18. K. Mimachi, "Connection problem in holonomic q-difference system associated with a Jackson integral of Jordan-Pochhammer type," *Nagoya Math. J.* **116** (1989), 149–161.
19. T. Terasoma, "Determinants of q-hypergeometric functions and another proof of Askey conjecture," preprint, 1995.