Journal of Algebraic Combinatorics **8** (1998), 235–252

Symplectic Matroids

ALEXANDRE V. BOROVIK
Department of Mathematics, UMIST, PO Box 88, Manchester M60 1QD, United Kingdom

ISRAEL GELFAND
Department of Mathematics, Rutgers University, New Brunswick NJ 08903

NEIL WHITE white@math.ufl.edu
Department of Mathematics, University of Florida, Gainesville, FL 32611

Received September 30, 1996; Revised June 3, 1997

Abstract. A symplectic matroid is a collection $\mathcal{B}$ of k-element subsets of $J = \{1, 2, \ldots, n, 1^*, 2^*, \ldots, n^*\}$, each of which contains not both of i and i^* for every $i \le n$, and which has the additional property that for any linear ordering $\prec$ of J such that $i \prec j$ implies $j^* \prec i^*$ and $i \prec j^*$ implies $j \prec i^*$ for all $i, j \le n$, $\mathcal{B}$ has a member which dominates element-wise every other member of $\mathcal{B}$. Symplectic matroids are a special case of Coxeter matroids, namely the case where the Coxeter group is the hyperoctahedral group, the group of symmetries of the n-cube. In this paper we develop the basic properties of symplectic matroids in a largely self-contained and elementary fashion. Many of these results are analogous to results for ordinary matroids (which are Coxeter matroids for the symmetric group), yet most are not generalizable to arbitrary Coxeter matroids. For example, representable symplectic matroids arise from totally isotropic subspaces of a symplectic space very similarly to the way in which representable ordinary matroids arise from a subspace of a vector space. We also examine Lagrangian matroids, which are the special case of symplectic matroids where $k = n$, and which are equivalent to Bouchet's symmetric matroids or 2-matroids.

Keywords: symplectic matroid, Coxeter matroid, totally isotropic subspace, symmetric matroid, 2-matroid

1. Symplectic matroids

Hyperoctahedral group and admissible permutations. Let

$$[n] = \{1, 2, \ldots, n\} \quad \text{and} \quad [n]^* = \{1^*, 2^*, \ldots, n^*\}.$$

Define the map $* : [n] \to [n]^*$ by $i \to i^*$ and the map $* : [n]^* \to [n]$ by $i^* \to i$. Then $*$ is an involutive permutation of the set $J = [n] \sqcup [n]^*$.

We say that a subset $K \subset J$ is *admissible* if and only if $K \cap K^* = \emptyset$.

Let W be the group of all permutations of the set J which commute with the involution $*$, i.e., a permutation w belongs to W if and only if $w(i^*) = w(i)^*$ for all $i \in J$. We shall call permutations with this property *admissible*. The group W is known under the name of the *hyperoctahedral group* BC_n. It is easy to see that W is isomorphic to the group of symmetries of the n-cube $[-1, 1]^n$ in the n-dimensional real Euclidean space $\mathbf{R}^n$. Indeed, if

$e_1, e_2, \ldots, e_n$ is the standard orthonormal basis in $\mathbf{R}^n$, then W acts on $\mathbf{R}^n$ by the following orthogonal transformations: for $i \in [n]$ we set $e_{i^*} = -e_i$ and $we_i = e_{w(i)}$. Since w is an admissible permutation of $J = [n] \sqcup [n]^*$, the linear transformation is well-defined. Also it can be easily seen that W is exactly the group of all orthogonal transformations of $\mathbf{R}^n$ preserving the set of vectors $\{\pm e_1, \pm e_2, \ldots, \pm e_n\}$ and thus preserving the unit cube $[-1, 1]^n$. Indeed, the vectors $\pm e_i$, $i \in [n]$, are exactly the unit vectors normal to the $(n-1)$-dimensional faces of the cube (given, obviously, by the linear equations $x_i = \pm 1$, $i = 1, 2, \ldots, n$).

It is an easy exercise to check that W is the group generated by the following permutations of J, in cycle notation: (i, i^*) for all $i \in [n]$, and $(i, j)(i^*, j^*)$ for all $i, j \in J$, $i \neq j, j^*$.

The name 'hyperoctahedral' for the group W is justified by the fact that the group of symmetries of the n-cube coincides with the group of symmetries of its dual polytope, whose vertices are the centers of the faces of the cube. The dual polytope for the n-cube is known under the name of *n-cross polytope* or *n-dimensional hyperoctahedron*.

Admissible orderings. We shall order the set J in the following way:

$$n^* < n-1^* < \cdots < 2^* < 1^* < 1 < 2 < \cdots < n-1 < n.$$

Now if $w \in W$ then we define a new ordering $\leq^w$ of the set J by the rule

$$i \leq^w j \text{ if and only if } w^{-1}i \leq w^{-1}j.$$

Orderings of the form $\leq^w$, $w \in W$, are called *admissible* orderings of J. They can be characterized by the following property:

an ordering $\prec$ on J is admissible if and only if $\prec$ is a linear ordering and from $i \prec j$ it follows that $j^* \prec i^*$.

Denote by J_k the set of all admissible k-subsets in J. If $\prec$ is an arbitrary ordering on J, it induces the ordering (which we denote by the same symbol $\prec$) on J_k: if $A, B \in J_k$ and

$$A = \{a_1 \prec a_2 \prec \cdots \prec a_k\} \quad \text{and} \quad B = \{b_1 \prec b_2 \prec \cdots \prec b_k\}$$

we set $A \prec B$ if

$$a_1 \prec b_1, a_2 \prec b_2, \ldots, a_k \prec b_k.$$

Symplectic matroids. Now let $\mathcal{B} \subseteq J_k$ be a set of admissible k-element subsets of the set J. We say that the triple $M = (J, {}^*, \mathcal{B})$ is a *a symplectic matroid*, if it satisfies the following *Maximality Property*:

for every $w \in W$ the set $\mathcal{B}$ contains a unique w-maximal element, i.e., a subset $A \in \mathcal{B}$ such that $B \leq^w A$ for all $B \in \mathcal{B}$.

The set $\mathcal{B}$ is called the *collection of bases* of the symplectic matroid M, its elements are called *bases* of M, and the cardinality k of the bases is the *rank* of M.

Symplectic matroids were introduced by Gelfand and Serganova in [12, 13], in a slightly different, but obviously equivalent way: they used the minimality property in place of our maximality property. In the case of symplectic matroids the two properties can be obtained from each other in the most trivial way, by reversing the inequalities. However, the paper [1] explains why the maximality property fits better in the general theory of *WP*-, or Coxeter matroids, of which symplectic matroids represent only a special case.

2. Representable symplectic matroids

Now we wish to see how symplectic matroids arise naturally from symplectic geometry, in much the same way that ordinary matroids arise from projective geometry. We begin with a *standard symplectic space,* which is a vector space V over a field F with a basis

$$E = \{e_1, e_2, \ldots, e_n, e_{1^*}, e_{2^*}, \ldots, e_{n^*}\}$$

and which is endowed with an antisymmetric bilinear form $(\cdot, \cdot)$ such that $(e_i, e_j) = 0$ for all $i, j \in J, i \neq j^*$, whereas $(e_i, e_{i^*}) = 1 = -(e_{i^*}, e_i)$ for $i \in [n]$. A *totally isotropic subspace* of V is a subspace U such that $(u, v) = 0$ for all $u, v \in U$. Let U be a totally isotropic subspace of V of dimension k. Since $U \perp U$, and $\dim U^\perp = 2n - \dim U$, we see that $k \leq n$. Now choose a basis $\{u_1, u_2, \ldots, u_k\}$ of U, and expand each of these vectors in terms of the basis E: $u_i = \sum_{j=1}^n a_{i,j} e_j + \sum_{j=1}^n b_{i,j} e_{j^*}$. Thus we have represented the totally isotropic subspace U as the row-space of a $k \times 2n$ matrix (A, B), $A = (a_{i,j})$, $B = (b_{i,j})$, with the columns indexed by J, specifically, the columns of A by $[n]$ and those of B by $[n]^*$.

Let us first see what it means in terms of A and B that U is totally isotropic. Since $(u_i, u_i) = 0$ for all i is immediate from the definition of the bilinear form, U is totally isotropic if and only if $(u_l, u_m) = 0$ for all $l, m \leq k, l \neq m$. From the definition of standard symplectic space, this is equivalent to $\sum_j C_{j,j^*}^{l,m} = 0$ for all $l, m \leq k, l \neq m$, where $C_{j,j^*}^{l,m}$ is the 2×2 minor $a_{l,j} b_{m,j} - a_{m,j} b_{l,j}$ of $C = (A, B)$. In general, we will denote determinantal minors of a matrix by using subscripts for column indices and superscripts for row indices. If we denote the mth row of A by A^m, this becomes $A^l \cdot B^m - A^m \cdot B^l = 0$, where $\cdot$ denotes ordinary dot product of row vectors. This in turn is equivalent to $AB^t = BA^t$ where t denotes transpose. Thus we have proven the following lemma.

Lemma 1 *A subspace U of the standard symplectic space V is totally isotropic if and only if U is represented by the matrix (A, B) with AB^t symmetric.*

Now, given a $k \times 2n$ matrix $C = (A, B)$ with columns indexed by J, let us define a family $\mathcal{B} \subseteq J_k$ by saying $X \in \mathcal{B}$ if X is an admissible k-set and the $k \times k$ minor formed by taking the jth column of C for all $j \in X$ is nonzero.

Theorem 2 *If U is totally isotropic, then $\mathcal{B}$ is the set of bases of a symplectic matroid.*

To prove this theorem, we first need the following result.

Theorem 3 *Let $C = (A, B)$ be a $k \times 2n$ matrix of rank k with columns indexed by J such that AB^t is symmetric. Let $\prec$ be an admissible ordering of J. Let K be the unique maximal k-subset of J such that the corresponding $k \times k$ minor of C is nonsingular. Then K is admissible.*

Proof: Let us reorder the columns of C by the ordering $\prec$, starting with the largest column index. Now, K is uniquely determined by the greedy algorithm of ordinary matroid theory. In particular, $K = \{i_1 \succ i_2 \succ \cdots \succ i_k\}$, and if we denote by x_j the i_j-th column of C, then $x_1 = C_{i_1}$ is the first nonzero column of C, $x_2 = C_{i_2}$ is the first column of C which does not depend on C_{i_1}, and in general, $x_j = C_{i_j}$ is the first column of C which does not depend on $\{C_{i_1}, C_{i_2}, \ldots, C_{i_{j-1}}\}$. Note that for all l, all columns between $C_{i_{l-1}}$ and C_{i_l} are linear combinations of $\{x_1, x_2, \ldots, x_{l-1}\}$.

Suppose now that K is not admissible. Thus there exists m such that $m, m^* \in K$, say $\{m, m^*\} = \{i_g, i_h\}$, $g < h$. The hypothesis that AB^t is symmetric means that for any two rows, say those indexed by u and v, $0 = \sum_j C^{u,v}_{j,j^*} = [x_g, x_h]^{u,v} + \sum_{i,j \neq g,h} \alpha_{i,j}[x_i, x_j]^{u,v}$, where $[x_i, x_j]^{u,v}$ denotes the 2×2 minor of rows u, v from the pair of column vectors x_i, x_j. The coefficients $\alpha_{i,j}$ come from expanding out each column of $C^{u,v}_{j,j^*}$ as a linear combination of x_i. The term $[x_g, x_h]$ occurs with coefficient 1 because it does not arise in the expansion of any $C^{u,v}_{j,j^*}$ with $j \neq m, m^*$, since according to our admissible ordering, either both j and j^* occur between m and m^*, hence x_h does not occur at all in the expansion of $C^{u,v}_{j,j^*}$, or else m and m^* occur between j and j^*, say with j being the largest, in which case neither x_g nor x_h can occur in the expansion of the column C_j, hence both cannot occur in any term of the expansion of $C^{u,v}_{j,j^*}$. Now notice that the coefficients $\alpha_{i,j}$ are independent of u, v, since they depend only on how each column of C is written as a linear combination of the x_i.

Let $M = A_{i_1,i_2,\ldots,i_k} = \det(x_1, x_2, \ldots, x_k)$ be the $k \times k$ determinantal minor indexed by K, and let $N_{u,v}$ denote the cofactor of $[x_g, x_h]^{u,v}$ in M, that is, $N_{u,v}$ is the complementary $(k-2) \times (k-2)$ minor to $[x_g, x_h]^{u,v}$, with appropriate sign attached, so that $M = \sum_{u,v} N_{u,v}[x_g, x_h]^{u,v} = \det(x_1, x_2, \ldots, x_k)$. Thus

$$\begin{aligned} 0 &= \sum_{u,v} N_{u,v}\left([x_g, x_h]^{u,v} + \sum_{i,j \neq g,h} \alpha_{i,j}[x_i, x_j]^{u,v} \right) \\ &= \det(x_1, \ldots, x_k) + \sum_{i,j\neq g,h} \alpha_{i,j} \\ &\quad \times \det(x_1, \ldots, x_{g-1}, x_i, x_{g+1}, \ldots, x_{h-1}, x_j, x_{h+1}, \ldots, x_k) \\ &= \det(x_1, x_2, \ldots, x_k), \end{aligned}$$

since each of the other terms is a determinant having a repeated column. This contradicts $\det(x_1, x_2, \ldots, x_k) \neq 0$, completing the proof. □

Proof of Theorem 2: Let $\prec$ be an admissible ordering of J. We must show that $\mathcal{B}$ contains a unique maximal member. Let $\mathcal{A}$ be the set of all k-element subsets of J such that the corresponding $k \times k$ minor of C is nonzero. Thus $\mathcal{B}$ is just the set of admissible members of $\mathcal{A}$. The previous theorem showed that the unique maximal member K of $\mathcal{A}$ was actually in $\mathcal{B}$. Thus K is clearly also the unique maximal member of $\mathcal{B}$. $\square$

Notice that we have actually proven more than the Maximality Principle, namely:

Theorem 4 *If U is totally isotropic, and $\prec$ is an admissible ordering of J, then the unique maximal basis, picked from the columns of C by the greedy algorithm according to $\prec$, is admissible.*

A symplectic matroid $\mathcal{B}$ which arises from a matrix (A, B), with AB^t symmetric, is called a *representable symplectic matroid*, and (A, B) (with its columns indexed by J) is a *representation* or *coordinatization* of it (over F).

Example We now give an example of a nonrepresentable symplectic matroid, with $n = 3$, $k = 2$. Let $\mathcal{B} = \{12, 12^*, 1^*3, 1^*3^*, 23, 23^*, 2^*3, 2^*3^*\}$. It is not difficult to see that $\mathcal{B}$ is in fact a symplectic matroid, if we allow ourselves to use techniques from Section 5. According to Theorem 10, we have only to check that a certain polytope determined by $\mathcal{B}$ has all of its edges parallel either to edges of the cube or to diagonals of two-dimensional faces of the cube; see figure 1. Suppose that $\mathcal{B}$ is represented by a 2×6 matrix (A, B). Then the 2×2 minors indexed by $13, 13^*, 1^*2, 1^*2^*$ are all 0. It follows that the columns indexed by $1, 3$, and 3^* are all nonzero scalar multiples of the same nonzero vector α, and those indexed by $1^*, 2$, and 2^* are all indexed by nonzero scalar multiples of another nonzero vector β, linearly independent from α. Then $AB^t - BA^t = C_{1,1^*} + C_{2,2^*} + C_{3,3^*} = \gamma_1[\alpha\beta] + \gamma_2[\beta\beta] + \gamma_3[\alpha\alpha] = \gamma_1[\alpha\beta] \neq 0$, where γ_j are nonzero scalars. Thus AB^t cannot be symmetric.

Now let us consider which matrix operations preserve the symplectic matroid represented by (A, B). Let us write $(A, B) \sim (C, D)$ whenever (A, B) and (C, D) represent the same

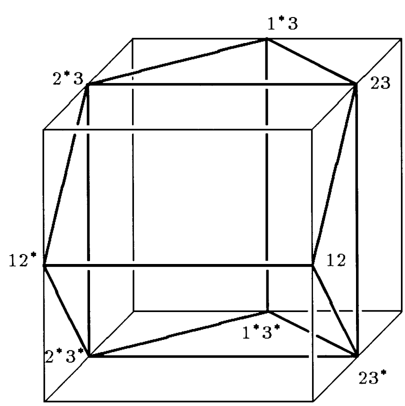

Figure 1. The matroid polytope of a nonrepresentable symplectic matroid.

symplectic matroid (with regard to the same indexing of the columns). Suppose that X is a nonsingular $k \times k$ matrix. Then

$$(A, B) \sim (XA, XB)$$

obviously, since row operations preserve dependencies among the columns. Note that here the row-space U is unchanged, as is the symmetry of AB^t, although the matrix AB^t itself may be changed via congruence, since $XA(XB)^t = X(AB^t)X^t$.

Secondly, let Λ be a nonsingular $n \times n$ diagonal matrix. Then

$$(A, B) \sim (A\Lambda^{-1}, B\Lambda),$$

since the collection of subsets of the columns which are linearly dependent is preserved. Now the row-space U is changed, whereas AB^t is unchanged. This type of transformation is referred to as the *torus action* on the representation (A, B).

The rank and signature of AB^t are invariants of both of the above types of transformations.

Thirdly, let us consider permuting columns of C. Any time we permute the columns of C, we permute the column indices in the same way, thus preserving the symplectic matroid represented by the matrix. Which column permutations are guaranteed to preserve the fact that the row space corresponds to an isotropic subspace of V? Well, the ith column of A and the ith column of B may be transposed, provided one of them is multiplied by -1. Furthermore, the ith and jth columns of A may be transposed provided the ith and jth columns of B are transposed at the same time. Thus, we see that all admissible permutations of the columns of C preserve the symmetry of AB^t.

3. Homogeneous symplectic matroids

A collection $\mathcal{B} \subseteq J_k$ is said to be m-*homogeneous* if for every two elements B_1 and B_2 of $\mathcal{B}$, $|B_1 \cap [n]| = |B_2 \cap [n]| = m$. In other words, all members of $\mathcal{B}$ have the same number of unstarred elements, and consequently also the same number of starred elements. We are going to show that a homogeneous symplectic matroid is equivalent to a flag matroid, which is a special kind of pair of ordinary matroids.

Let $\mathcal{F}_{k,l}$ denote the set of all pairs (A, B), where A is a k-subset of $[n]$, B is a l-subset of $[n]$ and $A \subseteq B$. When $k = l$, it will be convenient to identify $\mathcal{F}_{k,k}$ with $\mathcal{P}_k$, the set of all k-subsets in $[n]$. We shall call $\mathcal{F}_{k,l}$ the set of k, l-flags. For every ordering $\leq^w$, $w \in Sym_n$, of $[n]$ we define the ordering on $\mathcal{F}_{k,l}$ by setting

$$(A_1, A_2) \leq^w (B_1, B_2) \text{ if and only if } A_1 \leq^w B_1 \text{ and } A_2 \leq^w B_2.$$

Flag matroids. A subset $\mathcal{B} \subseteq \mathcal{F}_{k,l}$ is the set of bases of a *flag matroid* of rank (k, l) if it satisfies the Maximality Property: namely, if $w \in Sym_n$, there exists $F \in \mathcal{B}$ such that for all $G \in \mathcal{B}$, $G \leq^w F$. It is easy to see from the Maximality Property that the first components of $\mathcal{B}$ form an ordinary matroid of rank k, which we will denote $M_k(\mathcal{B})$, while the second components form a matroid $M_l(\mathcal{B})$ of rank l. We remark parenthetically that in [4] we show that the definition of flag matroid is equivalent to these two matroids being related by a strong map which is the identity on $[n]$.

If A is an admissible set in $[n] \sqcup [n]^*$, denote by $flag(A)$ a flag of two subsets built according to the following procedure. Denote $A_0 = A \cap [n]$ (the set of the nonstarred elements in A), $A_1 = A \cap [n]^*$ (the set of the starred elements in A) and take $A_1^* \subseteq [n]$ (the set of the starred elements with stars stripped off). Since A is admissible, $A \cap A_1^* = \emptyset$. Now $flag(A)$ is the pair $(A_0, \neg A_1^*)$ (where $\neg$ denotes the complement in $[n]$).

Theorem 5 *An m-homogeneous collection $\mathcal{B}$ of subsets of $[n] \sqcup [n]^*$ of cardinality $m + l$ is a symplectic matroid if and only if $flag(\mathcal{B})$ is the set of bases of a flag matroid on $[n]$ of rank $(m, n - l)$. A collection of flags $\mathcal{F} \subseteq \mathcal{F}_{k,l}$ is a flag matroid if and only if $flag^{-1}(\mathcal{F})$ is a k-homogeneous symplectic matroid of rank $k + n - l$ on $[n] \sqcup [n]^*$.*

Proof: The proof follows from the Maximality Property for symplectic matroids and flag matroids. First, let us assume that $\mathcal{B}$ is the collection of bases of a homogeneous symplectic matroid. Every ordering $\prec$ of $[n]$ induces an admissible ordering of $[n] \sqcup [n]^*$ (we denote it by the same symbol $\prec$): we set all starred elements to be $\prec$-smaller than nonstarred elements, and set $i^* \prec j^*$ if and only if $j \prec i$. Now let $A \prec B$ in $\mathcal{B}$. By homogeneity and our choice of ordering, $A_0 = A \cap [n] \prec B_0 = B \cap [n]$ and $A_1 = A \cap [n]^* \prec B_1 = B \cap [n]^*$, consequently $A_1^* \succ B_1^*$ and $\neg A_1^* \prec \neg B_1^*$. Thus $flag(A) \prec flag(B)$, and we have shown that $flag$ is an order preserving map. The Maximality Property for $flag(\mathcal{B})$ now follows from the Maximality Property for $\mathcal{B}$.

Conversely, suppose that $flag(\mathcal{B})$ is a flag matroid. Let an admissible ordering $\prec$ of $[n] \sqcup [n]^*$ be given. We now simply restrict $\prec$ to $[n]$, obtaining an ordering equal to $\leq^w$ for some $w \in Sym_n$. Suppose that $flag(A) \leq^w flag(B)$ for some $A, B \in \mathcal{B}$. Then $A_0 \leq^w B_0$ and $\neg A_1^* \leq^w \neg B_1^*$. It follows that $A_1^* \geq^w B_1^*$ and $A_1 \prec B_1$. Thus, the ith smallest element of A_1 is less than or equal to the ith smallest element of B_1 in $\prec$, and likewise for A_0 and B_0. It follows that $A \prec B$, and thus the Maximality Property of $flag(\mathcal{B})$ implies the Maximality Property of $\mathcal{B}$.

This proves the first statement of the theorem. The second statement is just a reformulation of the first. □

Now we proceed to show that the homogeneous symplectic matroid $\mathcal{B}$ of the previous theorem is representable if and only if the two ordinary matroids of $flag(\mathcal{B})$ are representable by a pair of subspaces related by containment. We say that an ordinary matroid M of rank k on the set $[n]$ is represented by a subspace U of F^n if U is the row-space of a $k \times n$ matrix with columns indexed by $[n]$ such that the bases of M are precisely the sets of k columns of the matrix which are nonsingular.

Theorem 6 *Let $\mathcal{B}$ be a symplectic matroid of rank $m + l$ represented by (A, B). Then the following are equivalent*:

(1) *$\mathcal{B}$ is m-homogeneous*,

(2) $\operatorname{rk} A = m$ *and* $\operatorname{rk} B = l$,

(3) *$\mathcal{B}$ may be represented by a matrix in block-diagonal form,*

$$\begin{pmatrix} Y & 0 \\ 0 & Z \end{pmatrix},$$

where Y is $m \times n$, Z is $l \times n$, and $YZ^t = 0$.

(4) *$M_m(flag(\mathcal{B}))$ is represented by $rowsp(Y)$ and $M_{n-l}(flag(\mathcal{B}))$ is represented by $(rowsp(Z))^{\perp}$, where $rowsp(Y) \subseteq (rowsp(Z))^{\perp}$.*

Proof: The equivalence of (2) and (3) is immediate by putting (A, B) into row-echelon form, where the condition that AB^t be symmetric is equivalent to $YZ^t = 0$. Furthermore, (3) implies (1) is obvious. The equivalence of (3) and (4) is immediate from the properties of *flag* above, and the well-known fact that a representation of a matroid is equivalent to a representation of the dual matroid (which is the matroid obtained by complementing the bases) via orthogonal complement of the row-space. Note that $rowsp(Y) \subseteq (rowsp(Z))^{\perp}$ is equivalent to $YZ^t = 0$.

Now we show that (1) implies (2). Suppose that $\mathcal{B}$ is m-homogeneous. Choose the admissible ordering $1 \succ 2 \succ \cdots \succ n \succ n^* \succ \cdots \succ 2^* \succ 1^*$, and select the maximal set K of columns which are linearly independent. By Theorem 3, K is an admissible set, and hence a basis in $\mathcal{B}$. Hence $|K \cap [n]| = m$, and it follows that A has rank m. A similar argument using the reverse ordering shows that B must have rank l. □

4. Root systems of type C_n

For a deeper study of symplectic matroids we have to introduce the system of vectors in $\mathbf{R}^n$ known as the *root system of type C_n*.

Roots. Let e_i, $i \in [n]$, be the standard orthonormal basis in $\mathbf{R}^n$, and again set $e_{i^*} = -e_i$ for $i^* \in [n]^*$. This defines the vectors e_j for all $j \in J = [n] \sqcup [n]^*$. Now the *roots* are the vectors $2e_j$, $j \in J$ (called *long roots*), together with the vectors $e_{j_1} - e_{j_2}$, where $j_1, j_2 \in J$, $j_1 \neq j_2$ or j_2^* (called *short roots*). Written in the standard basis $e_1, e_2, \ldots, e_n$, the roots take the form $\pm 2e_i$, $i = 1, 2, \ldots, n$, or $\pm e_i \pm e_j$, $i, j = 1, 2, \ldots, n$, $i \neq j$. Notice that both short and long roots can be written as $e_j - e_i$ for some $i, j \in J$. The set of all roots is denoted Φ.

Recall that if r is a nonzero vector in the Euclidean space then the *reflection* σ_r in the hyperplane perpendicular to r is the linear transformation of $\mathbf{R}^n$ determined by

$$\sigma_r(x) = x - \frac{2(x, r)}{(r, r)} r, \quad \text{for } x \in \mathbf{R}^n,$$

where $(\cdot, \cdot)$ is the standard scalar product in $\mathbf{R}^n$. Reflections can be characterized as linear orthogonal transformations of $\mathbf{R}^n$ with one eigenvalue -1 and $(n-1)$ eigenvalues 1; the vector r in this case may be chosen as an eigenvector corresponding to the eigenvalue -1. The set of points fixed by σ_r is the hyperplane $(x, r) = 0$ called the *mirror* of the *reflection* σ_r.

It is easy to see that when r is one of the long roots $\pm 2e_i$, $i \in [n]$, then σ_r is the linear transformation corresponding to the element $s_r = (i, i^*)$ of W in its canonical representation. Analogously, if $r = e_i - e_j$, $i, j \in J$, is a short root, then the reflection σ_r corresponds to the admissible permutation $s_r = (i, j)(i^*, j^*)$. Moreover, one can easily check (for example, by computing the eigenvalues of admissible permutations from W in their action on $\mathbf{R}^n$) that every reflection in the group of the symmetries of the unit cube $[-1, 1]^n$ is of one of these two types.

Now we see that use of the name 'root system' in regard to the set Φ is justified, because Φ satisfies the formal definition of a root system as given in [14]. A *root system* is a finite set Φ of nonzero vectors in $\mathbf{R}^n$ satisfying, for all $r \in \Phi$, the following two conditions.

(i) $\Phi \cap \mathbf{R}r = \{r, -r\}$.
(ii) $\sigma_r \Phi = \Phi$.

Simple systems of roots. Let $\alpha = \mathbf{R}^n \to \mathbf{R}$ be a linear function not vanishing on any element of Φ. Let $\Phi^+ = \{r \in \Phi \mid \alpha(r) > 0\}$ and $\Phi^- = \{r \in \Phi \mid \alpha(r) < 0\}$. Then $\Phi = \Phi^+ \sqcup \Phi^-$ and if $r \in \Phi^+$, then $-r \in \Phi^-$, and vice versa. We call Φ^+ and Φ^- *positive and negative systems of roots* (associated with α).

All roots in Φ^+ belong to the open halfspace $\alpha(x) > 0$ and thus span a convex polyhedral cone C. By definition, a *simple system* Π of roots is the set of all roots directed along the edges (i.e., one-dimensional faces) of C. It can be alternatively defined as a (unique) minimal set $r_1, \dots, r_n$ of roots in Φ^+ with the property that every root in Φ^+ is a nonnegative linear combination of $r_1, \dots, r_n$. It can be shown (see, for example, the proof of Theorem 1.3 in [14]) that every simple system in Φ is linearly independent and thus contains the same number of roots. This number is obviously the rank of Φ, i.e., the dimension of the subspace in $\mathbf{R}^n$ spanned by Φ.

Standard simple system of roots. In our system of roots Φ of type C_n, consider the linear functional

$$\alpha(x) = x_1 + 2x_2 + 3x_3 + \cdots + nx_n.$$

It is easy to see that a root $e_i - e_j$ is positive with respect to α if, in the ordering

$$n^* < n-1^* < \cdots < 1^* < 1 < 2 < \cdots < n$$

of the set J, we have $i > j$. The system of positive roots Φ^+ associated with α is called the *standard positive system of roots*. The set

$$\Pi = \{2e_1, e_2 - e_1, \dots, e_n - e_{n-1}\}$$

is obviously the simple system of roots contained in Φ^+; it is called the *standard simple system of roots*.

Vocabulary. We shall now describe natural one-to-one correspondences between the four classes of objects:

- admissible permutations of the set J;
- admissible orderings of the set J;
- systems of positive roots in Φ;
- systems of simple roots in Φ.

Indeed, for every admissible permutation $w \in W$ we have the admissible ordering $\leq^w$ of J. Vice versa, if $\prec$ is an admissible ordering of J, then the permutation

$$w = \begin{pmatrix} n^* & (n-1)^* & \cdots & 1^* & 1 & \cdots & n-1 & n \\ j_1 & j_2 & \cdots & j_n & j_{n+1} & \cdots & j_{2n-1} & j_{2n} \end{pmatrix}$$

where

$$j_1 \prec j_2 \prec \cdots \prec j_{2n-1} \prec j_{2n},$$

is admissible and the ordering $\prec$ coincides with $\leq^w$.

Now if

$$j_1 <^w j_2 <^w \cdots <^w j_{2n-1} <^w j_{2n},$$

is an admissible ordering of J, then the vectors $e_{j_{n+1}}, e_{j_{n+2}}, \ldots, e_{j_{2n}}$ form a basis in $\mathbf{R}^n$. Let $y_1, y_2, \ldots, y_n$ be the coordinates with respect to this basis and $\alpha(y) = y_1 + 2y_2 + 3y_3 + \cdots + ny_n$. Then, obviously, α does not vanish on roots in Φ, and, for a root $e_j - e_i$ in Φ, the inequality $\alpha(e_j - e_i) > 0$ is equivalent to $i \leq^w j$. Thus, the system of positive roots associated with α coincides with the system

$$w\Phi^+ = \{e_j - e_i \mid i <^w j\}$$

obtained from the standard system Φ^+ of positive roots by the action of the element w. Obviously, the simple system of roots contained in Φ^+ is exactly $w\Pi$.

Now if Π' is an arbitrary simple system of roots arising from an arbitrary linear function $\alpha : \mathbf{R}^n \to \mathbf{R}$ not vanishing on roots in Φ then the following objects are uniquely determined by our choice of Π':

- the system of positive roots $\Phi^{+\prime}$, which can be defined in two equivalent ways: as the set of all roots which are nonnegative linear combinations of roots from Π', and as the set $\{r \in \Phi \mid \alpha(r) > 0\}$;
- the (obviously admissible) ordering $\prec$ on J defined by the rule: $i \prec j$ if and only if $\alpha(e_i) \leq \alpha(e_j)$.

In particular, we immediately have the following lemma (which is a special case of a more general result about conjugacy of simple system of roots for arbitrary finite reflection groups, [14, Theorem 1.4]).

Lemma 7 *Any two simple systems of roots in Φ are conjugate under the action of W.*

If $w \in W$ then the sets $w\Phi^+$, $w\Phi^-$, $w\Pi$ will be called the system of w-positive, w-negative, w-simple roots.

5. Convex polytopes associated with symplectic matroids

For an admissible set $A \in J_k$ define the point $e_A \in \mathbf{R}^n$ as

$$e_A = e_{i_1} + e_{i_2} + \cdots + e_{i_k} \quad \text{where } A = \{i_1, i_2, \ldots, i_k\}.$$

The following lemma is obvious.

Lemma 8 *In this notation, let A and B be admissible k-subsets. Then $A \leq B$ implies that $e_B - e_A$ is a nonnegative linear combination of positive roots.*

The reverse statement,

if $e_B - e_A$ is a nonnegative linear combination of positive roots then $A \leq B$,

is not in general true, as the following simple example shows. Let

$$J = \{4^* < 3^* < 2^* < 1^* < 1 < 2 < 3 < 4\}$$

and $A = 2^*1$, $B = 3^*4$. Then

$$e_B - e_A = (e_4 - e_3) + (e_2 - e_1)$$

is the sum of positive roots, but it is not true that $A \leq B$. This example shows that the statement in the last two lines preceeding Theorem 8.1 in the paper by Gelfand and Serganova [13] is incorrect, which, in its turn, compromises the proof of Theorem 8.1 in the same paper; see [16, Theorem 3.4] for a complete proof of the Gelfand-Serganova Theorem. Theorem 10 is a special case of that result, for which we provide here a self-contained and elementary proof.

Our crucial tool is the following partial converse of Lemma 8.

Lemma 9 *Assume that A and B are admissible k-sets in J and $e_B - e_A = \lambda r$ for a positive root r and $\lambda > 0$. Then $A \leq B$.*

Proof: Notice first that, since the vectors e_A and e_B have equal lengths, by their construction, the points e_A and e_B are symmetric to each other with respect to the hyperplane H which contains the origin O of the coordinate system in $\mathbf{R}^n$ and which is perpendicular to the edge $[e_A e_B]$, since H must also contain the midpoint of the edge. Obviously, H is the mirror of reflection σ_r associated with the root r. This means $\sigma_r e_B = e_A$ and $s_r A = B$ for the permutation $s_r \in W$ corresponding to σ_r. We know already that s_r has one of the forms (i, i^*) or $(i, j)(i^*, j^*)$ for $i, j \in J$.

Consider the first case, $s_r = (i, i^*)$. We can assume without loss of generality that $i^* \in A$, then $B = (A \cup \{i\})\backslash\{i^*\}$ and $e_B - e_A = e_i - e_{i^*}$. This has to be a positive root, so $i > i^*$ and $B > A$.

Assume now that $s_r = (i, j)(i^*, j^*)$ and set $K = \{i, j, i^*, j^*\}$. Notice that $A \cap K$ is an admissible set. If $A \cap K$ is the empty set $\emptyset$ or one of the sets $\{i, j\}$ or $\{i^*, j^*\}$ then $s_r A = A$, which is impossible because $s_r A = B \neq A$. Therefore we can can further assume without loss of generality that we have one of the following subcases: $A \cap K = \{i\}$ or $A \cap K = \{i, j^*\}$ and, correspondingly, $B = (A \cup \{j\})\backslash\{i\}$ or $B = (A \cup \{i^*, j\})\backslash\{i, j^*\}$, and $e_B - e_A = e_j - e_i$ or $e_B - e_A = 2(e_j - e_i)$. But $e_B - e_A = \lambda r$ for a positive root r and scalar $\lambda > 0$. Hence, $r = e_j - e_i$, $i < j$, $j^* < i^*$, and then we easily see that $A < B$. □

Now we come to our special case of the Gelfand-Serganova Theorem.

Theorem 10 *Let $\mathcal{B} \subseteq J_k$ be a set of admissible k-sets in J. Let Δ be the convex hull of the points e_A with $A \in \mathcal{B}$.*

Then e_A are vertices of Δ for all $A \in \Delta$. Moreover, the set $\mathcal{B}$ is the collection of bases of a symplectic matroid on J if and only if all edges (i.e., one-dimensional faces) of Δ are parallel to roots in Φ.

Proof: Assume first that $\mathcal{B}$ is the collection of bases of a symplectic matroid M on the set J. Let l be an edge with vertices e_A and e_B that is not parallel to any root. Then there exists a linear function $\alpha : \mathbf{R}^n \to \mathbf{R}$ which is constant on l and takes smaller values on the other points of Δ. There is a unique simple system of roots $\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_n$ such that $\alpha(\tilde{r}_i) > 0$, $i = 1, 2, \ldots, n$. In view of Lemma 7, the group W acts transitively on the set of all simple root systems, so there is $w \in W$ sending $\{r_1, r_2, \ldots, r_n\}$ to $\{\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_n\}$. Then for any $C \in \mathcal{B}$ distinct from A we have $\alpha(e_C) \leq \alpha(e_A)$ and the vector $e_C - e_A$ has at least one negative coefficient with respect to $\{\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_n\}$. But this makes impossible the inequality $A \leq^w C$, because the latter implies, by Lemma 8, that $e_C - e_A$ is a nonnegative linear combination of the roots $\tilde{r}_i$. Therefore, by the maximality principle, A is the w-maximal element of $\mathcal{B}$. But the same arguments can be applied to the vertex e_B, and yield that B is also the w-maximal element of $\mathcal{B}$, a contradiction to the maximality principle.

Assume now that the edges of Δ are parallel to the roots. Fix $w \in W$ and the corresponding system $w\Pi = \{\tilde{r}_1, \ldots, \tilde{r}_n\}$ of w-simple roots.

Take $\alpha : \mathbf{R}^n \to \mathbf{R}$ a linear function such that $\alpha(\tilde{r}_i) > 0$ for all $i \in \{1, 2, \ldots, n\}$. Then α does not vanish on any root.

By properties of convex polytopes α attains a maximum at some vertex e_A of Δ. We want to prove that e_A is a unique. Indeed, if $\alpha(e_A) = \alpha(e_B)$ for some other $e_B \in \Delta$, the intersection Δ' of Δ and the hyperplane $\alpha(x) = \alpha(e_A)$ is a face of Δ containing two different vertices. Therefore Δ' contains some edge l of Δ. Since α is constant on the edge l, it vanishes on a root r parallel to l, a contradiction.

Thus we established the uniqueness of the α-maximal vertex e_A. Our next aim is to prove that A is the w-maximal base of M, i.e., $B \leq^w A$ for all bases $B \in \mathcal{B}$. Take an arbitrary base $B \in \mathcal{B}$. Let $e_{B_1}, \ldots, e_{B_m}$ be all vertices adjacent to e_B in Δ. The convex polytope Δ lies in the convex polyhedral cone Γ with the vertex at the point e_B, spanned by the edges

$e_{B_i} - e_B$:

$$\Gamma = \left\{ e_B + \sum_{i=1}^{m} \mu_i \left(e_{B_i} - e_B \right) \mid \mu_i \geq 0 \right\}.$$

By hypothesis, for all $i = 1, 2, \ldots, m$, we can write $e_{B_i} - e_B = \lambda_i r_i$ where $\lambda_i > 0$ and r_i is a root. If all roots r_i are w-negative, $\alpha(r_i) < 0$ and thus on Γ the function α reaches its maximum at the point e_B, which means $B = A$. Therefore, if B is distinct from A, at least one of the roots, r_p say, is w-positive and by Lemma 9, $B \leq^w B_p$. We can repeat the same argument for the vertex e_{B_p} (denote it $e_{B^{(1)}}$), and so on, until we get to e_A through the sequence of adjacent vertices $e_B = e_{B^{(0)}}, e_{B^{(1)}}, e_{B^{(2)}}, \ldots, e_{B^{(t)}} = e_A$, with $B^{(l)} \leq^w B^{(l+1)}$ for $l = 0, 1, \ldots, t-1$. But this means $B \leq^w A$. Therefore A is the w-maximal element in $\mathcal{B}$. □

Let $\mathcal{F} \subseteq \mathcal{F}_{k,l}$ be a set of flags. Assign to every $F \in \mathcal{F}_{k,l}$ the point e_F in $\mathbf{R}^n$ as follows: if $F = (A, B)$ then $e_F = e_A + e_B$, where e_X is defined by

$$e_X = \sum_{i \in X} e_i.$$

Let Δ be the convex hull of all e_F, $F \in \mathcal{F}_{k,l}$, and $\Delta_{\mathcal{F}}$ the convex hull of e_F for $F \in \mathcal{F}$. Let Φ be the system of roots of type C_n, let Φ_0 be the subsystem

$$\Phi_0 = \{e_i - e_j \mid i, j \in I, i \neq j\}$$

of type A_n, and let W_0 be the Weyl group corresponding to Φ_0, i.e., the group generated by the reflections corresponding to the roots.

Proposition 11 (Gelfand-Serganova [13], see also [4, Theorem 6.1]) *$\mathcal{F}$ is the set of bases of a flag matroid if and only if all edges of $\Delta_{\mathcal{F}}$ are parallel to roots $e_i - e_j$, $i \neq j$, of the root system of type A_n.*

Notice that $W_0 \simeq Sym_n$ leaves invariant the vector $v_0 = e_1 + e_2 + \cdots + e_n$.

Proposition 12 *The parallel translation $\Delta - v_0$ of Δ is exactly the convex polytope Γ associated with the full k-homogeneous symplectic matroid of rank $k + n - l$. Every convex polytope $\Delta_{\mathcal{F}}$ for a flag matroid $\mathcal{F}$ of rank (k, l) becomes, after the translation, the convex polytope for the corresponding homogeneous symplectic matroid $\mathcal{B} = flag^{-1}(\mathcal{F})$, obtained explicitly in the following way. If $(A, B) \in \mathcal{F}$, $(\neg B)^* \cup A \in \mathcal{B}$.*

Proof: The first statement can be checked by a direct computation, the second follows from the observation that if $\mathcal{F}$ is a flag matroid, the edges of $\Delta_{\mathcal{F}}$ are parallel to the roots in Φ_0 hence the translated polytope $\Delta_{\mathcal{F}} - v_0$ has the same property. Therefore it is a polytope of a symplectic matroid. Moreover, it is easy to see that this matroid is $\mathcal{B}$. □

6. Lagrangian matroids

A symplectic matroid of rank n on $J = [n] \sqcup [n]^*$ is called a *Lagrangian* matroid. A Lagrangian matroid is also called a *symmetric* matroid in [6] or a 2-*matroid* in [9]; these concepts are also equivalent to Δ-*matroids* [6] and to Dress and Havel's *metroids*, see [10]. For a proof that Lagrangian matroids and symmetric matroids are the same concept, see [15, Prop. 1.15], or [16, Section 6.2]. Bouchet's definition and Wenzel's or Zelevinski-Serganova's proof amount to the following characterization of Lagrangian matroids.

Theorem 13 *Let $\mathcal{B}$ be a collection of admissible n-subsets of $[n] \sqcup [n]^*$. If T is an admissible n-set, called a* transversal, *define*

$$\mathcal{I}_T = \{I \mid \text{there exists } B \in \mathcal{B} \text{ such that } I \subseteq B \cap T\}.$$

Then $\mathcal{B}$ is a Lagrangian matroid if and only if $\mathcal{I}_T$ is the collection of independent sets of an ordinary matroid for every transversal T.

This characterization gives a property of symplectic matroids, although it is no longer a characterization in this more general setting. We retain all notation from the preceeding theorem. In particular, a transversal is still an admissible n-set, although $\mathcal{B}$ now has rank k.

Theorem 14 *Let $\mathcal{B}$ be a symplectic matroid of rank k. Then $\mathcal{I}_T$ is the collection of independent sets of an ordinary matroid for every transversal T.*

Proof: Let $\mu : T \to \mathbf{R}$ be a given nonnegative function. Let $\prec$ denote any linear ordering on T which is compatible with μ. As is well known from ordinary matroid theory, it suffices to show that the greedy algorithm with respect to $\prec$ on $\mathcal{I}_T$ always returns an optimal member with respect to μ. We extend $\prec$ to an admissible ordering of J, also denoted $\prec$, by saying $i \succ i^*$ for all $i \in T$, and $i, j \in T$, $j \prec i$ implies $i^* \prec j^*$. Now we extend μ to a map on J by setting $\mu(i^*) = 0$ whenever $i \in T$. Let $\mathcal{B}$ be a symplectic matroid of rank k on J. By the Maximality Principle, there exists $B_0 \in \mathcal{B}$ such that $B_0 \succeq B$ for all $B \in \mathcal{B}$. This means that $\mu(B_0 \cap T) = \mu(B_0) \geq \mu(B) = \mu(B \cap T)$. Thus $\mu(I)$ for $I \in \mathcal{I}_T$ is optimized by $I = B_0 \cap T$. But $B_0 \cap T$ is clearly the member of $\mathcal{I}_T$ returned by the greedy algorithm with respect to $\prec$. □

Example The converse of the preceeding Theorem is false, as can be seen from the counterexample $\{12, 1^*3, 23\}$. It is easy to see that this is not a symplectic matroid by way of the Gelfand-Serganova Theorem.

Bouchet also gives an exchange axiom for Lagrangian matroids. Unfortunately, it does not generalize in any straightforward way to symplectic matroids. Bouchet, in addition, provides a very interesting way to construct Lagrangian matroids from Eulerian tours of 4-regular graphs. He provides as well, in [7], a notion of representation of Lagrangian matroids which is similar to ours. We show in the next section that Bouchet's version of the greedy algorithm for Lagrangian matroids generalizes to symplectic matroids.

Now let us see how some of the considerations of represented symplectic matroids specialize to the Lagrangian case. Suppose that M is a represented Lagrangian matroid. Thus we are given $C = (A, B)$, where A and B are now both $n \times n$. Let B_0 be a basis of M. By an allowable permutation of the columns, we can bring the columns indexed by B_0 to the leftmost n positions, replacing C by (R, S), where R is nonsingular, since B_0 is a basis of M. Hence $(R, S) \sim (I_n, R^{-1}S) = (I_n, T)$, where T is a symmetric matrix. The rank and the signature of the symmetric matrix T are invariants of the basis B_0 of M, independent of the ordering of B_0, and also preserved by the torus action. These invariants may be thought of as generalizing the orientation derived from a representation of an ordinary matroid over the reals, wherein each ordered basis is assigned a sign according to the sign of the corresponding determinant of the representation. In the case of an orientation of an ordinary matroid, however, the sign is dependent upon the ordering of the basis, and is also not invariant under the torus action.

Let us now specialize our work on homogeneous symplectic matroids to the Lagrangian case. Given $\mathcal{B}$, a collection of m-element subsets of $[n]$, we define $\Phi(B) = B \cup ([n]\backslash B)^*$, and $\Phi(\mathcal{B}) = \{\Phi(B) \mid B \in \mathcal{B}\}$. Then $\Phi(\mathcal{B})$ is an m-homogeneous collection of admissible n-element subsets of J.

Theorem 15 *Members of $\mathcal{B}$ are the bases of an ordinary matroid if and only if $\Phi(\mathcal{B})$ is a homogeneous Lagrangian matroid. Furthermore, $\mathcal{B}$ is a representable ordinary matroid if and only if $\Phi(\mathcal{B})$ is a representable homogeneous Lagrangian matroid.*

Proof: An immediate corollary of the results in Section 3. □

The first sentence of the preceeding result is also equivalent to Corollary 4.2 in [8], although the terminology is very different.

Bouchet [6, Corollary 7.3] considers a second way of imbedding an ordinary matroid into a Lagrangian matroid. Let $\mathcal{I}$ be the collection of independent sets of a matroid. Then $\Phi(\mathcal{I})$ still makes sense, and is a (nonhomogeneous) Lagrangian matroid. This seems less important than the above Theorem.

7. Greedy algorithm

Let us define an *admissible weight function* to be a function $\omega : J \to \mathbf{R}$ such that for some admissible order $\prec$ on J, $i \succ j$ for $i, j \in J$ implies $\omega(i) \geq \omega(j)$. We will say in this situation that ω is *compatible* with $\prec$. If $\mathcal{B}$ is any collection of subsets of J, we say that $B_0 \in \mathcal{B}$ is *optimal* if $\omega(B_0) \geq \omega(B)$ for all $B \in \mathcal{B}$, where, as usual, $\omega(B)$ denotes $\sum_{b \in B} \omega(b)$.

We now take essentially Bouchet's definition of a greedy algorithm, modified for the fact that k does not necessarily equal n, except that we cannot assume that the weight function is symmetric (i.e., $\omega(i) = -\omega(i^*)$ for all $i \in J$), as Bouchet does. Let $i_1 \succ i_2 \succ \cdots i_{2n}$ denote the elements of J in decreasing order, where, of course, $i_{n+l} = (i_{n-l+1})^*$, since $\prec$ is admissible. If $\mathcal{B}$ is a collection of admissible k-element subsets of J, define the *greedy solution* of $\mathcal{B}$ with respect to ω and $\prec$ to be the set B_0 returned by the following procedure:

```
begin
    B_0 := ∅
    for l = 1 to 2n do
        if B_0 ∪ {i_l} ⊆ B for some B ∈ 𝓑
        then B_0 := B_0 ∪ {i_l}
end
```

Clearly the set B_0 returned is a member of $\mathcal{B}$, for if i_l is selected by virtue of $B_0 \cup \{i_l\} \subseteq B$, then all larger elements of B must already be in B_0.

Theorem 16 *Let $\mathcal{B}$ be a collection of admissible k-element subsets of J. Then $\mathcal{B}$ is a symplectic matroid if and only if for every admissible ordering $\prec$ on J and every weight function ω compatible with $\prec$, the greedy solution in $\mathcal{B}$ is optimal.*

Proof: If $\mathcal{B}$ is a symplectic matroid, then by the Maximality Property for a given admissible $\prec$, there is a member of $\mathcal{B}$ which dominates every other element of $\mathcal{B}$ elementwise. Thus, it is clear that this member is the greedy solution and is also optimal for any ω compatible with $\prec$.

Conversely, if $\mathcal{B}$ is not a symplectic matroid, then there exists an admissible ordering $\prec$ under which $\mathcal{B}$ has two distinct maximal members. If B_m is a maximal member, we write $B_m = \{b_1^{(m)} \succ b_2^{(m)} \succ \cdots \succ b_k^{(m)}\}$, and let $B_m(l)$ denote $\{b_1^{(m)}, b_2^{(m)}, \ldots, b_l^{(m)}\}$, and $\mathcal{B}(l) = \{B_m(l) \mid B_m \in \mathcal{B}\}$. Clearly, there exists a maximal $l < k$ such that $\mathcal{B}_l$ has a unique maximal member, since $\mathcal{B}_1$ obviously does. Let q be one more than that maximal l. It follows that there exist B_1 and B_2 in $\mathcal{B}$ such that $B_1(q-1) \succeq B(q-1)$ for all $B \in \mathcal{B}$, $b_1^{(p)} \succ b_2^{(p)}$ for some $p < q$, and $b_1^{(q)} \prec b_2^{(q)}$. Furthermore, if there exist more than one B so that $B_1(q-1) = B(q-1)$, we may assume we have chosen the lexicographically greatest one for B_1, that is, for any such $B \neq B_1$, for the first l such that $B_1(l) \neq B(l)$, we have $B_1(l) \succ B(l)$.

Now let us choose the weight function ω, clearly compatible with $\prec$, by

$$\omega(x) = \begin{cases} 1 & \text{if } x \succeq b_q^{(2)} \\ 0 & \text{otherwise.} \end{cases}$$

Then the greedy algorithm selects B_1, but clearly $\omega(B_2) = q > q - 1 = \omega(B_1)$. □

8. Symplectic matroid constructions

One of the striking features of the theory of ordinary matroids is the large number of constructions, which allow one to derive new matroids from old; see, for example, [11]. In this section, we investigate whether some of these constructions may have analogues for symplectic matroids.

Unfortunately, the simplest and most important construction, that of submatroid, does not have such an analogue. To see this, let us examine the symplectic matroid $\mathcal{B}$ represented by the matrix

$$(A \mid B) = \begin{pmatrix} 0 & 1 & 0 & 1 & & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let us now "delete" $\{4, 4^*\}$, that is delete the last column of both A and B, resulting in $(A' \mid B')$, say. Although AB^t was symmetric, $A'(B')^t$ is not, which does not in itself prove that $\mathcal{B}' = \{B \in \mathcal{B} \mid B \subseteq [3] \cup [3]^*\}$ is not a symplectic matroid. However, note that $\mathcal{B}'$ is the example of a nonsymplectic matroid considered following Theorem 14. Since deletion of the pair $\{4, 4^*\}$ destroyed the property of being a symplectic matroid, it is clear the deletion of single elements cannot always preserve that property, either.

Contraction, however, is a different story. Let $\mathcal{B}$ be a symplectic matroid of rank k on J, and let $a \in J$. Then $\mathcal{B}' = \{B \backslash \{a\} \mid a \in B \text{ and } B \in \mathcal{B}\}$ is a symplectic matroid of rank $k - 1$, which is most easily seen by noting that the polytope $\Delta_{\mathcal{B}'}$ is a face (although not necessarily a facet) of the symplectic matroid polytope $\Delta_{\mathcal{B}}$, and hence satisfies the Gelfand-Serganova criterion.

Direct sum of matroids also has the obvious analogue in symplectic matroids. If $\mathcal{B}_1$ and $\mathcal{B}_2$ are symplectic matroids on disjoint sets J_1 and J_2, then $\mathcal{B} = \{B_1 \cup B_2 \mid B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\}$ is a symplectic matroid, as is easily seen from the Maximality Property.

The only other constructions which we have found to have symplectic analogues are truncation and Higgs lift. If $\mathcal{B}$ is a symplectic matroid of rank k on J, and $l < k$, then the truncation of $\mathcal{B}$ to rank l is $\mathcal{B}' = \{A \in J_l \mid \text{there exists } B \in \mathcal{B} \text{ such that } A \subseteq B\}$. For $l > k$, Higgs lift is defined in similar fashion, except for reversing the containment. The proofs that these are again symplectic matroids lie beyond the scope of this paper, and will be presented in a future paper in a more general setting.

Acknowledgment

Supported in part by NSA grant MDA904-95-1-1056.

References

1. A.V. Borovik and I.M. Gelfand, "*WP*-matroids and thin Schubert cells on Tits systems," *Adv. Math.* **103** (1994), 162–179.
2. A.V. Borovik, I.M. Gelfand, and N. White, "On exchange properties for Coxeter matroids and oriented matroids," *J. Discrete Math.* (to appear).
3. A.V. Borovik, I.M. Gelfand, and N. White, *Coxeter Matroids*, Birkhauser, in preparation.
4. A.V. Borovik, I.M. Gelfand, and N. White, *Flag Matroids*, to appear.
5. A.V. Borovik and K.S. Roberts, "Coxeter groups and matroids," in *Groups of Lie Type and Geometries*, W.M. Kantor and L. Di Martino (Eds.), Cambridge University Press, pp. 13–34, 1995.
6. A. Bouchet, "Greedy algorithm and symmetric matroids," *Mathematical Programming* **38** (1987), 147–159.
7. A. Bouchet, "Representability of Δ-matroids," *Proc. 6th Hungarian Colloquium of Combinatorics*, Colloquia Mathematica Societas Janos Bolyai, pp. 167–182, 1987.
8. A. Bouchet, "Matchings and Δ-matroids," *Discrete Applied Mathematics* **24** (1989), 55–62.
9. A. Bouchet, "Multimatroids I. Coverings by independent sets," *SIAM J. Discrete Math.* (to appear).

10. A. Bouchet, A. Dress, and T. Havel, “Δ-matroids and metroids,” *Advances in Math.* **91** (1992), 136–142.
11. T. Brylawski, “Constructions,” in *Theory of Matroids*, N. White (Ed.), Cambridge University Press, 1986.
12. I.M. Gelfand and V.V. Serganova, “On a general definition of a matroid and a greedoid,” *Soviet Math. Dokl.* **35** (1987), 6–10.
13. I.M. Gelfand and V.V. Serganova, “Combinatorial geometries and torus strata on homogeneous compact manifolds,” *Russian Math. Surveys* **42** (1987), 133–168; see also I.M. Gelfand, *Collected Papers*, Springer-Verlag, New York, Vol. III, pp. 926–958, 1989.
14. J.E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge University Press, 1990.
15. W. Wenzel, “Geometric algebra of Δ-matroids and related combinatorial geometries,” Habilitationsschrift, Bielefeld, 1991.
16. A.V. Zelevinski and V.V. Serganova, “Combinatorial optimisation on Weyl groups, greedy algorithms and generalised matroids,” Acad. Sci. USSR, Scientific Council for complex problem “Cybernetics”, 1989, 24 pp. (in Russian).