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Chip-Firing and the Critical Group of a Graph

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Abstract. A variant of the chip-firing game on a graph is defined. It is shown that the set of configurations that are stable and recurrent for this game can be given the structure of an abelian group, and that the order of the group is equal to the tree number of the graph. In certain cases the game can be used to illuminate the structure of the group.

Keywords: chip-firing, discrete Laplacian, tree number, invariant factor

1. Introduction

A *chip-firing game* on a graph G starts with a pile of tokens (chips) at each vertex. At each step of the game a vertex v is 'fired', that is, chips move from v to the adjacent vertices, one chip going along each edge incident with v. A vertex v can be fired if and only if the number of chips currently held at v is at least $deg(v)$, the degree of v.

Let s be a *configuration* of the game. By this we mean that s is a function defined on the vertices such that $s(v)$ is the number of chips at vertex v. Suppose that $\mathcal{S}$ is a non-empty finite sequence of (not necessarily distinct) vertices of G, such that starting from s, the vertices can be fired in the order of $\mathcal{S}$. If v occurs $x(v)$ times, we shall refer to x as the *representative vector* for $\mathcal{S}$. The configuration s' after the sequence of firings $\mathcal{S}$ is given by

$$s'(v) = s(v) - x(v)\, deg(v) + \sum_{w \neq v} x(w)\nu(v, w).$$

This is because each time v is fired it loses $deg(v)$ chips, and each time a vertex $w \neq v$ is fired v gains $\nu(v, w)$ chips, where $\nu(v, w)$ is the number of edges joining v and w. The relationship between s and s' can be written more concisely if we define the *Laplacian* matrix Q as follows:

$$(Q)_{vw} = \begin{cases} -\nu(v, w), & \text{if } v \neq w; \\ deg(v), & \text{if } v = w. \end{cases}$$

In terms of Q the relationship between s and s' is

$$s' = s - Qx.$$

In this paper we shall study a variant of the chip-firing game in which just one vertex q is allowed to go into debt—indeed we shall require that it is always in debt. It may help to think of the game as being played with *dollars*, rather than chips, and q as the government, which will issue more dollars if and only if the 'economy' gets stuck. In other words, q is fired if and only if no other firing is possible. There is no loss in assuming that, taking into account the debt at q, the total number of dollars is zero. Thus, in this variant, a configuration s is an integer-valued function satisfying

$$s(v) \geq 0 \quad (v \neq q), \qquad s(q) = -\sum_{v \neq q} s(v) \leq 0.$$

We define a *stable* configuration to be one for which

$$0 \leq s(v) < deg(v) \quad (v \neq q),$$

and we say that a sequence of firing is q-*legal* if and only if each occurrence of a vertex $v \neq q$ follows a configuration t with $t(v) \geq deg(v)$ and each occurrence of q follows a stable configuration. In the literature this game is often described in terms of 'snowfall' and 'avalanches', but we shall call it the *dollar game*.

A configuration r for the dollar game on a graph is said to be *recurrent* if there is a q-legal sequence for r which leads to the same configuration. We define a *critical* configuration to be one which is both stable and recurrent. Note that not all stable configurations are critical—for example, the configuration with zero dollars at every vertex is stable but not recurrent (except in a few special cases).

The first result of this paper is that the set of critical configurations on G can be given the structure of an abelian group $K(G)$. Then it is shown that the order of the $K(G)$ is κ, the number of spanning trees of G. These results are implicit in some earlier papers on the subject; see, for example, Gabrielov [10, 11], and the outline of his approach in [12].

The general theory of finite abelian groups tells us that there is a direct sum decomposition of $K(G)$, and that the associated *invariant factors* are indeed invariants of G. Using a quite different approach it has been shown [1, 9] that the invariant factors are finer invariants than κ, and so there is some interest in computing them. We shall show that the dollar game provides a calculus for analysing the structure of $K(G)$, and that it can be used effectively to compute the invariant factors in certain cases. For example, we shall prove that for a wheel graph W_n, with n odd, $K(W_n)$ is the direct sum of two cyclic groups of order l_n, where l_n is nth Lucas number. We shall also prove that, when G is a strongly regular graph, the group $K(G)$ has a subgroup of a specific kind.

2. The incidence matrix and the Laplacian

The most appropriate setting for this theory is a finite *multigraph without loops*, with an arbitrary *orientation*. A multigraph without loops G consists of a set V of vertices, a set E of edges, and an incidence function $i : E \to V^{(2)}$, where $V^{(2)}$ is the set of unordered pairs of vertices. An orientation of $G = (V, E, i)$ is a function $h : E \to V$ such that $h(e) \in i(e)$, for all edges e. In other words $h(e)$ is one of the vertices incident with e, which we shall refer to as its *head*. The *tail* of e, denoted by $t(e)$, is defined by the condition that $i(e) = \{h(e), t(e)\}$. When we speak of a 'graph' we shall mean a finite multigraph without loops which has been given a fixed, but arbitrary, orientation. All the important results turn out to be independent of the orientation—it is a technical device used in the construction of some of the matrices needed in the proofs. We shall usually pass over this point without comment. We shall also assume that G is connected.

Let $n = |V|$ and $m = |E|$, and define an $n \times m$ matrix $D = (d_{ve})$, the *incidence matrix* of G, as follows:

$$d_{ve} = \begin{cases} 1, & \text{if } v = h(e); \\ -1, & \text{if } v = t(e); \\ 0, & \text{if } v \notin i(e). \end{cases}$$

Denote by D^t the transpose of D. A simple calculation shows that DD^t is the Laplacian matrix Q defined in Section 1.

Suppose that α is a numerical function defined on V, regarded as a column vector. If α is such that $D^t\alpha = 0$ then, since $(D^t\alpha)(e) = \alpha(h(e)) - \alpha(t(e))$, it follows that α takes the same value of the head and tail of any edge. If (as we assume throughout) G is connected, then for any two vertices v and w there is a walk starting at v and ending at w. It follows that $\alpha(v) = \alpha(w)$. In other words, α is constant. Conversely, if α is constant then it satisfies $D^t\alpha = 0$. In other words, the kernel of the matrix D^t consists of scalar multiples of the function u given by $u(v) = 1$ for all $u \in V$.

Consider now the kernel of Q. Clearly, if $D^t\alpha = 0$ then $Q\alpha = DD^t\alpha = 0$. Conversely, if $Q\alpha = 0$ then $\alpha^t Q\alpha = \|D^t\alpha\|^2 = 0$, and so $D^t\alpha = 0$. Thus, the kernel of Q is also consists of the constant functions, that is, the multiples of u.

3. Theory of the dollar game

The theory of chip-firing games [6, 7] is based on a 'confluence' property: if we start with a given configuration s then there may be many different sequences which are possible starting from s, but it turns out that (in a sense) they all lead to the same 'outcome'. We shall outline the theory as it applies to the dollar game, using direct counting arguments instead of the more abstract ones given in earlier papers.

Let us say that a sequence $\mathcal{S}$ is *proper* if it does not contain q.

Lemma 3.1 *Given a configuration s, there is an upper bound on the length of a proper sequence $\mathcal{S}$ which is q-legal for s.*

Proof: Throughout the firing of $\mathcal{S}$ the total number of dollars held at vertices $v \neq q$ cannot exceed its initial value, and in particular there is an upper bound on the number of dollars held at any one of these vertices.

Suppose $\mathcal{S}$ can be arbitrarily long, so that, since the number of vertices is finite, there is a vertex w which can be fired as often as we please. Since G is connected there is a path from

q to w, and we may suppose that it is a geodesic: in particular, the penultimate vertex w' is closer to q than w is. Since w can be fired as often as we please, w' (as a neighbour of w) can receive an unbounded number of extra dollars. However, the number of dollars held at w' is bounded, so w' must be fired as often as we please. Repeating this argument we obtain a sequence of vertices $w, w', w'', \ldots$, each of which is closer to q than its predecessor, and each of which can be fired as often as we please. This contradicts the assumption that q does not occur, and so there must be a bound on the length of $\mathcal{S}$. □

Lemma 3.2 *Let s be a configuration of the dollar-firing game on a connected graph G. Then there is a critical configuration c which can be reached by a q-legal sequence of firings starting from s.*

Proof: By Lemma 3.1, if we start from s and fire the vertices other than q in any q-legal sequence, then we must eventually reach a configuration where no vertex except q can be fired—that is, a stable configuration. If we then fire q and repeat the process, we reach another stable configuration. This procedure can be repeated as often as we please, whereas the number of stable configurations is finite. So at least one of them must recur, and this is a critical configuration. □

We shall prove that there is only one critical configuration satisfying Lemma 3.2. The following construction is central to the argument.

Let $\mathcal{X}$ be a sequence of vertices and y be a vector such that $y(v) \geq 0$ for every $v \in V$. Construct a sequence $\mathcal{X}^y$ as follows: delete an occurrence of any vertex v from $\mathcal{X}$ if it is not preceded by at least $y(v)$ occurrences of v in $\mathcal{X}$. In other words, if there are more than $y(v)$ occurrences of v the first $y(v)$ of them are deleted, and if there are fewer than $y(v)$ occurrences, all of them are deleted.

The following results, Lemma 3.3, Theorem 3.4, and Corollary 3.5, deal with proper sequences (that is, the firing of q is not involved). Since q plays no part, we shall use the word *legal* rather than q-legal throughout this discussion.

Lemma 3.3 *Let $\mathcal{X}$ and $\mathcal{Y}$ be proper sequences, with representative vectors x and y, which are legal for the configuration s. Then the sequence $\mathcal{Z} = (\mathcal{Y}, \mathcal{X}^y)$ is also legal for s, and its representative vector z is given by*

$$z(v) = \max\{x(v), y(v)\}.$$

Proof: Clearly, it is enough to show that that $\mathcal{X}^y$ is legal for the configuration $s_2 = s - Qy$. Assume that $\mathcal{X}^y$ is legal for s_2 up to the point where a vertex $v \neq q$ is about to be fired for the ith time in $\mathcal{X}^y$, and that the configuration at that point is k^y. Let k be the configuration which occurs immediately before the corresponding occurrence of v in $\mathcal{X}$, which is the $(y(v) + i)$th. Let x_0 and x_0^y be the representative vectors of the initial segments of $\mathcal{X}$ and $\mathcal{X}^y$ up to these points, so that

$$k = s - Qx_0, \quad k^y = (s - Qy) - Qx_0^y = s - Qz_0$$

where $z_0 = y + x_0^y$. Evaluating at v we have

$$k(v) = s(v) - x_0(v)\, deg(v) + \sum_{w \neq v} x_0(w)\nu(w, v),$$

$$k^y(v) = s(v) - z_0(v)\, deg(v) + \sum_{w \neq v} z_0(w)\nu(w, v).$$

Since v is about to be fired for the ith time in $\mathcal{X}^y$ we have $z_0(v) = y(v) + (i - 1)$ and similarly $x_0(v) = (y(v) + i) - 1$. Hence $x_0(v) = z_0(v)$.

More generally, if w does occur in $\mathcal{X}^y$, suppose that it has occurred j times in $\mathcal{X}^y$ up to this point, so that $z_0(w) = y(w) + j$. If $j = 0$ then $x_0(w) \leq y(w)$, and $z_0(w) = y(w)$, so $x_0(w) \leq z_0(w)$. If $j > 0$ then $x_0(w) = y(w) + j = z_0(w)$. In both cases $x_0(w) \leq z_0(w)$. The same result holds if w does not occur in $\mathcal{X}^y$, because in that case the definitions imply that $z_0(w) = y(w) \geq x(w) \geq x_0(w)$.

Since $x_0(v) = z_0(v)$ and $z_0(w) \geq x_0(w) (w \neq v)$, the expressions for $k^y(v)$ and $k(v)$ show that $k^y(v) \geq k(v)$. But we are given that the firing of v in $\mathcal{X}$ is legal, that is, $k(v) \geq deg(v)$. Hence $k^y(v) \geq deg(v)$ and the corresponding firing of v in $\mathcal{X}^y$ is also legal.

It remains to check the formula for z. If $x(v) > y(v)$ then the first $y(v)$ occurrences of v are deleted from $\mathcal{X}$ to form $\mathcal{X}^y$. So v occurs $x(v) - y(v)$ times in $\mathcal{X}^y$, and the number of times it occurs in $\mathcal{Z} = (\mathcal{Y}, \mathcal{X}^y)$ is

$$y(v) + (x(v) - y(v)) = x(v).$$

On the other hand, if $x(v) \leq y(v)$ then v does not occur in $\mathcal{X}^y$, and hence it occurs $y(v)$ times in $\mathcal{Z}$. □

Theorem 3.4 *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are proper sequences as in Lemma* 3.3, *and that they produce configurations s_1 and s_2, respectively. Then there is a configuration s_3 which can be derived from both s_1 and s_2 by legal sequences.*

Proof: Lemma 3.3 tells us that $\mathcal{X}^y$ is legal for $s_2 = s - Qy$, and similarly $\mathcal{Y}^x$ is legal for $s_1 = s - Qx$. Furthermore, the sequences $(\mathcal{Y}, \mathcal{X}^y)$ and $(\mathcal{X}, \mathcal{Y}^x)$ have the same representative vector z, and hence they lead to the same configuration s_3. □

Corollary 3.5 *Let $\mathcal{X}$ and $\mathcal{Y}$ be as in Theorem* 3.4 *Then*:
(i) *if s_1 is stable, $\mathcal{Z} = (\mathcal{Y}, \mathcal{X}^y)$ also leads to s_1;*
(ii) *if s_1 and s_2 are both stable then $s_1 = s_2$.*

Proof:

(i) Lemma 3.3 shows that $\mathcal{Y}^x$ is legal for $s_1 = s - Qx$. But if s_1 is stable, no vertex $v \neq q$ can be fired. Hence, $\mathcal{Y}^x$ must be empty, and its construction implies that $x(v) \geq y(v)$ for all v. In this case $\mathcal{Z} = (\mathcal{Y}, \mathcal{X}^y)$ has representative vector $z = x$, and so it produces s_1 also.

(ii) If s_2 is also stable, a parallel argument shows that $y(v) \geq x(v)$ for all v. So $x = y$ and $s_1 = s_2$. □

Theorem 3.4 is the confluence property of the dollar game for proper sequences. Using Corollary 3.5 the result can be extended to general q-legal sequences, as follows.

Consider the structure of a general sequence $\mathcal{X}$ which is q-legal for s. It begins with a (possibly empty) sequence $\mathcal{Q}_0$ of firings of q followed by a proper sequence $\mathcal{X}_1$, after which q must fired again, and so on. The sequence can therefore be split into segments

$$\mathcal{Q}_0, \mathcal{X}_1, \mathcal{Q}_1, \ldots, \mathcal{Q}_{a-1}, \mathcal{X}_a,$$

where each $\mathcal{X}_i$ is proper, and each $\mathcal{Q}_i$ is a sequence of q's. Let $\mathcal{Y}$ be another q-legal sequence for s, with decomposition

$$\mathcal{Q}'_0, \mathcal{Y}_1, \mathcal{Q}'_1, \ldots, \mathcal{Q}'_{b-1}, \mathcal{Y}_b.$$

The q-legality condition means that the initial segments $\mathcal{Q}_0$ and $\mathcal{Q}'_0$ are the same. If $a = b = 1$ there are no other firings of q, and Lemma 3.3 establishes the confluence property. If $a > b = 1$, Corollary 3.5(i) shows that following $\mathcal{Y}_1$ by $\mathcal{X}_1^{y_1}$ leads to the same (stable) configuration as the one which follows $\mathcal{X}_1$. Hence the outcome of $\mathcal{X}$ can be obtained by starting with $\mathcal{Y}$. If $a \geq b > 1$, it follows from Corollary 3.5(ii) that the two sequences produce the same stable configurations t_i on the completion of $\mathcal{X}_i$ and $\mathcal{Y}_i$, for $i \leq b$. Starting from t_b and applying the previous argument gives the required result.

Lemma 3.6 *If the configuration c is recurrent then there is a q-legal sequence $\mathcal{U}$ for c which has representative vector u, the all-*1 *vector.*

Proof: Since c is recurrent there is a q-legal sequence $\mathcal{R}$ for c which produces c. Its representative vector r satisfies $c - Qr = c$. In Section 2 we observed that the kernel of Q consists of the constant functions, so r is a multiple λu of the all-1 vector u. By the proof of Lemma 3.3 the sequence $\mathcal{R}^u$ is q-legal for $c - Qu = c$; and its representative vector is $(\lambda - 1)u$. Repeating this process $\lambda - 1$ times in all we obtain a sequence $\mathcal{U}$ with the required properties. □

Lemma 3.7 *Suppose that c is a critical configuration and that there is a q-legal sequence $\mathcal{S}$ for c which produces a critical configuration d. Then $d = c$.*

Proof: Let $\mathcal{U}$ be as in Lemma 3.6. By Corollary 3.5(i) $(\mathcal{U}, \mathcal{S}^u)$ produces d, which means that $\mathcal{S}^u$ is also a q-legal sequence leading from c to d. Thus, if x and x^u are the representative vectors for $\mathcal{S}$ and $\mathcal{S}^u$, we have $d = c - Qx = c - Qx^u$. It follows that $x - x^u$ is in the kernel of Q, so $x - x^u$ is a multiple of u which, by the construction of $\mathcal{S}^u$, must be u. In other words, we can replace $\mathcal{S}$ by $\mathcal{S}^u$, and in this process one occurrence of each vertex is deleted. Repeating the argument we can reduce $\mathcal{S}$ to the empty sequence, so $\mathcal{S}$ must contain every vertex the same number of times, which implies that $d = c$. □

Theorem 3.8 *Let s be a configuration of the dollar game on a connected graph G. Then there is a unique critical configuration which can be reached by a q-legal sequence of firings starting from s.*

Proof: We have already shown (Lemma 3.2) that at least one such critical configuration exists. Suppose c_1 and c_2 are two of them. By confluence there is a configuration which can be reached from both c_1 and c_2. Using Lemma 3.2 again, there is a critical configuration d with this property. But Lemma 3.7 implies that $c_1 = d$ and $c_2 = d$, hence $c_1 = c_2$ as required. □

4. The group of critical configurations

Let $C^0(G; \mathbf{Z})$ and $C^1(G; \mathbf{Z})$ denote the abelian groups of integer-valued functions defined on V and E, respectively. Interpreting the elements of these spaces as column vectors, the incidence matrix D and its transpose D^t can be regarded as homomorphisms

$$D : C^1(G; \mathbf{Z}) \to C^0(G; \mathbf{Z}), \quad \text{and} \quad D^t : C^0(G; \mathbf{Z}) \to C^1(G; \mathbf{Z}).$$

We can also regard $Q = DD^t$ as a homomorphism $C^0(G; \mathbf{Z}) \to C^0(G; \mathbf{Z})$. Denote by $\sigma : C^0(G; \mathbf{Z}) \to \mathbf{Z}$ the homomorphism defined by $\sigma(f) = \sum_v f(v)$.

Lemma 4.1 *The image of Q is a normal subgroup of the kernel of σ.*

Proof: We observe first that $\sigma D = 0$, which follows directly from the fact that the matrix D has just two non-zero entries in each column, 1 and -1. Suppose that $x \in \mathrm{Im}\, Q$, say $x = Qy = DD^t y$. Then $\sigma(x) = \sigma DD^t y = \sigma D(D^t y) = 0$, that is, $x \in \mathrm{Ker}\, \sigma$. Thus the image of Q is a subgroup of Ker σ, and since the groups are abelian, it is a normal subgroup. □

Denote by $K(G)$ the set of critical configurations on a graph G, and for each configuration s let $\gamma(s) \in K(G)$ be the unique critical configuration determined by Theorem 3.8.

Theorem 4.2 *The set $K(G)$ of critical configurations on a connected graph G is in bijective correspondence with the abelian group* Ker $\sigma/$Im Q.

Proof: We show first that every coset $[f]$ in Ker $\sigma/$Im Q contains a configuration. Given $f \in \mathrm{Ker}\, \sigma$ let l be the configuration defined on vertices $u \neq q$ by

$$l(u) = \begin{cases} deg(u) - 1 & \text{if } f(u) \geq 0, \\ deg(u) - 1 - f(u) & \text{if } f(u) < 0, \end{cases}$$

and such that $l(q) = -\sum_{u \neq q} l(u)$. It follows from Lemma 3.1 that there is a finite sequence of firings which reduces l to a stable configurations k. If this sequence has representative vector x, we have $k = l - Qx$. Let $z = f + l - k$; then $z = f + Qx$ so $[z] = [f]$, and

$$z(u) = f(u) + l(u) - k(u) \geq deg(u) - 1 - k(u) \geq 0.$$

Hence z is a configuration representing the given coset $[f]$.

Next, we show that there is a well-defined function

$$h : \mathrm{Ker}\,\sigma/\mathrm{Im}\,Q \to K(G),$$

given by $h(\alpha) = \gamma(s)$, where s is any configuration in the coset α. Suppose that s_1 and s_2 are configurations such that $[s_1] = [s_2] = \alpha$. In that case $s_1 - s_2 = Q\phi, \phi \in C^0(G, \mathbf{Z})$. We can write $\phi = f_1 - f_2$ where $f_1(v)$ and $f_2(v)$ are non-negative for all v. Let $s_0 = s_1 - Qf_1 = s_2 - Qf_2$.

Suppose that $\gamma(s_1) = c_1$, and that $\mathcal{S}_1$ is a q-legal sequence for s_1 which produces c_1. Since c_1 is recurrent we can choose $\mathcal{S}_1$ so that any vertex v occurs at least $f_1(v)$ times. Now the proof of Lemma 3.3 shows that the sequence $\mathcal{S}_1^{f_1}$ is q-legal for s_0, and by construction it is obtained from $\mathcal{S}_1$ by deleting exactly $f_1(v)$ occurrences of v, for each vertex v. Hence $\mathcal{S}_1^{f_1}$ applied to s_0 produces c_1. It follows that $\gamma(s_0) = c_1 = \gamma(s_1)$. The same argument shows that $\gamma(s_0) = \gamma(s_2)$. Hence h is well-defined.

To show that h is a surjection, we simply observe that given $c \in K(G)$, we have $h[c] = \gamma(c) = c$. To show h is an injection, suppose that $h[s_1] = h[s_2]$. Then $\gamma(s_1) = \gamma(s_2) = c$, say, where the configuration c can be reached starting from s_1 and from s_2. Thus, there are vectors x_1 and x_2 such that $s_1 - Qx_1 = c$ and $s_2 - Qx_2 = c$. Hence $s_1 - s_2 = Q(x_1 - x_2)$, and so $[s_1] = [s_2]$. $\square$

There is an abelian group structure on Ker σ/Im Q, defined by $[s_1] + [s_2] = [s_1 + s_2]$. It follows that $K(G)$ is an abelian group under the operation $\bullet$, where $h[s_1] \bullet h[s_2] = h[s_1 + s_2]$, that is, $\gamma(s_1) \bullet \gamma(s_2) = \gamma(s_1 + s_2)$. Equivalently, for any two critical configurations c_1 and c_2, we have

$$c_1 \bullet c_2 = \gamma(c_1 + c_2).$$

We shall refer to $K(G)$ as the *critical group* of G.

For example, consider the complete graph K_4. A configuration is determined by a vector $(s(a), s(b), s(c))$ denoting the numbers of dollars at the vertices a, b, c other than q. A configuration is stable if and only if $0 \le s(v) \le 2$ for $v = a, b, c$, and so there are $3^3 = 27$ stable configurations. However, only 16 of them are recurrent. The zero element is (2, 2, 2) and the critical group is the direct sum of two cyclic groups of order 4, whose generators may be taken as (1, 1, 2) and (2, 1, 1). (These results are a special case of the analysis given in Section 9 for the family of wheel graphs, since K_4 is the wheel graph W_3.)

5. Flows, cuts, and the orthogonal projection

The aim of the next three sections is to prove that the critical group $K(G)$ is isomorphic to several other groups associated with G, and that its order is κ, the number of spanning trees of G. It will be necessary to outline parts of the algebraic theory of graphs, some of which is ‘classical’ and some of which is recent. More details can be found in [3].

The dollar game naturally involves the set of integer-valued functions defined on the vertices of a graph, but, as we shall see in Section 6, it is convenient to regard this set as being imbedded in the vector space of *real-valued* functions. We shall denote the vector spaces of real-valued functions defined on the vertices and edges of a graph by $C^0(G;\mathbf{R})$ and $C^1(G;\mathbf{R})$, respectively.

In this context the matrices D and D^t define linear mappings between the vector spaces. A function f in the subspace $Z = \operatorname{Ker} D$ is called a *flow* on G. There is a standard inner product $\langle x, y\rangle = \sum_e x(e)y(e)$ on $C^1(G;\mathbf{R})$, and we define B to be the orthogonal complement of Z with respect to this inner product. According to the general theory of vector spaces, there is a direct-sum decomposition

$$C^1(G;\mathbf{R}) = Z \oplus B = \operatorname{Ker} D \oplus (\operatorname{Ker} D)^\perp.$$

The dimensions of the summands are determined by theorems of elementary linear algebra, given the fact (Section 2), that the kernel of D^t is one-dimensional. It turns out that

$$\dim Z = m - n + 1, \quad \dim B = n - 1.$$

Let U be a non-empty proper subset of V. Define a function b_U in $C^1(G;\mathbf{R})$ by the rule

$$b_U(e) = \begin{cases} 1, & \text{if the intersection of } i(e) \text{ and } U \text{ is } h(e) \text{ only;} \\ -1, & \text{if the intersection of } i(e) \text{ and } U \text{ is } t(e) \text{ only;} \\ 0, & \text{otherwise.} \end{cases}$$

The set of edges which have exactly one vertex in U is called a *cut* in G, and $b_U(e) \neq 0$ precisely when e belongs to this cut. Thus b_U is the 'characteristic function' of the cut defined by U, except that there are $\pm$ signs according to the orientation.

Note that the cut corresponding to a single vertex v consists of the edges incident with v, and b_v (considered as column vector) is simply the transpose of row v of D. The equation

$$b_U = \sum_{v\in U} b_v$$

expreses b_U as a linear combination of rows of D. If $z \in Z$ we have $Dz = 0$, that is, $\langle b_v, z\rangle = 0$ for all $v \in V$. Consequently $\langle b_U, z\rangle = 0$, so b_U is in B.

Let T be a spanning tree in G, that is, a subset T of E which forms a connected acyclic subgraph containing every vertex of G. We know that $|T| = n-1$. If $f \in T$ the removal of f from T leaves two components, one containing $h(f)$ and the other $t(f)$. We shall denote these components by T_f^+ and T_f^-, respectively. Let $U(T, f)$ denote the set of vertices of T_f^+, so that the associated cut contains f and some other edges which are not in T; we call this the *fundamental cut* determined by T and f. The number of fundamental cuts associated with T is $n-1$, and for each one of them we have an element

$$b = b_{U(T,f)}$$

of the cut space. If $f' \in T$ then $b(f') = 1$ when $f' = f$ and $b(f') = 0$ when $f' \neq f$. It follows that the set of functions $b_{U(T,f)}$ $(f \in T)$ is linearly independent. Since $\dim B = n-1$ we immediately deduce:

Lemma 5.1 *For a given spanning tree T, the set $\mathcal{B}_T$ of functions $b_{U(T,f)}$ $(f \in T)$ is a basis for B.*

Let q be a given vertex and let $\mathcal{D}_q$ denote the set of functions b_v determined by the rows of D, excepting the one for which $v = q$. Then $\mathcal{D}_q$ is also a basis for B.

Lemma 5.2 *Let q and T be a given vertex and spanning tree of G.*

(i) *The change of basis matrix which expresses $\mathcal{D}_q$ in terms of $\mathcal{B}_T$ is $D(q, T)$, the submatrix of D formed by the intersection of the rows corresponding to all vertices except q and the columns corresponding to edges in T.*

(ii) *The inverse of $D(q, T)$ is the matrix $Y = (y_{ev})$ given by*

$$y_{ev} = \begin{cases} 1, & \text{if } v \in T_e^+ \text{ and } q \in T_e^-; \\ -1, & \text{if } v \in T_e^- \text{ and } q \in T_e^+; \\ 0, & \text{otherwise.} \end{cases}$$

(iii) *det $D(q, T) = \pm 1$.*

Proof:

(i) Suppose that

$$b_v = \sum_{f \in T} \alpha_{vf} b_{U(T,f)} \quad (v \neq q).$$

Evaluating both sides on an edge $e \in T$ we have

$$b_v(e) = \sum_{f \in T} \alpha_{vf} b_{U(T,f)}(e) = \alpha_{ve}.$$

It follows that $\alpha_{ve} = b_v(e) = d_{ve}$, as claimed.

(ii) The definition of $U(T, e)$ implies that

$$b_{U(T,e)} = \sum_{v \in T_e^+} b_v.$$

The equation $\sum_{v \in V} b_v = 0$ allows us to rewrite the displayed equation as a sum over $v \neq q$, in which the coefficients turn out to be y_{ev} as given.

(iii) It follows from (ii) that det Y det $D(q, T) = 1$. Both matrices have integer entries, so their determinants are integers and the result follows. □

The orthogonal decomposition $C^1(G; \mathbf{R}) = Z \oplus B$ implies that any $c \in C^1(G; \mathbf{R})$ can be uniquely expressed in the form $c = z + b$, with $z \in Z$, $b \in B$, so that $\langle z, b \rangle = 0$. There is an explicit formula for the unique b corresponding to a given c, or (equivalently) the orthogonal projection $P : C^1 \to B$.

Given a spanning tree T, define an $m \times m$ matrix N_T as follows: if f is not in T then column f of N_T is zero, while if f is in T, it is $b_{U(T,f)}$. Explicitly, the entries n_{ef} of N_T are given by:

$$n_{ef} = \begin{cases} 0, & \text{if } f \notin T; \\ b_{U(T,f)}(e), & \text{if } f \in T. \end{cases}$$

Theorem 5.3 *Let κ be the number of spanning trees of G (sometimes called the tree-number). Then* [3, *Proposition* 6.3]

$$P = (1/\kappa) \sum_T N_T$$

is the orthogonal projection $C^1 \to B$. That is, $Pz = 0$ $(z \in Z)$ and $Pb = b$ $(b \in B)$.

6. Lattices, determinants, and the tree number

For brevity, we shall use the subscript I to denote a set of integer-valued functions defined on the edges of a graph. Thus, we denote $C^1(G; \mathbf{Z})$, the abelian group of all integer-valued functions defined on the edges, by C_I. Since C_I is naturally imbedded in the vector space $C^1(G; \mathbf{R})$, we shall often speak of it as a *lattice*. Similarly, we define

$$Z_I = Z \cap C_I, \quad B_I = B \cap C_I,$$

so that Z_I and B_I are lattices (abelian groups) naturally imbedded in the vector spaces Z and B.

A fundamental observation is that the direct sum $Z_I \oplus B_I$ is a proper sublattice of C_I; that is, not every integer-valued function on the edges can be decomposed into an integer flow and an integer cut. Specifically:

Theorem 6.1 *A function $c \in C_I$ is in $Z_I \oplus B_I$ if and only if Pc is in B_I, where P is the orthogonal projection $C^1(G; \mathbf{R}) \to B$. Equivalently, if we let P_I denote the restriction to C_I of P, then the function*

$$\frac{C_I}{Z_I \oplus B_I} \to \frac{\operatorname{Im} P_I}{B_I},$$

which takes the coset $[c]$ (with respect to $Z_I \oplus B_I$) to the coset $[Pc]$ (with respect to B_I), is an isomorphism.

Proof: The first statement follows from the identity $c = (c - Pc) + Pc$.

The only non-trivial part of the second statement is to show that the function is an injection. This is simply another way of saying that if $[Pc]$ is the zero coset, then $[c]$ is the zero coset, which follows directly from the first statement. □

The *dual* of the lattice B_I in the vector space B is the lattice $(B_I)^\sharp$ defined by

$$(B_I)^\sharp = \{x \in B \mid \langle x, b\rangle \in \mathbf{Z} \text{ for all } b \in B_I\}.$$

A fundamental result of Bacher et al. [1] is that the dual lattice $(B_I)^\sharp$ is the image of P_I. Together with Theorem 6.1 this implies that the map $[c] \mapsto [Pc]$ defines a group isomorphism

$$C_I/(Z_I \oplus B_I) \to B_I^\sharp/B_I.$$

A parallel argument establishes the existence of an isomorphism between $C_I/(Z_I \oplus B_I)$ and $Z_I^\sharp/Z_I$, a finite abelian group which, in other contexts, is known as the *Jacobian* group. In the theory of integer lattices the *determinant* of a lattice Λ, written as $\det \Lambda$, is defined to be the index of Λ in its dual $\Lambda^\sharp$. Thus the index of $Z_I \oplus B_I$ in C_I is given by

$$\left|\frac{C_I}{Z_I \oplus B_I}\right| = \det B_I = \det Z_I.$$

In order to compute the determinant of the lattices Z_I and B_I we need a standard result. Let Λ be any lattice in a euclidean space, and let $\mathcal{B} = \{e_1, e_2, \ldots, e_\kappa\}$ be a $\mathbf{Z}$-basis for Λ. Then the determinant of Λ is equal to the determinant of the *Gram matrix* H of Λ, that is,

$$\det H, \quad \text{where } (H)_{ij} = \langle e_i, e_j\rangle.$$

It can be shown that this is independent of the chosen $\mathbf{Z}$-basis.

Theorem 6.2 *If G is a connected graph, the common value of* $\det Z_I$ *and* $\det B_I$ *is κ, the number of spanning trees of G.*

Proof: In Section 5 we noted that both $\mathcal{B}_T$ and $\mathcal{D}_q$ are bases for the vector space B. It is easy to see that $\mathcal{B}_T$ is also a $\mathbf{Z}$-basis for B_I, as follows. Given $b \in B_I$, we can use the fact that $\mathcal{B}_T$ is a vector space basis for B to write

$$b = \sum_{f \in T} \beta_f b_{U(T,f)} \quad \text{where } \beta_f \in \mathbf{R}.$$

Evaluating both sides on any edge $e \in T$, we get $b(e) = \beta_e$, since e is in the cut $U(T, e)$ but not in $U(T, f)$ when $f \neq e$. Since $b(e)$ is an integer, so is β_e.

We also showed (Lemma 5.2) that the change of basis from $\mathcal{B}_T$ to $\mathcal{D}_q$ is unimodular. It follows that $\mathcal{D}_q$ is also a $\mathbf{Z}$-basis for B_I.

The Gram matrix for $\mathcal{D}_q$ is DD^t with the row and column corresponding to q deleted, which is just Q_q, the Laplacian matrix Q with that row and column deleted. It is a classic result (see, for example, [2, p. 39]), that the determinant of Q_q is the tree number κ. $\square$

7. The Picard group

Recall that $\sigma : C^0(G; \mathbf{Z}) \to \mathbf{Z}$ is defined by $\sigma(f) = \sum_v f(v)$. The following result is a strengthening of Lemma 4.1.

Lemma 7.1 *If G is a connected graph, the image of $D : C_I \to C^0(G; \mathbf{Z})$ is equal to the kernel of σ.*

Proof: We have already noted that $\sigma D = 0$, so that $\text{Im } D \subseteq \text{Ker } \sigma$.

Conversely, suppose that $f \in \text{Ker } \sigma$, that is, $\sum_v f(v) = 0$. For $v \in V$ let $\delta_v \in C^0(G; \mathbf{Z})$ be the function defined by $\delta_v(w) = 0$ if $w \neq v$, and $\delta_v(v) = 1$, and for $e \in E$ define $\delta_e \in C_I$ similarly. Clearly, if e is an edge whose vertices are a and b, $\delta_a - \delta_b = D(\pm\delta_e)$, where the sign depends on the orientation.

Choosing any vertex x, and remembering that $\sigma(f) = 0$, we have

$$f = \sum_{v \in V} f(v)\delta_v = \sum_{v \neq x} f(v)(\delta_v - \delta_x).$$

There is a path in G from x to v, consisting (say) of the vertices and edges $x = v_0, e_1, v_1, \ldots, v_{r-1}, e_r, v_r = v$. Consequently,

$$\delta_v - \delta_x = \left(\delta_{v_r} - \delta_{v_{r-1}}\right) + \cdots + \left(\delta_{v_1} - \delta_{v_0}\right) = D\left(\pm\delta_{e_r}\right) + \cdots + D\left(\pm\delta_{e_1}\right).$$

This equations shows that $\delta_v - \delta_x$ is in the image of D, and it follows that $f \in \text{Im } D$. $\square$

Lemma 7.2 *If G is a connected graph, the image of $D^t : C^0(G; \mathbf{Z}) \to C_I$ is B_I.*

Proof: Suppose $y = D^t x$, where $x \in C^0(G; \mathbf{Z})$. For any $z \in Z$ we have $Dz = 0$ and so

$$\langle y, z\rangle = \langle D^t x, z\rangle = \langle x, Dz\rangle = \langle x, 0\rangle = 0.$$

Hence y is in B, and clearly it takes integer values, so y is in B_I.

Conversely, recall from the proof of Theorem 6.2 that $\mathcal{D}_q$ is a $\mathbf{Z}$-basis for B_I. Consequently, given $y \in B_I$ we have $y = \sum \alpha_v b_v$, where $\alpha_v \in \mathbf{Z}$. If we define $\alpha \in C^0(G; \mathbf{Z})$ in the obvious way (with $\alpha_q = 0$), the equation is equivalent to $y = D^t\alpha$, from which it follows that y is in $\text{Im } D^t$. $\square$

In Algebraic Geometry the image group $D(C_I)$ is known as the group of *divisors of degree* 0 of G. Its subgroup $D(B_I)$ is known as the group of *principal divisors* of G, and the *Picard group*, *Pic*(G) is defined to be the quotient $D(C_I)/D(B_I)$.

The preceding Lemmas provide a more familiar interpretation of *Pic*(G). According to Lemma 7.1, $D(C_I)$ is the kernel of σ. According to Lemma 7.2, B_I is the image of D^t, so $D(B_I)$ is the image of $DD^t = Q$. Thus

$$Pic(G) = \frac{D(C_I)}{D(B_I)} = \frac{\text{Ker } \sigma}{\text{Im } Q},$$

and Theorem 4.2 asserts that $Pic(G)$ is naturally isomorphic to the critical group $K(G)$. On the other hand, it can be shown [1, 3] that the function which takes a coset $[Dc]$ in $Pic(G)$ to the coset $[Pc]$ in $(B_I)^\sharp/B_I$ is an isomorphism. (Recall the result, mentioned above, that $\mathrm{Im}\ P_I = (B_I^\sharp)$.) Thus $Pic(G)$ is a group of order κ. Putting all this together we have:

Theorem 7.3 *If G is a connected graph the critical group $K(G)$ has order κ, the tree number of G.*

8. Structure of the critical group

In the preceding sections it has been shown that the critical group $K(G)$ associated with a connected graph G is isomorphic to a number of 'classical' abelian groups of order κ, the tree number of G. These groups are associated with the group of 'indecomposable' integral cochains $C_I/(Z_I \oplus B_I)$, and one of them, the Picard group $Pic(G) = D(C_I)/D(B_I)$, is precisely the group we used in Section 4 to define a group structure on the set of critical configurations.

The classification theorem for finite abelian groups asserts that $K(G)$ has a direct sum decomposition

$$K(G) = (\mathbf{Z}/n_1\mathbf{Z}) \oplus (\mathbf{Z}/n_2\mathbf{Z}) \oplus \cdots \oplus (\mathbf{Z}/n_r\mathbf{Z}),$$

where the integers n_i are known as *invariant factors*, and they satisfy $n_i \mid n_{i+1}$, $(1 \leq i < r)$. Since $|K(G)| = \kappa$, it follows that

$$n_1 n_2 \cdots n_r = \kappa.$$

The invariant factors can be used to distinguish pairs of non-isomorphic graphs which have the same κ (see Section 10), and so there is considerable interest in their properties. The standard method of computing them is to use a presentation of the group, and the definition of the Picard group $Pic(G)$ as $D(C_I)/D(B_I)$ provides just that.

Theorem 8.1 *Given a connected graph G, generators and relations for $Pic(G)$ can be chosen so that the matrix of relations is the reduced Laplacian matrix Q_q.*

Proof: Choose a vertex q in G. The proof of Lemma 7.1 shows that the set of functions $\zeta_v = \delta_v - \delta_q (v \neq q)$ is a $\mathbf{Z}$-basis for $D(C_I)$.

On the other hand, in the proof of Theorem 6.2 we observed that a $\mathbf{Z}$-basis for B_I is $\mathcal{D}_q$, the set of rows $b_v = D^t\delta_v$ of D for which $v \neq q$. Since b_q is a $\mathbf{Z}$-linear combination of the members of $\mathcal{D}_q$, it follows trivially that the set of functions $b_v - b_q = D^t(\delta_v - \delta_q) = D^t\zeta_v(v \neq q)$ is a $\mathbf{Z}$-basis for B_I.

The quotient group $D(C_I)/D(B_I)$ is generated by the images $\bar{\zeta}_v$ of the ζ_v. Since B_I is generated by the functions $D^t\zeta_v$, the 'relations group' $D(B_I)$ is generated by the functions $D(D^t\zeta_v)$. In other words, in the quotient group the following relations hold:

$$D(D^t\bar{\zeta}_v) = 0, \quad \text{that is, } Q\bar{\zeta}_v = 0 \ (v \neq q).$$

Since the rank of Q is $n-1$, one relation is redundant, let us say the one given by row q of Q. Also since there is no generator $\bar{\zeta}_q$ we may omit column q of Q. Thus Q_q is a relations matrix for the Picard group. $\square$

The standard technique for obtaining the invariant factors of a finitely-generated abelian group from a presentation is to reduce the matrix of relations M to *Smith normal form* *Sm*(M). Algorithmically, this is done by applying row and column operations to obtain a diagonal matrix, whose diagonal entries are the invariant factors. Formally, we require matrices R_1 and R_2 in $GL(r, \mathbf{Z})$ such that

$$R_1 M R_2 = Sm(M) = \mathrm{diag}(n_1, n_2, \ldots, n_r).$$

For example, the reduced Laplacian Q_q for K_n is the $(n-1) \times (n-1)$ matrix $nI - J$ (where J is the all-1 matrix). Partition Q_q as follows:

$$\begin{pmatrix} n-1 & -u^t \\ -u & nI - J \end{pmatrix},$$

where u is the all-1 column vector and I, J are now $(n-2) \times (n-2)$ matrices. Let

$$R_1 = \begin{pmatrix} 1 & u^t \\ u & I + J \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & -u^t \\ 0 & I \end{pmatrix}.$$

Then $\det R_1 = \det R_2 = 1$ so R_1 and R_2 are in $GL(n-1, \mathbf{Z})$. Furthermore,

$$R_1 Q_q R_2 = \begin{pmatrix} 1 & 0 \\ 0 & nI \end{pmatrix}.$$

It follows that the invariant factors of Q_q are 1 and n ($n-2$ times), and so the critical group $K(K_n)$ is the direct product of $n-2$ copies of $\mathbf{Z}/n\mathbf{Z}$. Since κ is the product of the invariant factors, this is a refinement of Cayley's formula $\kappa(K_n) = n^{n-2}$.

One feature of the dollar game is that it provides an alternative calculus of determining the invariant factors. Calculations with critical configurations can be regarded as the basic algorithmic steps underlying the matrix operations required to find the Smith normal form. We shall give some examples of this alternative calculus in the following sections.

9. Analysis of the wheel graphs

The *wheel graph* W_n has $n+1$ vertices, which we shall denote by q and the integers modulo n. The vertex q is adjacent to all other vertices, and those vertices form the *rim* of the wheel, a cycle in which i is adjacent to $i-1$ and $i+1$. Note that $W_3 = K_4$.

The wheel graphs form what has been called [5] a *recursive family*. This means that, in particular, the tree-numbers of the family are determined by a linear recursion. In this case

$\kappa_n = \kappa(W_n) = |K(W_n)|$ satisfies

$$\kappa_{n+3} = 4\kappa_{n+2} - 4\kappa_{n+1} + \kappa_n,$$

with the initial conditions $\kappa_2 = 5$, $\kappa_3 = 16$, $\kappa_4 = 45$, and the resulting formula is:

$$\kappa_n = \left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n - 2.$$

The first few values of κ_n are as follows.

2	3	4	5	6	7	8	9	10	11
5	16	45	121	320	841	2205	5776	15125	39602

Let (f_n) and (l_n) be the sequences of Fibonacci numbers and Lucas numbers, respectively. These sequences are defined by the initial conditions $f_0 = 1$, $f_1 = 1$ and $l_0 = 2, l_1 = 1$, respectively, and the recursion $x_n = x_{n-1} + x_{n-2}$. There are many relationships between these numbers, the basic one being $l_n = f_n + f_{n-2}$.

For our purposes the significant fact is that the numbers κ_n are given in terms of the Fibonacci and Lucas numbers by

$$\kappa_n = \begin{cases} l_n \times l_n, & \text{if } n \text{ is odd;} \\ 5 \times f_{n-1} \times f_{n-1}, & \text{if } n \text{ is even.} \end{cases}$$

We shall show that these factorisations of κ_n are closely related to the structure of the critical group $K(W_n)$.

We begin with the case when n is odd. Let $n = 2r + 1$ and denote the vertices on the rim of W_n by the residue classes $-r, -(r-1), \ldots, -1, 0, +1, +2, \ldots, +r \pmod n$. Define a configuration b on W_n as follows:

$$b(v) = \begin{cases} 1, & \text{if } v = \pm r; \\ 2, & \text{otherwise.} \end{cases}$$

Lemma 9.1 *The configuration b is critical and has order l_n in the abelian group $K(W_n)$.*

Proof: Clearly b is stable. It is easy to check that the sequence of vertices

$$q, 0, -1, +1, -2, +2, \ldots, -r, +r,$$

is q-legal for b, and since each vertex is fired once the resulting configuration is $b - Qu = b$. So b is recurrent, and therefore critical.

For any positive integer t let $t \cdot b$ denote $b \bullet b \bullet \cdots \bullet b$, the $\bullet$-sum of t copies of b in $K(W_n)$. Equivalently, $t \cdot b$ is the unique critical configuration $\gamma(b + b + \cdots + b)$, where the $+$ sign represents vector addition. We shall obtain explicit expressions for $t \cdot b$ when t is a Fibonacci number; specifically we shall show that, for $i = 1, 2, \dots, r$,

$$(f_{2i-2} \cdot b)(v) = \begin{cases} 1, & \text{if } v = \pm(r - i + 1); \\ 2, & \text{otherwise.} \end{cases}$$

$$(f_{2i-1} \cdot b)(v) = \begin{cases} 1, & \text{if } v = \pm(r - j) \text{ and } j = 0, 1, \dots, i - 1; \\ 2, & \text{otherwise.} \end{cases}$$

When $i = 1$ the expressions for $f_0 \cdot b$ and $f_1 \cdot b$ both reduce to that for b, which is correct since $f_0 = f_1 = 1$. Assume that the formulae hold when $i = k$, and suppose that $2 \leq k + 1 \leq r$. Then $f_{2(k+1)-2} = f_{2k} = f_{2k-1} + f_{2k-2}$. Hence

$$f_{2k} \cdot b = (f_{2k-1} \cdot b) \bullet (f_{2k-2} \cdot b) = \gamma(f_{2k-1} \cdot b + f_{2k-2} \cdot b).$$

Using the induction hypothesis we have

$$(f_{2k-1} \cdot b + f_{2k-2} \cdot b) = \begin{cases} 2, & \text{if } v = \pm(r - k + 1); \\ 3, & \text{if } v = \pm(r - j) \text{ and } j = 0, 1, \dots, k - 2; \\ 2, & \text{otherwise.} \end{cases}$$

It can be checked that the following sequence is q-legal for this configuration and results in the stated formula for $f_{2k} \cdot b$.

$$0, -1, +1, -2, +2, \dots, -r, +r, 0, -1, +1, -2, +2, \dots, -(r - k), r - k.$$

Similarly, the expression for $f_{2k+1} \cdot b$ can be verified. Hence we have verified the formulae for $f_j \cdot b$ when $j = 0, 1, \dots, 2r - 1$.

Using the same methods, the following expressions for $f_{2r} \cdot b$ and $f_{2r+1} \cdot b$ can be obtained.

$$(f_{2r} \cdot b)(v) = \begin{cases} 0, & \text{if } v = 0; \\ 2, & \text{otherwise.} \end{cases} \qquad (f_{2r+1} \cdot b)(v) = \begin{cases} 1, & \text{if } v = 0; \\ 2, & \text{otherwise.} \end{cases}$$

Since $n = 2r + 1$, we have $l_n = f_n + f_{n-2} = f_{2r+1} + f_{2r-1}$. Using the formulae obtained above, the configuration $s = f_{2r+1} \cdot b + f_{2r-1} \cdot b$ has $s(v) = 3$ for each vertex $v \neq q$. Firing each vertex except q once, we get the configuration $\gamma(s) = o$, where $o(v) = 2$ $(v \neq q)$. Clearly o is the zero element of $K(W_n)$, and so

$$l_n \cdot b = f_{2r+1} \cdot b \bullet f_{2r-1} \cdot b = \gamma(f_{2r+1} \cdot b + f_{2r-1} \cdot b) = \gamma(s) = o. \qquad \square$$

For any vertex $w \neq q$ let b_w be the configuration defined by $b_w(v) = b(v - w)$. In other words, b_w is obtained from b by rotating the rim of the wheel through w steps. We have $b_0 = b$, and for convenience we write $b_{+1} = b_1$. Each b_w is an element of order l_n in $K(W_n)$.

Theorem 9.2 *When n is odd the group $K(W_n)$ is the direct sum*

$$K(W_n) = (\mathbf{Z}/l_n\mathbf{Z}) \oplus (\mathbf{Z}/l_n\mathbf{Z}),$$

where the cyclic groups of order l_n are generated by b_0 and b_1.

Proof: Let π_0 and π_1 denote the permutations of the rim vertices defined as follows:

$$\pi_0(0) = 0, \pi_0(+i) = -i, \pi_0(-i) = +i \quad (i = 1, 2, \ldots, r);$$
$$\pi_1(1) = 1, \pi_1(-r) = -(r-1), \pi_1(-(r-1)) = -r,$$
$$\pi_1(+i) = -(i-2), \pi_1(-(i-2)) = +i, \quad (i = 1, 2, \ldots, r).$$

Every multiple of b_0 is symmetrical with respect to π_0, that is, $(\alpha \cdot b_0)(v) = (\alpha \cdot b_0)(\pi_0 v)$. On the other hand, every multiple of b_1 is symmetrical with respect to π_1. Since π_0 and π_1 generate a group which acts transitively on the rim, the only configurations which are symmetrical with respect to both permutations are those in which every rim vertex has the same number of dollars. So, if $\alpha \cdot b_0 = \beta \cdot b_1$, both configurations are in fact the critical configuration in which each rim vertex has two dollars, which is the zero element of $K(W_n)$. It follows that the subgroup generated by b_0 and b_1 is the direct sum of the cyclic groups generated by b_0 and b_1. This has order l_n^2, which we know to be the order of $K(W_n)$, and so the result follows. □

It is easy to express the configurations b_w $(w \neq 0, +1)$ in terms of b_0 and b_1. A simple computation shows that $b_{-1} \bullet b_1 = \gamma(b_{-1} + b_1) = 3 \cdot b_0$, and in general for any w we have

$$b_{w-1} \bullet b_{w+1} = 3 \cdot b_w.$$

This observation is relevant to the observation made at the end of Section 8, that calculations with critical configurations are in effect equivalent to finding the Smith normal form of the reduced Laplacian. The reduced Laplacian Q_q of a wheel graph consists of a main diagonal of 3's, with -1's in adjacent positions, and by Theorem 8.1 this is a matrix of relations for $K(W_n)$.

We now turn to the case when n is even. Define configurations

$$b(j) = \begin{cases} 1, & \text{if } j = 0; \\ 2, & \text{otherwise.} \end{cases}$$

$$c(j) = \begin{cases} 1, & \text{if } j = 0, 2, 4, \ldots, n-2; \\ 2, & \text{otherwise.} \end{cases}$$

By methods like those used for the odd case, we can verify that $K(W_n)$ is generated by configurations b_0 and b_1, where $b_0 = b$ and b_1 is obtained from b by a unit rotation. However, in this case b_0 and b_1 are not independent generators. It can be checked that $f_{n-1} \cdot b_0 = c$ and $f_{n-1} \cdot b_1 = -c$, where $-c$ is obtained from c by switching the values

1 and 2. Furthermore $5 \cdot c = o$. Hence both generators have order $5 f_{n-1}$, but the cyclic groups they generate intersect in a group of order 5, generated by c.

The direct sum decomposition and invariant factors of $K(W_n)$ in the even case can be determined from the foregoing observations. It should be noted that the Fibonacci number f_{n-1} is divisible by 5 if and only if n is a multiple of 5, and this affects the form of the invariant factorisation when n is multiple of 10.

10. Strongly regular graphs

A connected graph is *strongly regular* with parameters (k, a, c) if: (i) it is regular, with degree $k \geq 2$; (ii) any two adjacent vertices have the same number $a \geq 0$ of common neighbours; (iii) any two non-adjacent vertices have the same number $c \geq 1$ of common neighbours.

Strongly regular graphs are a subset of the class of *distance-regular* graphs. The general case will be studied in another paper [4], but it is convenient to emphasise the relationship by using a more general form of parametrisation. Denote by $d(v, w)$ the distance between two vertices v and w, and let b_1 be the number of vertices x such that $d(x, v) = 1$ and $d(x, w) = 2$, given that $d(v, w) = 1$. Similarly, denote by c_2 the number of vertices x such that $d(x, p) = 1$ and $d(x, q) = 1$, given that $d(p, q) = 2$. Then $b_1 = k - a - 1$ and $c_2 = c$. The *intersection array* of a strongly regular graph is defined to be $\{k, b_1; 1, c_2\}$.

It is known that the tree number κ of a strongly regular graph is determined by its intersection array. This follows from the fact that for any connected graph, we have the formula [2, p. 40]

$$\kappa = n^{-1}\mu_1\mu_2 \cdots \mu_{n-1},$$

where $\mu_1, \mu_2, \ldots, \mu_{n-1}$ are the non-zero eigenvalues of the Laplacian matrix Q. For a regular graph of degree k, $Q = kI - A$, where A is the adjacency matrix. If, in addition, the graph is strongly regular the spectrum of A is completely determined by the intersection array [2, 8].

However, the invariant factors are not determined by the intersection array. For example, there are two strongly regular graphs with intersection array $\{6, 3; 1, 2\}$, the lattice graph $L(4)$ and the Shrikhande graph *Shr*. For both graphs $\kappa = 2^{35}$, but the invariant factorisations of κ are different [1]:

$$\kappa(L(4)) = 8^5 \cdot 32^4, \quad \kappa(Shr) = 2 \cdot 8^2 \cdot 16^2 \cdot 32^4.$$

(The Smith normal forms for the Laplacian matrices of $L(4)$ and *Shr* are also given in [9].) This observation shows that the structure of the critical group $K(G)$ can be used to distinguish graphs in cases where other algebraic invariants, such as those derived from the spectrum, fail. In the case of a strongly regular graph, the parameters do not determine the structure of $K(G)$ but, as we shall now explain, they do determine a subgroup of it. Given a strongly regular graph G and a specified vertex q, let us say that a configuration s of the dollar game on G is *layered* if $s(v)$ depends only on $d(q, v)$. Thus, a layered configuration is defined by an ordered pair (s_1, s_2), where s_j is the value of $s(v)$ for any vertex v at distance j from q. Let $\mathcal{F}_j$ $(j = 0, 1, 2)$ denote the operation of firing all vertices at distance j from q once (in any order). Note that $\mathcal{F}_0$ denotes the firing of q only, and clearly $\mathcal{F}_1$ or $\mathcal{F}_2$ can be applied to a layered configuration s if and only if $s_1 \geq k$ or $s_2 \geq k$, respectively. The application of these operations to a layered configuration results in another layered configuration, and the following rules are easily verified:

$$(s_1, s_2) \overset{\mathcal{F}_0}{\mapsto} (s_1 + 1, s_2);$$

$$(s_1, s_2) \overset{\mathcal{F}_1}{\mapsto} (s_1 - b_1 - 1, s_2 + c_2);$$

$$(s_1, s_2) \overset{\mathcal{F}_2}{\mapsto} (s_1 + b_1, s_2 - c_2).$$

Lemma 10.1 *The layered configuration $s = (s_1, s_2)$ is critical if and only if*

$$s_1 = k - 1 \quad \textit{and} \quad k - c_2 \leq s_2 \leq k - 1.$$

Proof: Suppose that s satisfies the conditions. Then clearly s is stable. Consider what happens when we attempt to fire the vertices in the sequence $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2$.

Since s is stable the condition for firing q (that is, $\mathcal{F}_0$) holds. After $\mathcal{F}_0$ the new configuration s' has $s'_1 = k$, so that $\mathcal{F}_1$ can be applied. After $\mathcal{F}_1$ we have a configuration s'' with $s''_2 = s_2 + c_2$, and the given conditions imply that $s_2 + c_2 \geq k$, so $\mathcal{F}_2$ can be applied. So the sequence is q-legal, and the final result is s again. This shows that s is recurrent, and consequently critical.

Conversely, suppose s is critical. Since s is stable the condition $s_j \leq k - 1$ certainly holds for $j = 1, 2$. Thus, it is sufficient to prove that if $s_1 < k - 1$ or $s_2 < k - c_2$ then s is not recurrent.

If $s_1 < k - 1$ then we can use $\mathcal{F}_0$ $k - 1 - s_1$ times to obtain a configuration $(k - 1, s_2)$. If $s_2 \geq k - c_2$ then this configuration is critical and we stop. If $s_2 < k - c_2$ the sequence $\mathcal{F}_0, \mathcal{F}_1$ is q-legal and results in the configuration $(k - 1 - b_1, s_2 + c_2)$. By firing $\mathcal{F}_0 b_1$ times again we obtain $(k - 1, s_2 + c_2)$. If $s_2 + c_2$ is in the critical range then we stop. If not, by repeating this process we can increase the second component, say f times in all, until $s_2 + fc_2$ is in the critical range, and then restore the value $k - 1$ of the first component. This is a configuration s^* which, by the first part, is critical and which can be reached from s by a q-legal sequence of firings. It follows from Theorem 3.8 that s is not critical. □

We now investigate the effect of the $\bullet$ operation on the set of c_2 basic critical configurations specified in Lemma 10.1, which we denote by

$$\langle i \rangle = (k - 1, i) \quad i = k - c_2, \ldots, k - 1.$$

We can calculate $\langle i \rangle \bullet \langle j \rangle$ as follows. Consider $\langle i \rangle + \langle j \rangle = (2k - 2, i + j)$. The operation $\mathcal{F}_1$ can be applied to this configuration and results in $(2k - 3 - b_1, i + j + c_2)$; now the operation $\mathcal{F}_2$ can be applied and results in $(2k - 3, i + j)$. Let $\mathcal{R}$ denote $\mathcal{F}_1$ followed by $\mathcal{F}_2$. Repeating the foregoing argument, we can apply $\mathcal{R}(k - 1)$ times in all, until we reach $(k - 1, i + j)$. If $i + j \leq k - 1$, this is the critical configuration $\langle i + j \rangle$. If $i + j \geq k$, then

$\mathcal{F}_2$ is legal and results in the configuration $(k-1+b_1, i+j-c_2)$. Now we can apply $\mathcal{R}$ b_1 times, which yields $(k-1, i+j-c_2)$. Either this is critical, or $\mathcal{F}_2$ is legal and we can repeat the process.

The conclusion is that the critical configuration $\langle i\rangle \bullet \langle j\rangle = \gamma(\langle i\rangle + \langle j\rangle)$ is $\langle h\rangle$, where h is the (unique) integer in the range $k-c_2 \le h \le k-1$ which is congruent to $i+j \bmod c_2$. This rule also shows that the zero element of $K(G)$ is $\langle o_2\rangle$, where o_2 is the unique multiple of c_2 in the range $k-c_2 \le o_2 \le k-1$.

We can express these results algebraically as follows. Given a congruence class r in $\mathbf{Z}/c_2\mathbf{Z}$, let r_2 denote the unique representative of r which satisfies $k-c_2 \le r_2 \le k-1$. Then the map from $\mathbf{Z}/c_2\mathbf{Z}$ to $K(G)$ defined by $r \mapsto \langle r_2\rangle$ is a monomorphism. Thus we have proved the following result.

Theorem 10.2 *Let G be strongly regular graph with intersection array* $\{k, b_1; 1, c_2\}$. *Then the layered critical configurations form a cyclic subgroup of* $K(G)$, *of order* c_2.

For the vast majority of strongly regular graphs $c_2 > 1$, and the subgroup of order c_2, although relatively small, is nevertheless significant. Consider the *Paley graph* of order q where q is a prime power of the form $4c+1$. This is strongly regular with intersection array

$$\{2c, c; 1, c\}, \quad \text{and} \quad |K(G)| = \kappa = q^{2c-1}c^{2c}.$$

Since q and c are coprime, the arithmetical facts imply a direct summand of order c^{2c}, but they do not force a subgroup of order c. For example, when $q=49$ and $c=12$ the summand of order 12^{24} must, in the light of Theorem 10.2, contain elements of order 12, although this is not forced by the numerical information.

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