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A Twodimensional Variational Model for the Equilibrium Configuration of an Incompressible, Elastic Body with a Three-Well Elastic Potential

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We consider a geometrically linear variational model in two space dimensions for an incompressible, elastic body whose elastic potential has exactly three wells corresponding to one austenitic and to two martensitic phases. Passing to the dual problem we show that the stress tensor is weakly differentiable on the interior of the domain and in addition Hölder continuous on any subset of the union of the pure phases.

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1. Introduction

In this paper we are concerned with the mathematical analysis of the variational problem which corresponds to the physical situation that a multiphase elastic body is in equilibrium state under the action of a given system of forces. In order to obtain such a variational formulation, we have to assume that the temperature as well as the loads are fixed. We are also not going to consider the most general situation which means that we restrict ourselves to the geometrically linear case, for a description of the nonlinear setting and its comparison with the linear one we refer the reader to [4, 5], [7] and [12, 13]. Another restriction is that we consider an elastic potential with three wells corresponding to one austenitic and to two martensitic phases. Even under these assumptions the analysis of the problem turns out to be quite difficult for the following reason: by definition the elastic energy is the pointwise infimum of the different phase energies and so in general not quasiconvex. Hence one has to consider the quasiconvex envelope for which Dacorogna's formula (see [8]) is available. Unfortunately this representation is not very explicit and so only very few examples for the computation of the envelope are known, in particular this concerns the case of three wells. For two wells with the same elastic moduli explicit formulas for the quasiconvex envelope can be found in the papers [13], [16] and [19], the case of two isotropic wells with well-ordered elastic moduli is solved in principle in [1, 2] which means that the problem is reduced to the minimization of suitable functions depending only on a finite number of variables. We wish to mention that for incompressible

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bodies there is some analogue of Dacorogna's formula obtained in [22] (see also [18] for a more general setting), and in the case of two isotropic wells explicit representations of the quasiconvex envelope are given in [14] and [22].

For completeness we would like to mention that in the paper [13, Section 8, p. 232] the reader will find some comments concerning the computation of the envelope in the case of N-wells but no explicit formula is given.

The importance of explicit formulas is evident: as a matter of fact one should expect degeneracy of the relaxed functional, and since Dacorogna's formula is not local, it is quite difficult to decide where degeneracy occurs and what the precise behaviour of the energy is.

However, there are several cases for which the quasiconvex envelope turns out to be a convex or "almost" convex function, e.g. the case of two wells with the same elastic moduli. Here the quasiconvex envelope is convex provided the stress-free strains are compatible (see [13] for details), or it can be replaced by a convex integrand in the case of incompatible stress-free strains (see [21]). It is then possible to apply the powerful methods of duality theory (compare [20] and [21]) to this particular situation.

Another setting recently has been studied by the second author in [22]: this paper addresses the case of incompressible bodies in two spatial variables and it is shown that the notions of convexity and quasiconvexity coincide. Motivated by this result we also impose incompressibility as a further restriction, and we will limit ourselves to the twodimensional case. For two space dimensions one may also use an alternative approach for the construction of a suitable relaxed variational problem. If, for example, Ω is simply connected, then we may look for solutions of our original variational problem which are the "curl" of some scalar function, thus we arrive at a scalar variational problem of higher order. For the relaxation of first order scalar problems we refer to [9], the case of higher order problems is treated in [18]. Besides the topological constraint it is also not immediate how to incorporate the boundary conditions in this setting.

Our paper is organized as follows: In Section 2 we fix our notation and introduce the basic variational problem together with its relaxation. We also formulate the natural dual variational problem whose unique solution σ has the meaning of the stress tensor. Lemma 2.1 contains the so-called effective stress-strain relation, in Theorem 2.2 we show weak differentiability of σ, and our main result Theorem 2.4 states that σ is Hölder continuous on any region where no microstructure occurs. In Section 3 Lemma 2.1 is established, in Section 4 we prove Theorem 2.2. The proof of Theorem 2.4 presented in Section 7 is based on some local estimates of Caccioppoli-type (see Section 5) and on a decay lemma for the squared mean oscillation of σ (see Section 6) being valid at centers x where the formation of microstructure is excluded.

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2. Formulation of the problem and statement of the main results

Let $\mathbb{M}^d$ denote the space of all real $d \times d$ matrices. $\mathbb{S}^d$ is the subspace consisting of symmetric matrices. We will use the following notation

$$\begin{aligned} u \cdot v &= u_i v_i,\ |u| = \sqrt{u \cdot u}, \\ u \otimes v &= (u_i v_j) \in \mathbb{M}^d, \\ u \odot v &= \frac{1}{2}(u \otimes v + v \otimes u) \in \mathbb{S}^d \quad \text{for } u = (u_i), v = (v_i) \in \mathbb{R}^d, \\ A : B &= \operatorname{tr} A^T B = A_{ij} B_{ij},\ A^T = (A_{ji}) \in \mathbb{M}^d,\ |A| = \sqrt{A : A}, \\ Aa &= (A_{ij} a_j) \in \mathbb{R}^d \quad \text{for } A = (A_{ij}), B = (B_{ij}) \in \mathbb{M}^d, a \in \mathbb{R}^d, \end{aligned}$$

where the convention of summation over repeated Latin indices running from 1 to d is adopted. Further we let

$$\begin{aligned} &\overset{\circ}{\mathbb{M}}{}^d = \left\{ A \in \mathbb{M}^d \,:\, \operatorname{tr} A = 0 \right\}, \quad \overset{\circ}{\mathbb{S}}{}^d = \mathbb{S}^d \cap \overset{\circ}{\mathbb{M}}{}^d \\ &\mathbb{M} = \mathbb{M}^2, \quad \overset{\circ}{\mathbb{M}} = \overset{\circ}{\mathbb{M}}{}^2, \quad \mathbb{S} = \mathbb{S}^2 \quad \text{and} \quad \overset{\circ}{\mathbb{S}} = \overset{\circ}{\mathbb{S}}{}^2. \end{aligned}$$

We consider an elastic body with three different phases, an austenitic one and two martensitic phases. Moreover, we restrict ourselves to the twodimensional case ($d = 2$) and assume in addition that the body is incompressible. The energy density (or elastic potential or just energy) of the austenitic (the 1st) phase is given by

$$g_1(\varepsilon) = \mu|\varepsilon|^2 + g_0,\ \varepsilon \in \overset{\circ}{\mathbb{S}},$$

where μ is a positive constant, and g_0 denotes a constant depending on the temperature T. For the martensitic phases the densities are of the form

$$g_2(\varepsilon) = |\varepsilon - \varepsilon_0|^2,\ g_3(\varepsilon) = |\varepsilon + \varepsilon_0|^2,\ \varepsilon \in \overset{\circ}{\mathbb{S}};$$

here ε_0 and $-\varepsilon_0$ are the stress-free strains of the martensitic phases. In this setting it is assumed that the energies of the martensitic phases have the same minima and that the common value is equal to zero. Of course this is no further restriction since we may add any constant to the energies. On the other hand it follows from our notation that the elastic moduli of the martensitic phases are the same, and just for simplicity we put them equal to 1. As mentioned above the constant g_0 depends on the temperature, precisely we have

$$\begin{cases} g_0 > 0 & \text{if } T < T_0, \\ g_0 = 0 & \text{if } T = T_0, \\ g_0 < 0 & \text{if } T > T_0 \end{cases}$$

which means that for $T > T_0$ the stress-free state of the austenitic phase is preferred whereas for $T < T_0$ the stress-free states of the martensitic phases are favoured. Here T_0 denotes the transition temperature.

Now the energy density of this three-phase body is given by

$$g(\varepsilon) = \min\{g_1(\varepsilon), g_2(\varepsilon), g_3(\varepsilon)\},\ \varepsilon \in \overset{\circ}{\mathbb{S}},$$

and according to the general theory the state of phase equilibrium is described in terms of the following variational problem:

Problem $\mathcal{P}$: Find a vector-valued function $u \in \overset{\circ}{J}{}^1_2(\Omega) + u_0$ such that

$$I(u) = \inf\left\{I(v) : v \in \overset{\circ}{J}{}^1_2(\Omega) + u_0\right\}.$$

Here Ω is a bounded Lipschitz domain in $\mathbb{R}^2$ (the undeformed state of the body is represented by the set $\overline{\Omega}$); we further let ($\varepsilon(v)$ denoting the symmetric derivative)

$$I(v) = \int\limits_\Omega (g(\varepsilon(v)) - f \cdot v)\ dx$$

and assume

$$f \in L^2(\Omega;\mathbb{R}^2) \quad , \quad u_0 \in J^1_2(\Omega). \tag{2.1}$$

As usual $L^p(\Omega;\mathbb{R}^2)$ denotes the Lebesgue space of all vectorfields from Ω into $\mathbb{R}^2$ being p-integrable, the Sobolev space $W^1_p(\Omega;\mathbb{R}^2)$ is defined as the subspace of $L^p(\Omega;\mathbb{R}^2)$ consisting of those fields whose first weak derivatives are generated by L^p-functions. Finally, we let

$$\begin{aligned} J^1_p(\Omega) &= \{v \in W^1_p(\Omega;\mathbb{R}^2) \ : \operatorname{div} v = 0 \text{ on } \Omega\},\\ \overset{\circ}{J}{}^1_p(\Omega) &= \text{closure of } \overset{\bullet}{C}{}^\infty(\Omega) \text{ in } W^1_p(\Omega;\mathbb{R}^2),\\ \overset{\bullet}{C}{}^\infty(\Omega) &= \{v \in C^\infty_0(\Omega;\mathbb{R}^2) \ : \operatorname{div} v = 0 \text{ on } \Omega\}. \end{aligned}$$

The energy density g is continuous and bounded from below. Moreover, there exist constants $\nu > 0$, $c_1, c_2 \geq 0$ such that

$$\nu|\varepsilon|^2 - c_1 \ \leq \ g(\varepsilon) \ \leq \ \frac{1}{\nu}|\varepsilon|^2 + c_2 \tag{2.2}$$

is true for any $\varepsilon \in \overset{\circ}{\mathbb{S}}$. Clearly estimate (2.2) combined with Korn's inequality implies boundedness of any minimizing sequence in the space $J^1_2(\Omega)$, and one may pass to weakly convergent subsequences. But unfortunately our functional I is not sequentially weakly lower semicontinuous on the space $J^1_2(\Omega)$ and examples show that in fact problem $\mathcal{P}$ may fail to have solutions. The question of weak lower semicontinuity for functionals like our energy I defined on the space $J^1_2(\Omega)$ has been investigated in the paper [22] with the result that this property of the energy is (under some additional assumptions on the density) equivalent to J^1_2 - quasiconvexity of the integrand. This notion is a natural analogue of the definition of quasiconvexity introduced by Morrey [17] (or the concept of W^1_p - quasiconvexity due to Ball and Murat [6]) for the case of solenoidal vectorfields. We say that a continuous function $h : \overset{\circ}{\mathbb{S}} \to \mathbb{R}$ is J^1_p - quasiconvex iff

$$\int_\omega h(A)\,dx \le \int_\omega h\left(A+\varepsilon(u)\right)dx$$

holds for any $A \in \overset{\circ}{\mathbb{S}}$, for any bounded open set $\omega \subset \mathbb{R}^2$ and for all functions $u \in \overset{\circ}{J}{}^1_p(\omega)$. Clearly we have the same definition for any dimension $d \ge 2$.

Since problem $\mathcal{P}$ has in general no solution, we now pass to a suitable relaxed variational problem, i.e. we look at

Problem $\mathcal{QP}$: Find a function $u \in u_0 + \overset{\circ}{J}{}^1_2(\Omega)$ such that

$$QI(u) = \inf\{QI(v) : v \in u_0 + \overset{\circ}{J}{}^1_2(\Omega)\}.$$

Here the relaxed energy is given by the formula

$$QI(v) = \int_\Omega \left(Qg(\varepsilon(v)) - f\cdot v\right)\,dx$$

and Qg denotes the J^1_2-quasiconvex envelope of g for which in [22] the following representation formula has been established (compare also [18] for a more general setting):

$$Qg(\kappa) = \inf\left\{ \fint_B g(\kappa+\varepsilon(v))\,dx \;: v \in \overset{\circ}{J}{}^1_2(B)\right\},$$
$$\kappa \in \overset{\circ}{\mathbb{S}}\,,\; B = \{x \in \mathbb{R}^2 : |x| < 1\}.$$

Moreover, the next statements are also due to [22]:

- Problem $\mathcal{QP}$ has at least one solution.
- Any weak limit of any subsequence of a minimizing sequence of problem $\mathcal{P}$ is termed a general solution of problem $\mathcal{P}$. These generalized solutions are exactly the solutions of $\mathcal{QP}$.
- For our particular integrand we have $Qg(\kappa) = g^{**}(\kappa), \kappa \in \mathbb{S}$, g^{**} denoting the second Young transform, hence Qg is a convex function.

Let us recall the definitions

$$\begin{aligned} g^{**}(\kappa) &= \sup\{\kappa : \tau - g^*(\tau) : \tau \in \overset{\circ}{\mathbb{S}}\}, \kappa \in \overset{\circ}{\mathbb{S}},\\ g^*(\tau) &= \sup\{\kappa : \tau - g(\kappa) : \kappa \in \overset{\circ}{\mathbb{S}}\}, \tau \in \overset{\circ}{\mathbb{S}}, \end{aligned}$$

where g^* is the first Young transform of g. In our case we have

$$\begin{aligned} g^*(\tau) &= \max\{g_1^*(\tau), g_2^*(\tau), g_3^*(\tau)\},\\ g_1^*(\tau) &= \frac{1}{4\mu}|\tau|^2 - g_0,\\ g_2^*(\tau) &= \frac{1}{4}\,|\tau|^2 + \varepsilon_0 : \tau,\\ g_3^*(\tau) &= \frac{1}{4}\,|\tau|^2 - \varepsilon_0 : \tau\,,\; \tau \in \overset{\circ}{\mathbb{S}}{}^2. \end{aligned} \tag{2.3}$$

We are not going to compute g^{**}, in place of this we discuss the so-called effective stress-strain relation

$$\sigma = \frac{\partial g^{**}}{\partial \kappa}(\kappa),\ \kappa \in \overset{\circ}{\mathbb{S}}, \tag{2.4}$$

which has the following physical meaning: suppose that $g^{**}(\kappa) < g(\kappa)$ with $\kappa = \varepsilon(u(x))$ for some point $x \in \Omega$, u denoting a solution of problem $\mathcal{QP}$. Then we say that at $x \in \Omega$ a microstructure occurs which on a macro-level is described by the stress-strain relation (2.4).

In Lemma 2.1 below we give explicit formulas for (2.4) in all possible cases which means that the results of our computation heavily depend on the various choices for the parameters μ and g_0. Throughout the lemma we use latin numbers in brackets to indicate the region of strains or stresses where microstructure can appear. For example, (I, III) means that we have microstructure generated by the first and third phase.

Lemma 2.1. *Let* $\sigma = \frac{\partial g^{**}}{\partial \varepsilon}(\varepsilon), \varepsilon \in \overset{\circ}{\mathbb{S}}$.

(a) *Suppose that*

$$\mu > 1. \tag{2.5}$$

Then we have

$$\sigma = 2\begin{cases} \varepsilon - \varepsilon_0 & \text{if } \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2} > \ 1 \quad \text{(II)} \\ \varepsilon + \varepsilon_0 & \text{if } \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2} < -1 \quad \text{(III)} \\ \varepsilon - \frac{\varepsilon_0:\varepsilon}{|\varepsilon_0|^2}\varepsilon_0 & \text{if } \frac{|\varepsilon:\varepsilon_0|}{|\varepsilon_0|^2} \le \ 1 \quad \text{(II, III)} \end{cases} \tag{2.6}$$

for $g_0 > 0$,

$$\sigma = 2\begin{cases} \varepsilon - \varepsilon_0 & \text{if } \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2} > \ 1 & \text{(II)} \\ \varepsilon + \varepsilon_0 & \text{if } \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2} < -1 & \text{(III)} \\ \varepsilon - \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2}\varepsilon_0 & \text{if } \frac{|\varepsilon:\varepsilon_0|}{|\varepsilon_0|^2} \le \ 1, \varepsilon \ne \alpha\varepsilon_0 & \text{(II, III)} \\ 0 & \text{if } \varepsilon = \alpha\varepsilon_0, |\alpha| \le 1 & \text{(I, II, III)} \end{cases} \tag{2.7}$$

for $g_0 = 0$,

$$\sigma = 2 \begin{cases} \mu\varepsilon & \text{if } |\varepsilon + \frac{\varepsilon_0}{a\mu}| < R_2, |\varepsilon - \frac{\varepsilon_0}{a\mu}| < R_2 & \text{(I)} \\ \varepsilon - \varepsilon_0 & \text{if } \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2} > 1, |\varepsilon + \frac{\varepsilon_0}{a\mu}| > R_1 & \text{(II)} \\ \varepsilon + \varepsilon_0 & \text{if } \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2} < -1, |\varepsilon - \frac{\varepsilon_0}{a\mu}| > R_1 & \text{(III)} \\ \varepsilon - \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2}\varepsilon_0 & \text{if } \frac{|\varepsilon:\varepsilon_0|}{|\varepsilon_0|^2} \le 1, \left|\varepsilon - \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2}\varepsilon_0\right| > \sqrt{\frac{-g_0}{a}} & \text{(II, III)} \\ R_1 \frac{\varepsilon + \frac{\varepsilon_0}{a\mu}}{|\varepsilon + \frac{\varepsilon_0}{a\mu}|} - \frac{\varepsilon_0}{a} & \text{if } \begin{cases} R_2 \le |\varepsilon + \frac{\varepsilon_0}{a\mu}| \le R_1, \\ 1 - a + a\frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2} > \sqrt{-\frac{a}{g_0}} \left|\varepsilon - \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2}\varepsilon_0\right| \end{cases} & \text{(I, II)} \\ R_1 \frac{\varepsilon - \frac{\varepsilon_0}{a\mu}}{|\varepsilon - \frac{\varepsilon_0}{a\mu}|} + \frac{\varepsilon_0}{a} & \text{if } \begin{cases} R_2 \le |\varepsilon - \frac{\varepsilon_0}{a\mu}| \le R_1, \\ 1 - a - a\frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2} > \sqrt{-\frac{a}{g_0}} \left|\varepsilon - \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2}\varepsilon_0\right| \end{cases} & \text{(I, III)} \\ \sqrt{-\frac{g_0}{a}} \frac{\varepsilon - \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2}\varepsilon_0}{\left|\varepsilon - \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2}\varepsilon_0\right|} & \text{if } \begin{cases} \left|\varepsilon - \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2}\varepsilon_0\right| \le -\frac{g_0}{a}, \\ 1 - a + a\frac{|\varepsilon:\varepsilon_0|}{|\varepsilon_0|^2} \le \sqrt{-\frac{a}{g_0}} \left|\varepsilon - \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2}\varepsilon_0\right| \end{cases} & \text{(I, II, III)} \end{cases} \tag{2.8}$$

for $g_0 < 0$, *where*

$$a = 1 - \frac{1}{\mu},\ R_1 = \sqrt{\frac{|\varepsilon_0|^2}{a^2} - \frac{g_0}{a}},\ R_2 = \frac{1}{\mu} R_1. \tag{2.9}$$

(b) *Suppose that*

$$\mu = 1 \tag{2.10}$$

Then (2.6) *for* $g_0 > 0$ *and* (2.7) *for* $g_0 = 0$ *are valid. Further we have*

$$\sigma = 2 \begin{cases} \varepsilon & \text{if } |\varepsilon : \varepsilon_0| < -\frac{g_0}{2} & \text{(I)} \\ \varepsilon - \varepsilon_0 & \text{if } \varepsilon : \varepsilon_0 > -\frac{g_0}{2} + |\varepsilon_0|^2 & \text{(II)} \\ \varepsilon + \varepsilon_0 & \text{if } \varepsilon : \varepsilon_0 < \frac{g_0}{2} - |\varepsilon_0|^2 & \text{(III)} \\ \varepsilon - \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2}\varepsilon_0 - \frac{g_0}{4|\varepsilon_0|^2}\varepsilon_0 & \text{if } -\frac{g_0}{2} \le \varepsilon : \varepsilon_0 \le -\frac{g_0}{2} + |\varepsilon_0|^2 & \text{(I, II)} \\ \varepsilon - \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2}\varepsilon_0 + \frac{g_0}{4|\varepsilon_0|^2}\varepsilon_0 & \text{if } \frac{g_0}{2} - |\varepsilon_0|^2 \le \varepsilon : \varepsilon_0 \le \frac{g_0}{2} & \text{(I, III)} \end{cases} \tag{2.11}$$

for the case $g_0 < 0$.

(c) *Suppose that*

$$0 < \mu < 1. \tag{2.12}$$

Then we get

$$\sigma = 2\begin{cases} \mu\varepsilon & \text{if } |\varepsilon + \frac{\varepsilon_0}{b\mu}| > R_4,\ |\varepsilon - \frac{\varepsilon_0}{b\mu}| > R_4 & \text{(I)} \\ \varepsilon - \varepsilon_0 & \text{if } |\varepsilon - \frac{\varepsilon_0}{b\mu}| < R_3,\ \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2} > 1 & \text{(II)} \\ \varepsilon + \varepsilon_0 & \text{if } |\varepsilon + \frac{\varepsilon_0}{b\mu}| < R_3,\ \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2} < -1 & \text{(III)} \\ \varepsilon - \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2}\varepsilon_0 & \text{if } \frac{|\varepsilon:\varepsilon_0|}{|\varepsilon_0|^2} \le 1,\ \left|\varepsilon - \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2}\varepsilon_0\right| < \sqrt{\frac{g_0}{b}} & \text{(II, III)} \\ R_3\frac{\varepsilon - \frac{\varepsilon_0}{\mu b}}{|\varepsilon - \frac{\varepsilon_0}{\mu b}|} + \frac{\varepsilon_0}{b} & \text{if } \begin{cases} R_3 \le |\varepsilon - \frac{\varepsilon_0}{\mu b}| \le R_4, \\ \left|\varepsilon - \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2}\varepsilon_0\right| > \sqrt{\frac{g_0}{b}}\left(b + 1 - b\frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2}\right) \end{cases} & \text{(I, II)} \\ R_3\frac{\varepsilon + \frac{\varepsilon_0}{\mu b}}{|\varepsilon + \frac{\varepsilon_0}{\mu b}|} - \frac{\varepsilon_0}{b} & \text{if } \begin{cases} R_3 \le |\varepsilon + \frac{\varepsilon_0}{b\mu}| \le R_4, \\ \left|\varepsilon - \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2}\varepsilon_0\right| > \sqrt{\frac{g_0}{b}}\left(b + 1 + b\frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2}\right) \end{cases} & \text{(I, III)} \\ \sqrt{\frac{g_0}{b}}\frac{\varepsilon - \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2}\varepsilon_0}{\left|\varepsilon - \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2}\varepsilon_0\right|} & \text{if } \begin{cases} \sqrt{\frac{g_0}{b}} \le |\varepsilon - \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2}\varepsilon_0| \le \frac{1}{\mu}\sqrt{\frac{g_0}{b}}, \\ |\varepsilon - \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2}\varepsilon_0| \le \sqrt{\frac{g_0}{b}}(b + 1 - b\frac{|\varepsilon:\varepsilon_0|}{|\varepsilon_0|^2}) \end{cases} & \text{(I, II, III)} \end{cases} \tag{2.13}$$

for $g_0 > 0$, *where* $b = \frac{1}{\mu} - 1,\ R_3 = \frac{|\varepsilon_0|^2}{b^2} + \frac{g_0}{b},\ R_4 = \frac{1}{\mu}R_3,$

$$\sigma = 2\begin{cases} \mu\varepsilon & \text{if } |\varepsilon - \frac{\varepsilon_0}{b\mu}| > R_4, |\varepsilon + \frac{\varepsilon_0}{b\mu}| > R_4 & \text{(I)} \\ \varepsilon - \varepsilon_0 & \text{if } |\varepsilon - \frac{\varepsilon_0}{b\mu}| < R_3 & \text{(II)} \\ \varepsilon + \varepsilon_0 & \text{if } |\varepsilon + \frac{\varepsilon_0}{b\mu}| < R_3 & \text{(III)} \\ R_3\frac{\varepsilon - \frac{\varepsilon_0}{b\mu}}{|\varepsilon - \frac{\varepsilon_0}{b\mu}|} + \frac{\varepsilon_0}{b} & \text{if } \begin{cases} R_3 \le |\varepsilon - \frac{\varepsilon_0}{b\mu}| \le R_4, \\ \varepsilon \ne \gamma\varepsilon_0,\ 0 \le \gamma \le 1 \end{cases} & \text{(I, II)} \\ R_3\frac{\varepsilon + \frac{\varepsilon_0}{b\mu}}{|\varepsilon + \frac{\varepsilon_0}{b\mu}|} - \frac{\varepsilon_0}{b} & \text{if } \begin{cases} R_3 \le |\varepsilon + \frac{\varepsilon_0}{b\mu}| \le R_4, \\ \varepsilon \ne \gamma\varepsilon_0,\ -1 \le \gamma \le 0 \end{cases} & \text{(I, III)} \\ 0 & \text{if } \varepsilon = \gamma\varepsilon_0, |\gamma| \le 1 & \text{(I, II, III)} \end{cases} \tag{2.14}$$

for $g_0 = 0$,

$$\sigma = 2\mu\varepsilon \quad (I) \quad \textit{for } g_0 < 0 \textit{ and } \frac{|\varepsilon_0|^2}{b^2} + \frac{g_0}{b} < 0, \tag{2.15}$$

$$\sigma = 2\mu\begin{cases} \varepsilon & \text{if } \varepsilon \ne \pm\frac{\varepsilon_0}{b\mu} & \text{(I)} \\ \varepsilon & \text{if } \varepsilon = \frac{\varepsilon_0}{b\mu} & \text{(I, II)} \\ \varepsilon & \text{if } \varepsilon = -\frac{\varepsilon_0}{b\mu} & \text{(I, III)} \end{cases} \tag{2.16}$$

for $g_0 < 0$ *and* $\frac{|\varepsilon_0|^2}{|b|^2} + \frac{g_0}{b} = 0$,

$$\sigma = 2 \begin{cases} \mu\varepsilon & \textit{if } |\varepsilon - \frac{\varepsilon_0}{b\mu}| > R_4,\ |\varepsilon + \frac{\varepsilon_0}{b\mu}| > R_4 \quad \text{(I)} \\ \varepsilon - \varepsilon_0 & \textit{if } |\varepsilon - \frac{\varepsilon_0}{b\mu}| < R_3 \quad \text{(II)} \\ \varepsilon + \varepsilon_0 & \textit{if } |\varepsilon + \frac{\varepsilon_0}{b\mu}| < R_3 \quad \text{(III)} \\ R_3 \frac{\varepsilon - \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2}\varepsilon_0}{\left|\varepsilon - \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2}\varepsilon_0\right|} + \frac{\varepsilon_0}{b} & \textit{if } R_3 \le |\varepsilon - \frac{\varepsilon_0}{b\mu}| \le R_4 \quad \text{(I, II)} \\ R_3 \frac{\varepsilon + \frac{\varepsilon_0:\varepsilon}{|\varepsilon_0|^2}\varepsilon_0}{\left|\varepsilon + \frac{\varepsilon_0:\varepsilon}{|\varepsilon_0|^2}\varepsilon_0\right|} - \frac{\varepsilon_0}{b} & \textit{if } R_3 \le |\varepsilon + \frac{\varepsilon_0}{b\mu}| \le R_4 \quad \text{(I, III)} \end{cases} \tag{2.17}$$

for $g_0 < 0$ *and* $\frac{|\varepsilon_0|^2}{b^2} + \frac{g_0}{b} > 0$.

Next we are going to state our results concerning the regularity properties of solutions to problem $\mathcal{QP}$. Partially they could be obtained at least in a qualitative sense by an adoption of the techniques developed in [3]. The approach presented here is based on duality theory which already has been applied to problems of phase transition in the papers [20]–[22] and which also turned out useful in the context of plasticity theory (see, e.g.[10]). In our opinion duality methods are quite effective since they give more precise regularity results, moreover, it is possible to formulate integral conditions whether a microstructure occurs at some point of the body or not. The dual variational problem $\mathcal{P}^*$ is introduced as follows:

Problem $\mathcal{P}^*$: Find a tensor $\sigma \in Q_f$ such that

$$R(\sigma) = \sup\{R(\tau) : \tau \in Q_f\}.$$

Here R denotes the functional

$$R(\tau) = \int_\Omega (\varepsilon(u_0) : \tau - f \cdot u_0 - g^*(\tau))dx$$

defined for tensors τ from the set

$$Q_f = \left\{ \tau \in L^2(\Omega; \mathring{\mathbb{S}}) : \int_\Omega (\tau : \varepsilon(v) - f \cdot v)dx = 0 \text{ for all } v \in \mathring{J}^1_2(\Omega) \right\}.$$

We recall (see [9]) that $\mathcal{P}^*$ has a unique solution σ; if u denotes a solution of $\mathcal{QP}$, then we have the duality relation

$$\sigma(x) = \frac{\partial g^{**}}{\partial \kappa}(\varepsilon(u)(x)) \text{ for almost all } x \in \Omega \tag{2.18}$$

as well as the equation

$$QI(u) = R(\sigma). \tag{2.19}$$

Theorem 2.2. *Suppose in addition to* (2.1) *that* $f \in W^1_{2,\mathrm{loc}}(\Omega;\mathbb{R}^2)$.

Then the solution σ *of problem* $\mathcal{P}^*$ *has weak derivatives in the space* L^2_{loc}, *i.e. we have*

$$\sigma \in W^1_{2,\mathrm{loc}}(\Omega;\overset{\circ}{\mathbb{S}}). \tag{2.20}$$

Remark 2.3. The statement of Theorem 2.2 holds in a quite more general setting which means that the proof just uses convexity of g^{**} together with the boundedness of the second derivatives, in particular, no condition like strict convexity is needed to carry out the proof.

As before we let σ denote the unique maximizer of the functional R and consider the set of all Lebesgue points of σ, i.e. the set

$$\Omega_0 = \left\{ x \in \Omega : \lim_{R\downarrow 0} (\sigma)_{x,R} \text{ exists} \right\}.$$

Here we use the symbols

$$(f)_{x,R} = \fint_{B_R(x)} f\ dy = \frac{1}{|B_R|} \int_{B_R(x)} f\ dy$$

to denote the mean value of a function f w.r.t. the disc $B_R(x)$ with radius R and center at $x \in \mathbb{R}^2$. For the particular case $x = 0$ we just write B_R and $(f)_R$ in place of $B_R(0)$ and $(f)_{0,R}$.

Next we let

$$A = \{1,2,3\}, \quad A(\tau) = \{i \in A : \quad g^*(\tau) = g^*_i(\tau)\},$$
$$a(\sigma) = \{x \in \Omega_0 : \quad \text{card}\, A(\sigma(x)) = 1\}.$$

The physical meaning of the set $a(\sigma)$ is that it can be seen as the union of single phases and that at the points of $a(\sigma)$ no microstructure occurs. Our main regularity result reads as follows:

Theorem 2.4. *Suppose that all the conditions of Theorem* 2.2 *hold. If in addition* f *belongs to the space* $L^\infty_{\mathrm{loc}} \cap W^1_{2,\mathrm{loc}}(\Omega;\mathbb{R}^2)$, *then the set* $a(\sigma)$ *is open and* σ *is Hölder continuous on* $a(\sigma)$ *for any exponent* $0 < \alpha < 1$. *Moreover, card* $A(\sigma(x)) > 1$ *for almost all* $x \in \Omega - a(\sigma)$.

3. Proof of Lemma 2.1

We are not going to prove Lemma 2.1 for all possible cases, instead of this we restrict ourselves to a representative situation, for example $g_0 < 0$ together with $\mu > 1$, i.e. we are going to prove relation (2.8).

In order to compute $\frac{\partial g^{**}}{\partial \varepsilon}(\varepsilon)$ we let

$$\overset{\circ}{\mathbb{S}} \ni \tau \to F_\varepsilon(\tau) = g^*(\tau) - \tau : \varepsilon$$

and observe the relation

$$\sigma = \frac{\partial g^{**}}{\partial \varepsilon}(\varepsilon) \quad \Longleftrightarrow \quad 0 \in \partial F_\varepsilon(\sigma), \tag{3.1}$$

where $\partial F_\varepsilon(\sigma)$ is the subdifferential of the function F_ε at $\sigma \in \overset{\circ}{\mathbb{S}}$.

As in Section 2 we let

$$A = \{1,2,3\}, \quad A(\tau) = \{i \in A : g_i^*(\tau) = g^*(\tau)\}, \quad \tau \in \overset{\circ}{\mathbb{S}}. \tag{3.2}$$

Then, for any $\tau \in \overset{\circ}{\mathbb{S}}$, we have

$$\partial F_\varepsilon(\tau) = \sum_{i \in A(\tau)} \lambda_i \partial g_i^*(\tau) - \varepsilon \tag{3.3}$$

with suitable numbers $\lambda_i \geq 0, \quad i \in A(\tau)$, satisfying

$$\sum_{i \in A(\tau)} \lambda_i = 1. \tag{3.4}$$

It is easy to see that

$$\partial g_1^*(\tau) = \left\{\frac{1}{2\mu}\tau\right\}, \; \partial g_2^*(\tau) = \left\{\frac{1}{2}\tau + \varepsilon_0\right\}, \; \partial g_3^*(\tau) = \left\{\frac{1}{2}\tau - \varepsilon_0\right\}. \tag{3.5}$$

Finally, we introduce the functions

$$\mathcal{F}_{ij}(\tau) = g_i^*(\tau) - g_j^*(\tau), \; i,j = 1,2,3, \; \tau \in \overset{\circ}{\mathbb{S}}, \tag{3.6}$$

and observe

$$\begin{aligned} \mathcal{F}_{12}(\tau) &= \frac{1}{4\mu}|\tau|^2 - g_0 - \frac{1}{4}|\tau|^2 - \tau : \varepsilon_0 = \\ &\quad \frac{a}{4}(4R_1^2 - |\tau + \frac{2\varepsilon_0}{a}|^2), \\ \mathcal{F}_{31}(\tau) &= \frac{a}{4}(|\tau - \frac{2\varepsilon_0}{a}|^2 - 4R_1^2), \\ \mathcal{F}_{23}(\tau) &= 2\tau : \varepsilon_0 \end{aligned} \tag{3.7}$$

for any $\tau \in \overset{\circ}{\mathbb{S}}$, the quantities a and R_1 being defined in (2.9).

The following cases can occur (observe $\mathcal{F}_{12} + \mathcal{F}_{23} + \mathcal{F}_{31} = 0$):

$$\mathcal{F}_{12}(\tau) \; > \; 0, \; \mathcal{F}_{31}(\tau) \; < \; 0, \tag{3.8}$$
$$\mathcal{F}_{23}(\tau) \; > \; 0, \; \mathcal{F}_{12}(\tau) \; < \; 0, \tag{3.9}$$
$$\mathcal{F}_{31}(\tau) \; > \; 0, \; \mathcal{F}_{23}(\tau) \; < \; 0, \tag{3.10}$$
$$\mathcal{F}_{23}(\tau) \; = \; 0, \; \mathcal{F}_{12}(\tau) \; < \; 0 \; (\Longrightarrow \mathcal{F}_{31}(\tau) \; > \; 0), \tag{3.11}$$
$$\mathcal{F}_{23}(\tau) \; > \; 0, \; \mathcal{F}_{12}(\tau) \; = \; 0, \tag{3.12}$$
$$\mathcal{F}_{23}(\tau) \; < \; 0, \; \mathcal{F}_{31}(\tau) \; = \; 0, \tag{3.13}$$
$$\mathcal{F}_{12}(\tau) \; = \; 0, \; \mathcal{F}_{31}(\tau) \; = \; 0 \; (\Longrightarrow \mathcal{F}_{23}(\tau) \; = \; 0). \tag{3.14}$$

Next we recall that $\sigma = \frac{\partial g^{**}}{\partial \varepsilon}(\varepsilon)$ and suppose at first that (3.8) is valid for the tensor σ. From (3.6) it follows that $A(\sigma) = \{1\}$ and therefore, by (3.3)–(3.5), we get $\sigma = 2\mu\varepsilon$, hence the first line in (2.8) is proved. The next cases (3.9) and (3.10) are treated in the same way.

Now let us consider (3.11), i.e. $\mathcal{F}_{23}(\sigma) = 0$ together with $\mathcal{F}_{12}(\sigma) < 0$.

Then $A(\sigma) = \{2,3\}$ and (recall (3.3), (3.4))

$$\sigma = \frac{\varepsilon - (\lambda_2 - \lambda_3)\varepsilon_0}{\frac{\lambda_2}{2} + \frac{\lambda_3}{2}}$$

with $\lambda_2, \lambda_3 \geq 0,\ \lambda_2 + \lambda_3 = 1$. In order to calculate λ_2, λ_3 we observe that $\mathcal{F}_{23}(\sigma) = 0$ implies

$$\lambda_2 - \lambda_3 = \frac{\varepsilon : \varepsilon_o}{|\varepsilon_0|^2},$$

hence

$$\begin{aligned} \sigma &= 2(\varepsilon - \frac{\varepsilon_0 : \varepsilon}{|\varepsilon_0|^2}\varepsilon_0), \\ \lambda_2 &= \frac{1}{2}(1 + \frac{\varepsilon : \varepsilon_0}{|\varepsilon_0|^2}), \\ \lambda_3 &= \frac{1}{2}(1 - \frac{\varepsilon : \varepsilon_0}{|\varepsilon_0|^2}). \end{aligned} \tag{3.15}$$

Since $\lambda_2, \lambda_3 \geq 0$, we see that ε must satisfy the restriction

$$\frac{|\varepsilon : \varepsilon_0|}{|\varepsilon|^2} \leq 1. \tag{3.16}$$

Recalling $\mathcal{F}_{12}(\sigma) < 0$ and inserting (3.15), we get

$$\mathcal{F}_{12}\left(2\left(\varepsilon - \frac{\varepsilon : \varepsilon_0}{|\varepsilon_0|^2}\varepsilon_0\right)\right) = -a\left[\frac{g_0}{a} + \left|\varepsilon - \frac{\varepsilon : \varepsilon_0}{|\varepsilon_0|^2}\varepsilon_0\right|^2\right] < 0,$$

and this together with (3.15), (3.16) implies the fourth line in (2.8). Suppose next that σ satisfies (3.12). Then $A(\sigma) = \{1,2\}$, hence

$$\sigma = \frac{\varepsilon - \lambda_2\varepsilon_0}{\frac{\lambda_1}{2\mu} + \frac{\lambda_2}{2}},\ \lambda_1, \lambda_2 \geq 0,\ \lambda_1 + \lambda_2 = 1.$$

Rewriting $\mathcal{F}_{12}(\sigma) = 0$ as

$$\begin{aligned}\mathcal{F}_{12}\left(\frac{\varepsilon - \lambda_2\varepsilon_0}{\frac{\lambda_1}{2\mu} + \frac{\lambda_2}{2}}\right) &= \mathcal{F}_{12}\left(2\frac{\varepsilon - \lambda_2\varepsilon_0}{1 - a + a\lambda_2}\right) \\ &= a\left[R_1^2 - \left|\frac{\varepsilon - \lambda_2\varepsilon_0}{1 - a + a\lambda_2} + \frac{\varepsilon_0}{a}\right|^2\right] \\ &= a\left[R_1^2 - \frac{|\varepsilon + \frac{\varepsilon_0}{a\mu}|^2}{(1 - a + a\lambda_2)^2}\right] \\ &= 0\end{aligned}$$

we obtain

$$\begin{cases}\lambda_2 &= \frac{1}{a}(-\frac{1}{\mu} + \frac{1}{R_1}|\varepsilon + \frac{\varepsilon_0}{a\mu}|) \\ \sigma &= 2\left[R_1\frac{\varepsilon + \frac{\varepsilon_0}{a\mu}}{|\varepsilon + \frac{\varepsilon_0}{a\mu}|} - \frac{\varepsilon_0}{a}\right],\end{cases} \tag{3.17}$$

and $0 \le \lambda_2 \le 1$ implies the inequality

$$R_2 \le \left|\varepsilon + \frac{\varepsilon_0}{a\mu}\right| \le R_1. \tag{3.18}$$

In addition we know

$$\mathcal{F}_{23}(\sigma) > 0 \Longleftrightarrow \lambda_2 = \frac{1}{a}\left(-\frac{1}{\mu} + \frac{1}{R_1}\left|\varepsilon + \frac{\varepsilon_0}{a\mu}\right|\right) < \frac{\varepsilon : \varepsilon_0}{|\varepsilon_0|^2},$$

hence

$$1 - a + a\frac{\varepsilon : \varepsilon_0}{|\varepsilon_0|^2} > \sqrt{-\frac{a}{g_0}}\left|\varepsilon - \frac{\varepsilon : \varepsilon_0}{|\varepsilon_0|^2}\varepsilon_0\right|,$$

and this together with (3.17) and (3.18) completes the fifth line in formula (2.8). Case (3.13) is treated in the same way, and it remains to discuss (3.14). Now $A(\sigma) = \{1, 2, 3\}$ and

$$\sigma = \frac{\varepsilon - (\lambda_2 - \lambda_3)\varepsilon_0}{\frac{\lambda_1}{2\mu} + \frac{\lambda_2}{2} + \frac{\lambda_3}{2}},\ \lambda_1, \lambda_2, \lambda_3 \ge 0,\ \lambda_1 + \lambda_2 + \lambda_3 = 1.$$

$\mathcal{F}_{23}(\sigma) = 0$ gives

$$\lambda_2 - \lambda_3 = \frac{\varepsilon : \varepsilon_0}{|\varepsilon_0|^2}, \sigma = 2\frac{\varepsilon - \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2}\varepsilon_0}{1 - a + a(\lambda_2 + \lambda_3)}.$$

From $\mathcal{F}_{12}(\sigma) = \mathcal{F}_{23}(\sigma) = 0$ we deduce $|\sigma|^2 = -\frac{g_0}{a}$, so that

$$\lambda_2 + \lambda_3 = \frac{1}{a}\left(\sqrt{-\frac{a}{g_0}}\left|\varepsilon - \frac{\varepsilon_0 : \varepsilon}{|\varepsilon_0|^2}\varepsilon_0\right| - \frac{1}{\mu}\right).$$

This implies

$$2\lambda_2 = \frac{1}{a}\left(\sqrt{-\frac{a}{g_0}}\left|\varepsilon - \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2}\varepsilon_0\right| - \frac{1}{\mu}\right) + \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2},$$
$$2\lambda_3 = \frac{1}{a}\left(\sqrt{-\frac{a}{g_0}}\left|\varepsilon - \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2}\varepsilon_0\right| - \frac{1}{\mu}\right) - \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2},$$
$$\sigma = 2\sqrt{-\frac{g_0}{a}}\frac{\varepsilon - \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2}\varepsilon_0}{\left|\varepsilon - \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2}\varepsilon_0\right|}.$$

Taking into account the restrictions $\lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_2 + \lambda_3 \leq 1$, the last line of (2.8) is established which completes the proof of Lemma 2.1. □

4. Proof of Theorem 2.2

From Lemma 2.1 we deduce that $\frac{\partial g^{**}}{\partial \varepsilon}$ is Lipschitz continuous on $\overset{\circ}{\mathbb{S}}$, hence there exists $c_1 \geq 0$ such that

$$\left|D^2 g^{**}(\varepsilon)\right| \ \leq \ c_1 \tag{4.1}$$

for all $\varepsilon \in \overset{\circ}{\mathbb{S}}$. Quoting [15] we find a pressure function $p \in L^2(\Omega)$ with the property

$$\int\limits_\Omega \sigma : \varepsilon(v)\, dx = \int\limits_\Omega p \ \operatorname{div} v\, dx + \int\limits_\Omega f \cdot v\, dx \tag{4.2}$$

being valid for all $v \in \overset{\circ}{W}{}^1_2(\Omega;\mathbb{R}^2)$. Consider a disc $B_R(x_0)$ with compact closure in Ω and choose $\varphi \in C_0^\infty(\Omega)$ according to

$$\begin{aligned} \varphi = 0 &\quad \text{outside of } B_r(x_0),\\ \varphi = 1 &\qquad \text{on} \quad B_q(x_0),\\ |\nabla\varphi| \leq \frac{c_2}{r-q}&, 0 \leq \varphi \leq 1 \text{ in } \Omega. \end{aligned} \tag{4.3}$$

Here $q < r$ denote arbitrary numbers in the interval $\left[\frac{R}{2}, R\right]$. For $h \in \mathbb{R}^2$ with sufficiently small norm we have $\pm h + B_R(x_0) \subset \Omega$. For functions w let us define $\Delta_h w(x) = w(x+h) - w(x)$. Then we obtain from (4.2)

$$\begin{aligned} \int\limits_{B_R(x_0)} \Delta_h\sigma : \varepsilon(\varphi^2\Delta_h\overline{u})\, dx = &\int\limits_{B_R(x_0)} \Delta_h f \cdot \varphi^2 \Delta_h \overline{u}\, dx \\ &+ \int\limits_{B_R(x_0)} \widetilde{\Delta_h p}\ \operatorname{div}(\varphi^2\Delta_h\overline{u})\, dx, \end{aligned} \tag{4.4}$$

where

$$\overline{u}(x) = u(x) - A(x-x_0) - u_0,\ A \in \overset{\circ}{\mathbb{M}},\ u_0 \in \mathbb{R}^2,$$
$$\widetilde{\Delta_h p} = \Delta_h p - (\Delta_h p)_{x_0,r}.$$

Next we introduce the parameter dependent bilinear form

$$L(x)=\int_0^1 D^2 g^{**}\left(\varepsilon(u)(x)+\Theta\varepsilon(\Delta_h u)(x)\right)d\Theta,\ x\in\Omega.$$

We have

$$(L(x)\varkappa):\varkappa\geq 0, \varkappa\in\overset{\circ}{\mathbb{S}}, x\in\Omega, |L(x)|\ \leq\ c_1. \tag{4.5}$$

Note that $L(x)$ is defined only at almost all points x of Ω. Observing $\Delta_h\sigma = L\,\varepsilon(\Delta_h u)$ we find that

$$\begin{aligned}
&|\Delta_h\sigma|^2 = L\,\varepsilon(\Delta_h u):\Delta_h\sigma\leq\\
&(\varepsilon(\Delta_h u):L\ \varepsilon(\Delta_h u))^{\frac12}\,(\Delta_h\sigma:L\Delta_h\sigma)^{\frac12}\leq\\
&(\varepsilon(\Delta_h u):\Delta_h\sigma)^{\frac12}\ \sqrt{c_1}\,|\Delta_h\sigma|\,,\ \text{i.e.}\\
&|\Delta_h\sigma|^2\leq c_1\Delta_h\sigma:\varepsilon(\Delta_h u).
\end{aligned}$$

Using this estimate in equation (4.4) and recalling (4.3) we get

$$\begin{aligned}
\int\limits_{B_R(x_0)}\varphi^2|\Delta_h\sigma|^2\,dx &\leq c_3\left\{\frac{1}{(r-q)^2}\int\limits_{B_R(x_0)}|\Delta_h\overline{u}|^2\,dx\right.\\
&\left.+\ (r-q)^2\int\limits_{B_R(x_0)}|\Delta_h f|^2\,dx+\int\limits_{B_R(x_0)}\widetilde{\Delta_h p}\nabla\varphi^2\cdot\Delta_h\overline{u}\,dx\right\}\leq\\
&c_4\left\{\frac{1}{(r-q)^2}\int\limits_{B_R(x_0)}|\Delta_h\overline{u}|^2dx+\frac{1}{r-q}\left(\int\limits_{B_r(x_0)}|\widetilde{\Delta_h p}|^2\,dx\right)^{\frac12}\right.\\
&\left.\cdot\quad\left(\int\limits_{B_R(x_0)}|\Delta_h\overline{u}|^2\,dx\right)^{\frac12}+R^2\int\limits_{B_R(x_0)}|\Delta_h f|^2dx\right\}.
\end{aligned}\tag{4.6}$$

By results of [15] there exists $w\in\overset{\circ}{W}{}^1_2(B_r(x_0);\mathbb{R}^2)$ such that

$$\begin{aligned}
&\operatorname{div} w=\widetilde{\Delta_h p}\text{ in }B_r(x_0),\\
&\|\nabla w\|_{L^2(B_r(x_0))}\ \leq\ c_5\|\widetilde{\Delta_h p}\|_{L^2(B_r(x_0))},
\end{aligned}\tag{4.7}$$

the constant c_5 being independent of x_0 and r. (4.2) together with (4.7) implies

$$\int\limits_{B_r(x_0)}\Delta_h\sigma:\varepsilon(w)\,dx=\int\limits_{B_r(x_0)}\Delta_h f\cdot w\,dx+\int\limits_{B_r(x_0)}|\widetilde{\Delta_h p}|^2\,dx$$

so that

$$\int_{B_r(x_0)} |\widetilde{\Delta_h p}|^2\, dx \le c_6 \left[\int_{B_r(x_0)} |\Delta_h \sigma|^2\, dx + R^2 \int_{B_R(x_0)} |\Delta_h f|^2 dx \right]. \tag{4.8}$$

Inserting (4.8) into (4.6) and using Young's inequality we get

$$\int_{B_q(x_0)} |\Delta_h \sigma|^2\, dx \le \frac{1}{2} \int_{B_r(x_0)} |\Delta_h \sigma|^2\, dx + c_7 \left[\frac{1}{(r-q)^2} \int_{B_R(x_0)} |\Delta_h \overline{u}|^2\, dx + R^2 \int_{B_R(x_0)} |\Delta_h f|^2\, dx \right]$$

for all $q < r$ in $\left[\frac{R}{2}, R\right]$. As in [11] this implies

$$\int_{B_{\frac{R}{2}}(x_0)} |\Delta_h \sigma|^2\, dx \le c_8 \left[\frac{1}{R^2} \int_{B_R(x_0)} |\Delta_h \overline{u}|^2\, dx + R^2 \int_{B_R(x_0)} |\Delta_h f|^2\, dx \right]$$

for any $B_R(x_0) \subset \Omega$. If we replace h by λe for some unit vector $e \in \mathbb{R}^2$ and divide the above inequality by $|\lambda|$, we see that the right-hand side has a limit as $\lambda \to 0$ which can be bounded by

$$c_8 \left[\frac{1}{R^2} \int_{B_R(x_0)} |\varepsilon(u) - \varkappa|^2\, dx + R^2 \int_{B_R(x_0)} |\nabla f|^2\, dx \right]$$

where $\varkappa$ is some matrix in $\overset{\circ}{\mathbb{S}}$. (To prove this choose A and u_0 in an appropriate way and use Korn's inequality.) Putting together our results we have shown that σ has weak derivatives in $L^2_{\mathrm{loc}}(\Omega)$, moreover, we have the estimate

$$\int_{B_{\frac{R}{2}}(x_0)} |\nabla \sigma|^2\, dx \le c_8 \left[\frac{1}{R^2} \int_{B_R(x_0)} |\varepsilon(u) - \varkappa|^2\, dx + R^2 \int_{B_R(x_0)} |\nabla f|^2\, dx \right] \tag{4.9}$$

being valid for any disc $B_R(x_0)$ with compact closure in Ω and for any matrix $\varkappa \in \overset{\circ}{\mathbb{S}}$. This completes the proof of Theorem 2.2. □

5. Local estimates of Caccioppoli-type

We are going to consider the unique solution σ of the dual problem $\mathcal{P}^*$ and introduce the following quantities related to this tensor:

$$\begin{aligned}
\Delta_x^{12} &= 2R_1 - \left|\sigma(x) + \frac{2\varepsilon_0}{a}\right|, \\
\Delta_{x,\rho}^{12} &= 2R_1 - \left|(\sigma)_{x,\rho} + \frac{2\varepsilon_0}{a}\right|, \\
\Delta_x^{23} &= \sigma(x) : \frac{\varepsilon_0}{|\varepsilon_0|}, \Delta_{x,\rho}^{23} = (\sigma)_{x,\rho} : \frac{\varepsilon_0}{|\varepsilon_0|}, \\
\Delta_x^{31} &= \left|\sigma(x) - \frac{2\varepsilon_0}{a}\right| - 2R_1, \\
\Delta_{x,\rho}^{31} &= \left|(\sigma)_{x,\rho} - \frac{2\varepsilon_0}{a}\right| - 2R_1.
\end{aligned} \tag{5.1}$$

Here x is taken from the Lebesgue set Ω_0 of σ and $\rho > 0$ is a radius such that $B_\rho(x) \subset \Omega$. We further recall the definition of the functions $\mathcal{F}_{ij} : \mathbb{S} \to \mathbb{R}$ (see (3.6) and (3.7)) and observe

$$\begin{aligned}
\mathcal{F}_{ij}\left(\sigma(x)\right) \quad &> (<) 0 \iff \Delta_x^{ij} \quad > (<)\, 0, \\
\mathcal{F}_{ij}\left((\sigma)_{x,\rho}\right) \quad &> (<) 0 \iff \Delta_{x,\rho}^{ij} \quad > (<)\, 0
\end{aligned} \tag{5.2}$$

for $\quad i = 1,\ j = 2,\ i = 2,\ j = 3$ and $i = 3,\ j = 1$.

Lemma 5.1. *Suppose that all the hypotheses of Theorem* 2.4 *are valid. Then we have the following statements:*

(i) *If* $\quad \Delta_{x_0,R}^{12} > 0 \quad$ *and* $\quad \Delta_{x_0,R}^{31} < 0, \quad$ *then*

$$\int\limits_{B_{\frac{R}{2}}(x_0)} |\nabla\sigma|^2\, dx \le c_1 \Big[\left(1 + |\Delta_{x_0,R}^{12}|^{-2} + |\Delta_{x_0,R}^{31}|^{-2}\right) \cdot \cdot \frac{1}{R^2} \int\limits_{B_R(x_0)} |\sigma - (\sigma)_{x_0,R}|^2 dx + R^2 \int\limits_{B_R(x_0)} |\nabla f|^2 dx \Big]. \tag{5.3}$$

(ii) *If* $\quad \Delta_{x_0,R}^{23} > 0 \quad$ *and* $\quad \Delta_{x_0,R}^{12} < 0, \quad$ *then*

$$\int\limits_{B_{\frac{R}{2}}(x_0)} |\nabla\sigma|^2\, dx \le c_2 \Big[\left(1 + |\Delta_{x_0,R}^{23}|^{-2} + |\Delta_{x_0,R}^{12}|^{-2}\right) \cdot \cdot \frac{1}{R^2} \int\limits_{B_R(x_0)} |\sigma - (\sigma)_{x_0,R}|^2 dx + R^2 \int\limits_{B_R(x_0)} |\nabla f|^2 dx \Big]. \tag{5.4}$$

(iii) *If* $\Delta^{31}_{x_0,R} > 0$ *and* $\Delta^{23}_{x_0,R} < 0$, *then*

$$\int\limits_{B_{\frac{R}{2}}(x_0)} |\nabla\sigma|^2\, dx \le c_3\Big[\big(1 + |\Delta^{31}_{x_0,R}|^{-2} + |\Delta^{23}_{x_0,R}|^{-2}\big)\cdot \cdot \frac{1}{R^2}\int\limits_{B_R(x_0)} |\sigma - (\sigma)_{x_0,R}|^2 dx + R^2 \int\limits_{B_R(x_0)} |\nabla f|^2 dx\Big]. \tag{5.5}$$

The estimates are valid for any $x_0 \in \Omega_0$ and any $R > 0$ s.t. $B_R(x_0) \subset\subset \Omega$. Finally, the positive constants c_1, c_2, c_3 do not depend on x_0 and R.

Proof. Our arguments closely follow the lines of the papers [20]–[22].

We let

$$\begin{aligned}
\omega^{ij}(x_0,R) &= \left\{x \in B_R(x_0) : \Delta^{ij}_x \le 0\right\},\\
\Omega^{ij}(x_0,R) &= \left\{x \in B_R(x_0) : \Delta^{ij}_x \ge 0\right\},\\
\omega^{ij}_-(x_0,R) &= \left\{x \in B_R(x_0) : \Delta^{ij}_x < 0\right\},\\
\omega^{ij}_0(x_0,R) &= \left\{x \in B_R(x_0) : \Delta^{ij}_x = 0\right\},\\
\omega^{ij}_+(x_0,R) &= \left\{x \in B_R(x_0) : \Delta^{ij}_x > 0\right\}
\end{aligned} \tag{5.6}$$

for $i = 1$ and $j = 2$, $i = 2$ and $j = 3$, $i = 3$ and $j = 1$ and recall formula (4.9) in which we have to estimate the term $\int\limits_{B_R(x_0)} |\varepsilon(u) - \varkappa|^2\, dx$. So let us first consider case (i) of Lemma 5.1. In order to prove (5.3) we define

$$\varkappa = \frac{1}{2\mu}(\sigma)_{x_0,R} \tag{5.7}$$

and observe that (5.1) together with (5.2) implies the bound

$$|(\sigma)_{x_0,R}| \le 2\left(R_1 + \frac{|\varepsilon_0|}{a}\right) \tag{5.8}$$

Next we use the decomposition

$$\int\limits_{B_R(x_0)} |\varepsilon(u) - \varkappa|^2\, dx = \frac{1}{(2\mu)^2} \int\limits_{\omega^{12}_+(x_0,R)\cap\omega^{31}_-(x_0,R)} |\sigma - (\sigma)_{x_0,R}|^2\, dx + \int\limits_{\omega^{12}(x_0,R)\cup\Omega^{31}(x_0,R)} \left|\varepsilon(u) - \frac{1}{2\mu}(\sigma)_{x_0,R}\right|^2 dx. \tag{5.9}$$

The second integral on the right-hand side of (5.9) can be splitted in the following way (recall that $\mathcal{F}_{12} + \mathcal{F}_{23} + \mathcal{F}_{31} = 0$):

$$I = \int\limits_{\omega^{12} \cup \Omega^{31}} \left| \varepsilon(u) - \frac{1}{2\mu}(\sigma)_{x_0,R} \right|^2 dx = I_2 + I_3 + \ldots + I_7, \tag{5.10}$$

where

$$I_2 = \int\limits_{\omega_-^{12} \cap \omega_+^{23}} \left| \varepsilon(u) - \frac{1}{2\mu}(\sigma)_{x_0,R} \right|^2 dx,$$

$$I_3 = \int\limits_{\omega_+^{31} \cap \omega_-^{23}} \left| \varepsilon(u) - \frac{1}{2\mu}(\sigma)_{x_0,R} \right|^2 dx,$$

$$I_4 = \int\limits_{\left(\omega_-^{12} \cup \omega_+^{31}\right) \cap \omega_0^{23}} \left| \varepsilon(u) - \frac{1}{2\mu}(\sigma)_{x_0,R} \right|^2 dx,$$

$$I_5 = \int\limits_{\omega_0^{12} \cap \omega_+^{23}} \left| \varepsilon(u) - \frac{1}{2\mu}(\sigma)_{x_0,R} \right|^2 dx,$$

$$I_6 = \int\limits_{\omega_0^{31} \cap \omega_-^{23}} \left| \varepsilon(u) - \frac{1}{2\mu}(\sigma)_{x_0,R} \right|^2 dx,$$

$$I_7 = \int\limits_{\omega_0^{31} \cap \omega_0^{12} \cap \omega_0^{23}} \left| \varepsilon(u) - \frac{1}{2\mu}(\sigma)_{x_0,R} \right|^2 dx\,.$$

Here and in what follows we will not explicitly indicate x_0 and R in the domain of definition. The numeration of the integrals $I_2, \cdots, I_7$ is in correspondence with the lines of formula (2.8)where the relation between σ and $\varepsilon(u)$ for the various cases is given. Using this together with (5.8) we find

$$\begin{aligned} I_2 &= \int\limits_{\omega_-^{12} \cap \omega_+^{23}} \left| \frac{1}{2}\sigma + \varepsilon_0 - \frac{1}{2\mu}(\sigma)_{x_0,R} \right|^2 dx \\ &\leq 2 \left[\frac{1}{4} \int\limits_{B_R(x_0)} |\sigma - (\sigma)_{x_0,R}|^2 \, dx + \left(|\varepsilon_0|^2 + \frac{1}{2} a \left| (\sigma)_{x_0,R} \right|^2 \right) \left| \omega_-^{12} \cap \omega_+^{23} \right| \right] \\ &\leq \frac{1}{2} \int\limits_{B_R(x_0)} |\sigma - (\sigma)_{x_0,R}|^2 \, dx + c_4 \left(\left|\omega^{12}\right| + \left|\Omega^{31}\right| \right), \end{aligned}$$

$$I_3 \leq \frac{1}{2} \int\limits_{B_R(x_0)} |\sigma - (\sigma)_{x_0,R}|^2 \, dx + c_5 \left(|\omega^{12}| + |\Omega^{31}|\right),$$

$$\begin{aligned} I_4 &= \int\limits_{(\omega_-^{12} \cup \omega_+^{31}) \cap \omega_0^{23}} \left| \frac{1}{2}\sigma + \frac{\varepsilon : \varepsilon_0}{|\varepsilon_0|^2}\varepsilon_0 - \frac{1}{2\mu}(\sigma)_{x_0,R} \right|^2 dx \\ &\leq \frac{1}{2} \int\limits_{B_R(x_0)} |\sigma - (\sigma)_{x_0,R}|^2 \, dx + c_6 \left(|\omega^{12}| + |\Omega^{31}|\right), \end{aligned}$$

$$\begin{aligned} I_5 &\leq 2 \int\limits_{\omega_0^{12} \cap \omega_+^{23}} \left[|\varepsilon(u)|^2 + \frac{1}{2\mu} |(\sigma)_{x_0,R}|^2 \right] dx \\ &\leq 2 \left[\left(R_1 + \frac{|\varepsilon_0|}{a\mu} \right)^2 + \frac{1}{4\mu^2} 4 \left(R_1 + \frac{|\varepsilon_0|}{a} \right)^2 \right] |\omega_0^{12} \cap \omega_+^{23}| \\ &\leq c_7 \left(|\omega^{12}| + |\Omega^{31}|\right), \end{aligned}$$

$$I_6 \leq c_8 \left(|\omega^{12}| + |\Omega^{31}|\right),$$

$$\begin{aligned} I_7 &\leq 2 \int\limits_{\omega_0^{12} \cap \omega_0^{23} \cap \omega_0^{31}} \left[|\varepsilon(u)|^2 + \frac{1}{4\mu^2} |(\sigma)_{x_0,R}|^2 \right] dx \\ &\leq 2 \left[\left(-\frac{g_0}{a} + |\varepsilon_0|^2 \right) + \frac{1}{\mu^2} \left(R_1 + \frac{|\varepsilon_0|}{a} \right)^2 \right] \left(|\omega^{12}| + |\Omega^{31}|\right) \end{aligned}$$

and therefore

$$I \leq c_9 \left[\int\limits_{B_R(x_0)} |\sigma - (\sigma)_{x_0,R}|^2 \, dx + |\omega^{12}| + |\Omega^{31}| \right]. \tag{5.11}$$

It remains to discuss the measures of the sets $\omega^{12}(x_0, R)$ and $\Omega^{31}(x_0, R)$.

We have

$$
\begin{aligned}
\left|\omega^{12}(x_0,R)\right| &= \left|\Delta^{12}_{x_0,R}\right|^{-1} \int\limits_{\omega^{12}(x_0,R)} \Delta^{12}_{x_0,R}\,dx \\
&\le \left|\Delta^{12}_{x_0,R}\right|^{-1} \int\limits_{\omega^{12}(x_0,R)} \left(\Delta^{12}_{x_0,R} - \Delta^{12}_{x}\right)\,dx \\
&\le \left|\Delta^{12}_{x_0,R}\right|^{-1} \int\limits_{\omega^{12}(x_0,R)} \left|\sigma - (\sigma)_{x_0,R}\right|\,dx \\
&\le \frac{\left|\omega^{12}(x_0,R)\right|^{\frac{1}{2}}}{\left|\Delta^{12}_{x_0,R}\right|} \left(\int\limits_{B_R(x_0)} \left|\sigma - (\sigma)_{x_0,R}\right|^2\,dx \right)^{\frac{1}{2}}, \\
\left|\Omega^{31}(x_0,R)\right| &= \left|\Delta^{31}_{x_0,R}\right|^{-1} \int\limits_{\Omega^{31}(x_0,R)} \left(-\Delta^{31}_{x_0,R}\right)\,dx \\
&\le \left|\Delta^{31}_{x_0,R}\right|^{-1} \int\limits_{\Omega^{31}(x_0,R)} \left(\Delta^{31}_{x} - \Delta^{31}_{x_0,R}\right)\,dx \\
&\le \left|\Delta^{31}_{x_0,R}\right|^{-1} \int\limits_{\Omega^{31}(x_0,R)} \left|\sigma - (\sigma)_{x_0,R}\right|\,dx \\
&\le \frac{\left|\Omega^{31}(x_0,R)\right|^{\frac{1}{2}}}{\left|\Delta^{31}_{x_0,R}\right|} \left(\int\limits_{B_R(x_0)} \left|\sigma - (\sigma)_{x_0,R}\right|^2\,dx \right)^{\frac{1}{2}}.
\end{aligned}
$$

From this together with (4.9), (5.9) and (5.11) part (i) of Lemma 5.1 will follow.

Now we are going to prove inequality (5.4). In this case we let

$$
\varkappa = \frac{1}{2}(\sigma)_{x_0,R} + \varepsilon_0
$$

and obtain

$$
\begin{aligned}
\int\limits_{B_R(x_0)} \left|\varepsilon(u) - \varkappa\right|^2\,dx = \frac{1}{(2\mu)^2} &\int\limits_{\omega^{23}_+ \cap \omega^{12}_-} \left|\sigma - (\sigma)_{x_0,R}\right|^2\,dx \\
&+ \int\limits_{\omega^{23} \cup \Omega^{12}} \left|\varepsilon(u) - \frac{1}{2}(\sigma)_{x_0,R} - \varepsilon_0\right|^2\,dx.
\end{aligned} \tag{5.12}
$$

As in the previous case we split

$$
\begin{aligned}
I' &= \int\limits_{\omega^{23} \cup \Omega^{12}} \left|\varepsilon(u) - \frac{1}{2}(\sigma)_{x_0,R} - \varepsilon_0\right|^2\,dx \\
&= I'_1 + I'_3 + I'_4 + I'_5 + I'_6 + I'_7,
\end{aligned} \tag{5.13}
$$

where

$$I_1' = \int\limits_{\omega_+^{12}\cap\omega_-^{31}} \left|\varepsilon(u) - \frac{1}{2}(\sigma)_{x_0,R} - \varepsilon_0\right|^2\, dx,$$

$$I_3' = \int\limits_{\omega_+^{31}\cap\omega_-^{23}} \left|\varepsilon(u) - \frac{1}{2}(\sigma)_{x_0,R} - \varepsilon_0\right|^2\, dx,$$

$$I_4' = \int\limits_{\left(\omega_-^{12}\cup\omega_+^{31}\right)\cap\omega_0^{23}} \left|\varepsilon(u) - \frac{1}{2}(\sigma)_{x_0,R} - \varepsilon_0\right|^2\, dx,$$

$$I_5' = \int\limits_{\omega_0^{12}\cap\omega_+^{23}} \left|\varepsilon(u) - \frac{1}{2}(\sigma)_{x_0,R} - \varepsilon_0\right|^2\, dx,$$

$$I_6' = \int\limits_{\omega_0^{31}\cap\omega_-^{23}} \left|\varepsilon(u) - \frac{1}{2}(\sigma)_{x_0,R} - \varepsilon_0\right|^2\, dx,$$

$$I_7' = \int\limits_{\omega_0^{12}\cap\omega_0^{23}\cap\omega_0^{31}} \left|\varepsilon(u) - \frac{1}{2}(\sigma)_{x_0,R} - \varepsilon_0\right|^2\, dx,$$

and the numeration of the integrals corresponds to the numeration of the lines in (2.8). Proceeding as before we find

$$\begin{aligned} I_1' &= \int\limits_{\omega_+^{12}\cap\omega_-^{31}} \left|\frac{1}{2\mu}\sigma - \frac{1}{2}(\sigma)_{x_0,R} - \varepsilon_0\right|^2\, dx \\ &\le 2\left[\frac{1}{4}\int\limits_{B_R(x_0)} \left|\sigma - (\sigma)_{x_0,R}\right|^2\, dx + \int\limits_{\omega_+^{12}\cap\omega_-^{31}} \left|\frac{1}{2}a\sigma - \varepsilon_0\right|^2\, dx\right]. \end{aligned}$$

On $\omega_+^{12}\cap\omega_-^{31}$ we have $|\sigma| \le 2R_1 + 2\frac{|\varepsilon_0|}{a}$, hence

$$I_1' \le \frac{1}{2}\int\limits_{B_R(x_0)} \left|\sigma - (\sigma)_{x_0,R}\right|^2\, dx + c_{10}\left(\left|\omega^{23}\right| + \left|\Omega^{12}\right|\right).$$

Further we have

$$
\begin{aligned}
I_3' &= \int\limits_{\omega_+^{31}\cap\omega_-^{23}} \left|\frac{1}{2}\sigma - \varepsilon_0 - \frac{1}{2}(\sigma)_{x_0,R} - \varepsilon_0\right|^2 dx \\
&\le \frac{1}{2}\int\limits_{B_R(x_0)} \left|\sigma - (\sigma)_{x_0,R}\right|^2 dx + 8|\varepsilon_0|^2\left(\left|\omega^{23}\right| + \left|\Omega^{12}\right|\right),
\end{aligned}
$$

$$
\begin{aligned}
I_4' &= \int\limits_{\left(\omega_-^{12}\cup\omega_+^{31}\right)\cap\omega_0^{23}} \left|\frac{1}{2}\sigma + \frac{\varepsilon:\varepsilon_0}{|\varepsilon_0|^2}\varepsilon_0 - \frac{1}{2}(\sigma)_{x_0,R} - \varepsilon_0\right|^2 dx \\
&\le \frac{1}{2}\int\limits_{B_R(x_0)} \left|\sigma - (\sigma)_{x_0,R}\right|^2 dx + 8|\varepsilon_0|^2\left(\left|\omega^{23}\right| + \left|\Omega^{12}\right|\right),
\end{aligned}
$$

$$
\begin{aligned}
I_5' &= \int\limits_{\omega_0^{12}\cap\omega_+^{23}} \left|\varepsilon(u) - \frac{1}{2}\sigma + \frac{1}{2}\left(\sigma - (\sigma)_{x_0,R}\right) - \varepsilon_0\right|^2 dx \\
&\le \frac{1}{2}\int\limits_{B_R(x_0)} \left|\sigma - (\sigma)_{x_0,R}\right|^2 dx + 2\int\limits_{\omega_0^{12}\cap\omega_-^{23}} \left(|\varepsilon(u)|^2 + |\sigma|^2 + |\varepsilon_0|^2\right) dx \\
&\le \frac{1}{2}\int\limits_{B_R(x_0)} \left|\sigma - (\sigma)_{x_0,R}\right|^2 dx + c_{11}\left(\left|\omega^{23}\right| + \left|\Omega^{12}\right|\right),
\end{aligned}
$$

$$
I_6' \le \frac{1}{2}\int\limits_{B_R(x_0)} \left|\sigma - (\sigma)_{x_0,R}\right|^2 dx + c_{12}\left(\left|\omega^{23}\right| + \left|\Omega^{12}\right|\right),
$$

$$
\begin{aligned}
I_7' &= \int\limits_{\left(\omega_0^{12}\cap\omega_0^{23}\right)\cap\omega_0^{31}} \left|\varepsilon(u) - \frac{1}{2}\sigma + \frac{1}{2}\left(\sigma - (\sigma)_{x_0,R}\right) - \varepsilon_0\right|^2 dx, \\
&\le \frac{1}{2}\int\limits_{B_R(x_0)} \left|\sigma - (\sigma)_{x_0,R}\right|^2 dx + c_{13}\left(\left|\omega^{23}\right| + \left|\Omega^{12}\right|\right),
\end{aligned}
$$

and we arrive at (recall (5.13))

$$
I' \le c_{14}\left[\int\limits_{B_R(x_0)} \left|\sigma - (\sigma)_{x_0,R}\right|^2 dx + \left|\omega^{23}\right| + \left|\Omega^{12}\right|\right]. \tag{5.14}
$$

It is easy to show that the estimates

$$\left|\omega^{23}(x_0,R)\right| \le \left|\Delta^{23}_{x_0,R}\right|^{-2} \int\limits_{B_R(x_0)} \left|\sigma-(\sigma)_{x_0,R}\right|^2\,dx,$$

$$\left|\Omega^{12}(x_0,R)\right| \le \left|\Delta^{12}_{x_0,R}\right|^{-2} \int\limits_{B_R(x_0)} \left|\sigma-(\sigma)_{x_0,R}\right|^2\,dx$$

hold true, and (5.4) is a consequence of (4.9) and (5.12) - (5.14).

The proof of Lemma 5.1 (iii) is very close to the previous case, the necessary adjustments are left to the reader. Altogether the claim of Lemma 5.1 is established with constants c_1, c_2 and c_3 not depending on x_0 and R. □

6. A decay estimate

We continue our discussion of the properties of the maximizer σ. As usual the behaviour of σ is described in terms of the squared mean oscillation which is given by

$$\Psi(x_0,R) = \left(\;\fint\limits_{B_R(x_0)} \left|\sigma-(\sigma)_{x_0,R}\right|^2\,dx\right)^{\frac{1}{2}}.$$

Recalling (5.1) we further let

$$\Gamma_{23}(x_0,R,p) = 1 + \left|\Delta^{12}_{x_0,R}\right|^{-p} + \left|\Delta^{12}_{x_0,\frac{R}{2}}\right|^{-p} + \left|\Delta^{31}_{x_0,R}\right|^{-p} + \left|\Delta^{31}_{x_0,\frac{R}{2}}\right|^{-p},$$

$$\Gamma_{31}(x_0,R,p) = 1 + \left|\Delta^{23}_{x_0,R}\right|^{-p} + \left|\Delta^{23}_{x_0,\frac{R}{2}}\right|^{-p} + \left|\Delta^{12}_{x_0,R}\right|^{-p} + \left|\Delta^{12}_{x_0,\frac{R}{2}}\right|^{-p},$$

$$\Gamma_{12}(x_0,R,p) = 1 + \left|\Delta^{31}_{x_0,R}\right|^{-p} + \left|\Delta^{31}_{x_0,\frac{R}{2}}\right|^{-p} + \left|\Delta^{23}_{x_0,R}\right|^{-p} + \left|\Delta^{23}_{x_0,\frac{R}{2}}\right|^{-p},$$

whenever these expressions make sense. The next result is an analogue of Lemma 7.3 in [22] where a decay estimate is established for the case of two wells.

Lemma 6.1. *Suppose that all the hypotheses of Theorem 2.4 are satisfied. Fix some number $p > 2$ and consider a domain Ω' with compact closure in Ω. Then, for any $x_0 \in \Omega'$ and any $0 < R \le \frac{1}{2}R_0 = \frac{1}{2}\operatorname{dist}(\partial\Omega,\Omega')$, the following statements are true:*

(i) *If*

$$\Delta^{12}_{x_0,R} > 0,\ \Delta^{12}_{x_0,\frac{R}{2}} > 0,\ \Delta^{31}_{x_0,R} < 0,\ \Delta^{31}_{x_0,\frac{R}{2}} < 0, \tag{6.1}$$

then

$$\Psi(x_0,\rho) \le c_1\left\{\left[\frac{\rho}{R} + \frac{R}{\rho}\,\Gamma_{23}(x_0,R,p)\Psi^{\frac{p}{2}-1}(x_0,R)\right]\cdot \right.$$
$$\left. \cdot\,\Psi(x_0,R) + \frac{R}{\rho}\,\Gamma_{23}(x_0,R,p)R\right\},\ 0<\rho\le R. \tag{6.2}$$

(ii) *If*

$$\Delta^{23}_{x_0,R} > 0,\ \Delta^{23}_{x_0,\frac{R}{2}} > 0,\ \Delta^{12}_{x_0,R} < 0,\ \Delta^{12}_{x_0,\frac{R}{2}} < 0, \tag{6.3}$$

then

$$\Psi(x_0,\rho) \le c_1 \left\{ \left[\frac{\rho}{R} + \frac{R}{\rho}\,\Gamma_{31}(x_0,R,p)\Psi^{\frac{p}{2}-1}(x_0,R) \right] \cdot \right.$$
$$\left. \cdot\ \Psi(x_0,R) + \frac{R}{\rho}\,\Gamma_{31}(x_0,R,p)R \right\},\ 0<\rho\le R. \tag{6.4}$$

(iii) *If*

$$\Delta^{31}_{x_0,R} > 0,\ \Delta^{31}_{x_0,\frac{R}{2}} > 0,\ \Delta^{23}_{x_0,R} < 0,\ \Delta^{23}_{x_0,\frac{R}{2}} < 0, \tag{6.5}$$

then

$$\Psi(x_0,\rho) \le c_1 \left\{ \left[\frac{\rho}{R} + \frac{R}{\rho}\,\Gamma_{12}(x_0,R,p)\Psi^{\frac{p}{2}-1}(x_0,R) \right] \cdot \right.$$
$$\left. \cdot\ \Psi(x_0,R) + \frac{R}{\rho}\,\Gamma_{12}(x_0,R,p)R \right\},\ 0<\rho\le R. \tag{6.6}$$

Here c_1 denotes a positive constant independent of x_0, ρ and R.

Proof. Let (6.1) hold and consider the following Neumann boundary value problem: to find $u_R \in J^1_2\left(B_{\frac{R}{2}}(x_0)\right)$ and $\sigma_R \in L^2\left(B_{\frac{R}{2}}(x_0); \overset{\circ}{\mathbb{S}}\right)$ such that

$$\begin{aligned} &\sigma_R = 2\,\varepsilon(u_R) \quad \text{in} \quad B_{\frac{R}{2}}(x_0),\\ &\int\limits_{B_{\frac{R}{2}(x_0)}} (\sigma_R - \sigma) : \varepsilon(w)\,dx = 0 \quad \text{for any} \quad w \in J^1_2\left(B_{\frac{R}{2}}(x_0)\right). \end{aligned} \tag{6.7}$$

Equations (6.7) determine σ_R in an unique way, whereas u_R is fixed up to an incompressible rigid motion. From [22] we get the estimate

$$\Big(\int\limits_{B_\rho(x_0)} \left|\sigma_R - (\sigma_R)_{x_0,\rho}\right|^2 dx \Big)^{\frac{1}{2}}$$
$$\le c_2 \Big[\Big(\frac{\rho}{R}\Big)^2 \left(\int\limits_{B_{\frac{R}{2}}(x_0)} \left|\sigma_R - (\sigma_R)_{x_0,\frac{R}{2}}\right|^2 dx \right)^{\frac{1}{2}} + R^2\|f\|_{L^\infty\left(B_{\frac{R}{2}}(x_0)\right)} \Big]$$

being valid for any $0 < \rho \le \frac{R}{2}$, hence

$$\left(\int_{B_\rho(x_0)} \left| \sigma - (\sigma)_{x_0,\rho} \right|^2 \, dx \right)^{\frac{1}{2}} \le c_3 \left[\left(\frac{\rho}{R} \right)^2 \left(\int_{B_R(x_0)} |\sigma - (\sigma)_{x_0,R}|^2 \, dx \right)^{\frac{1}{2}} + R^2 + \left(\int_{B_{\frac{R}{2}}(x_0)} |\sigma - \sigma_R|^2 \, dx \right)^{\frac{1}{2}} \right] \quad (6.8)$$

which is true for $\rho \le R$. We are going to discuss the second integral on the right-hand side of (6.8):

$$\begin{aligned}
\frac{1}{2\mu} \int_{B_{\frac{R}{2}}(x_0)} |\sigma - \sigma_R|^2 \, dx
&\overset{(6.7)}{=} \frac{1}{2\mu} \int_{B_{\frac{R}{2}}(x_0)} \sigma : (\sigma - \sigma_R) \, dx - \frac{1}{\mu} \int_{B_{\frac{R}{2}}(x_0)} \varepsilon(u_R) : (\sigma - \sigma_R) \, dx \\
&\overset{(6.7)}{=} \frac{1}{2\mu} \int_{B_{\frac{R}{2}}(x_0)} \sigma : (\sigma - \sigma_R) \, dx \\
&= \int_{B_{\frac{R}{2}}(x_0)} \varepsilon(u) : (\sigma - \sigma_R) \, dx + \int_{B_{\frac{R}{2}}(x_0)} \left(\frac{1}{2\mu}\sigma - \varepsilon(u) \right) : (\sigma - \sigma_R) \, dx \\
&\overset{(6.7)}{=} \int_{B_{\frac{R}{2}}(x_0)} \left(\frac{1}{2\mu}\sigma - \varepsilon(u) \right) : (\sigma - \sigma_R) \, dx \\
&= \int_{\omega^{12}(x_0,\frac{R}{2}) \cup \Omega^{31}(x_0,\frac{R}{2})} \left(\frac{1}{2\mu}\sigma - \varepsilon(u) \right) : (\sigma - \sigma_R) \, dx.
\end{aligned}$$

This implies

$$\int_{B_{\frac{R}{2}}(x_0)} |\sigma - \sigma_R|^2 \, dx \le (2\mu)^2 \int_{\omega^{12}(x_0,\frac{R}{2}) \cup \Omega^{31}(x_0,\frac{R}{2})} \left| \frac{1}{2\mu}\sigma - \varepsilon(u) \right|^2 \, dx. \quad (6.9)$$

As in Section 5 we decompose

$$\begin{aligned}
&\omega^{12}(x_0,\tfrac{R}{2}) \cup \Omega^{31}(x_0,\tfrac{R}{2}) = \\
&\{\omega^{12}_-(x_0,\tfrac{R}{2}) \cap \omega^{23}_+(x_0,\tfrac{R}{2})\} \cup \\
&\{\omega^{31}_+(x_0,\tfrac{R}{2}) \cap \omega^{23}_-(x_0,\tfrac{R}{2})\} \cup \\
&\{(\omega^{12}_-(x_0,\tfrac{R}{2}) \cup \omega^{31}_+(x_0,\tfrac{R}{2})) \cap \omega^{23}_0(x_0,\tfrac{R}{2})\} \cup \\
&\{\omega^{12}_0(x_0,\tfrac{R}{2}) \cap \omega^{23}_+(x_0,\tfrac{R}{2})\} \cup \\
&\{\omega^{31}_0(x_0,\tfrac{R}{2}) \cap \omega^{23}_-(x_0,\tfrac{R}{2})\} \cup \\
&\{\omega^{31}_0(x_0,\tfrac{R}{2}) \cap \omega^{12}_0(x_0,\tfrac{R}{2}) \cap \omega^{23}_0(x_0,\tfrac{R}{2})\}
\end{aligned} \tag{6.10}$$

and observe that according to (2.8) the quantity $\left|\frac{1}{2\mu}\sigma - \varepsilon(u)\right|$ is bounded on the last three sets which occur on the right-hand side of (6.10). Using (2.8) also on the remaining sets we find that

$$\begin{aligned}
&\int_{\omega^{12}(x_0,\frac{R}{2})\cup\Omega^{31}(x_0,\frac{R}{2})} \left|\frac{1}{2\mu}\sigma - \varepsilon(u)\right|^2 dx \\
&\qquad \leq c_4\left[\int_{\omega^{12}(x_0,\frac{R}{2})\cup\Omega^{31}(x_0,\frac{R}{2})} |\sigma|^2\, dx + \left|\omega^{12}(x_0,\frac{R}{2})\cup\Omega^{31}(x_0,\frac{R}{2})\right|\right] \\
&\leq c_5\left[\int_{\omega^{12}(x_0,\frac{R}{2})\cup\Omega^{31}(x_0,\frac{R}{2})} \left|\sigma - (\sigma)_{x_0,\frac{R}{2}}\right|^2 dx + \left(1+\left|(\sigma)_{x_0,\frac{R}{2}}\right|^2\right)\left(\left|\omega^{12}(x_0,\frac{R}{2})\right| + \left|\Omega^{31}(x_0,\frac{R}{2})\right|\right)\right].
\end{aligned}$$

From assumption (6.1) it follows that

$$\left|(\sigma)_{x_0,\frac{R}{2}}\right| \leq 2R_1 + \frac{2|\varepsilon_0|}{a}.$$

Now, by combining the last two estimates and using Hölder's inequality, we get

$$\begin{aligned}
&\int_{\omega^{12}(x_0,\frac{R}{2})\cup\Omega^{31}(x_0,\frac{R}{2})} \left|\frac{1}{2\mu}\sigma - \varepsilon(u)\right|^2 dx \\
&\qquad \leq c_6R^2\left[\frac{|\omega^{12}(x_0,\frac{R}{2})|}{|B_{\frac{R}{2}}(x_0)|} + \frac{|\Omega^{31}(x_0,\frac{R}{2})|}{|B_{\frac{R}{2}}(x_0)|}\right. \\
&\left. + \left(\frac{|\omega^{12}(x_0,\frac{R}{2})|}{|B_{\frac{R}{2}}(x_0)|} + \frac{|\Omega^{31}(x_0,\frac{R}{2})|}{|B_{\frac{R}{2}}(x_0)|}\right)^{1-\frac{2}{p}} \left(\fint_{B_{\frac{R}{2}}(x_0)} \left|\sigma - (\sigma)_{x_0,\frac{R}{2}}\right|^p dx\right)^{\frac{2}{p}}\right].
\end{aligned} \tag{6.11}$$

The measures $|\omega^{12}(x_0, \frac{R}{2})|$ and $|\Omega^{31}(x_0, \frac{R}{2})|$ have already been estimated in Section 5 with the result

$$\left|\omega^{12}(x_0, \frac{R}{2})\right| \le \left|\Delta^{12}_{x_0,\frac{R}{2}}\right|^{-1} \int\limits_{\omega^{12}(x_0,\frac{R}{2})} \left|\sigma - (\sigma)_{x_0,\frac{R}{2}}\right| \, dx,$$

$$\left|\Omega^{31}(x_0, \frac{R}{2})\right| \le \left|\Delta^{31}_{x_0,\frac{R}{2}}\right|^{-1} \int\limits_{\Omega^{31}(x_0,\frac{R}{2})} \left|\sigma - (\sigma)_{x_0,\frac{R}{2}}\right| \, dx.$$

From Hölder's inequality we infer

$$\left(\frac{|\omega^{12}(x_0,\frac{R}{2})|}{|B_{\frac{R}{2}}(x_0)|}\right)^{\frac{1}{p}} \le \left|\Delta^{12}_{x_0,\frac{R}{2}}\right|^{-1} \left(\;\;\fint\limits_{B_{\frac{R}{2}}(x_0)} \left|\sigma - (\sigma)_{x_0,\frac{R}{2}}\right|^p dx \right)^{\frac{1}{p}},$$

$$\left(\frac{|\Omega^{31}(x_0,\frac{R}{2})|}{|B_{\frac{R}{2}}(x_0)|}\right)^{\frac{1}{p}} \le \left|\Delta^{31}_{x_0,\frac{R}{2}}\right|^{-1} \left(\;\;\fint\limits_{B_{\frac{R}{2}}(x_0)} \left|\sigma - (\sigma)_{x_0,\frac{R}{2}}\right|^p dx \right)^{\frac{1}{p}}. \tag{6.12}$$

In a final step we apply Poincare's inequality and use (5.3) to get

$$\begin{aligned} \left(\;\;\fint\limits_{B_{\frac{R}{2}}(x_0)} \left|\sigma - (\sigma)_{x_0,\frac{R}{2}}\right|^p dx \right)^{\frac{2}{p}} &\le c_7 \int\limits_{B_{\frac{R}{2}}(x_0)} |\nabla\sigma|^2 dx \\ &\le c_8 \left[\left(1 + |\Delta^{12}_{x_0,R}|^{-2} + |\Delta^{31}_{x_0,R}|^{-2}\right) \Psi^2(x_o, R) + R^2 \right]. \end{aligned} \tag{6.13}$$

The desired claim (6.2) is then a consequence of (6.8), (6.9) and (6.11)–(6.13). Since the proofs of (6.4) and (6.6) are less difficult than the proof of (6.2), we just present the ideas for (6.4), the remaining claim (6.6) is left to the reader. Again we consider the auxiliary problem (6.7) and use estimate (6.8). Then we proceed as follows: we have

$$\frac{1}{2}\int\limits_{B_{\frac{R}{2}}(x_0)} |\sigma-\sigma_R|^2\ dx$$

$$= \frac{1}{2}\int\limits_{B_{\frac{R}{2}}(x_0)} \sigma:(\sigma-\sigma_R)\ dx - \int\limits_{B_{\frac{R}{2}}(x_0)} \varepsilon(u_R):(\sigma-\sigma_R)\ dx$$

$$= \frac{1}{2}\int\limits_{B_{\frac{R}{2}}(x_0)} \sigma:(\sigma-\sigma_R)\ dx$$

$$= \int\limits_{B_{\frac{R}{2}}(x_0)} (\varepsilon(u)-\varepsilon_0):(\sigma-\sigma_R)\ dx + \int\limits_{B_{\frac{R}{2}}(x_0)} \left(\frac{1}{2}\sigma-(\varepsilon(u)-\varepsilon_0)\right):(\sigma-\sigma_R)\,dx$$

$$= \int\limits_{B_{\frac{R}{2}}(x_0)} \left(\frac{1}{2}\sigma-(\varepsilon(u)-\varepsilon_0)\right):(\sigma-\sigma_R)\,dx.$$

Applying Hölder's inequality and using (2.8) we find that

$$\int\limits_{B_{\frac{R}{2}}(x_0)} |\sigma-\sigma_R|^2\,dx \le 4\int\limits_{B_{\frac{R}{2}}(x_0)} \left|\frac{1}{2}\sigma-(\varepsilon(u)-\varepsilon_0)\right|^2\ dx$$

$$= 4\int\limits_{\omega^{23}(x_0,\frac{R}{2})\cup\Omega^{12}(x_0,\frac{R}{2})} \left|\frac{1}{2}\sigma-(\varepsilon(u)-\varepsilon_0)\right|^2\ dx.$$

From (2.8) we deduce boundedness of $\left|\frac{1}{2}\sigma-(\varepsilon(u)-\varepsilon_0)\right|$ on $\omega^{23}(x_0,\frac{R}{2})\cup\Omega^{12}(x_0,\frac{R}{2})$, hence

$$\int\limits_{B_{\frac{R}{2}}(x_0)} |\sigma-\sigma_R|^2\,dx \quad\le\quad c_9\left(\left|\omega^{23}(x_0,\frac{R}{2})\right| + \left|\Omega^{12}(x_0,\frac{R}{2})\right|\right). \tag{6.14}$$

Analogous to estimate (6.12) we get

$$\left(\frac{|\omega^{23}(x_0,\frac{R}{2})|}{|B_{\frac{R}{2}}(x_0)|}\right)^{\frac{1}{p}} \le \left|\Delta^{23}_{x_0,\frac{R}{2}}\right|^{-1}\left(\ \fint\limits_{B_{\frac{R}{2}}(x_0)} \left|\sigma-(\sigma)_{x_0,\frac{R}{2}}\right|^p dx\right)^{\frac{1}{p}},$$

$$\left(\frac{|\Omega^{12}(x_0,\frac{R}{2})|}{|B_{\frac{R}{2}}(x_0)|}\right)^{\frac{1}{p}} \le \left|\Delta^{12}_{x_0,\frac{R}{2}}\right|^{-1}\left(\ \fint\limits_{B_{\frac{R}{2}}(x_0)} \left|\sigma-(\sigma)_{x_0,\frac{R}{2}}\right|^p dx\right)^{\frac{1}{p}}.$$

Using Poincare's inequality together with the estimates (6.8), (6.14), (6.15) and recalling the Caccioppoli-type inequality (5.4), the proof of (6.4) and hence of Lemma 6.1 is complete. □

7. Proof of Theorem 2.4

In this section we are going to prove Theorem 2.4 for a representative case. So let us assume that $\mu > 1$ and $g_0 < 0$ which means that formula (2.8) of Lemma 2.1 is valid. All other cases can be treated in the same way.

Lemma 7.1. *Consider numbers $p > 2, \nu \in (0,1), \gamma > 0$ and let Ω' denote a subdomain with compact closure in Ω.*

Consider $t \in (0,1)$ such that

$$2\, c_1\, t^{1-\nu} \le 1 \tag{7.1}$$

where the constant c_1 is defined in Lemma 6.1. Suppose that for some point $x_0 \in \Omega'$ and some radius $R < \frac{1}{2} R_0 = \frac{1}{2}\,\mathrm{dist}(\Omega', \partial\Omega)$ the following statements are true:

$$\Psi(x_0, R) + c_1 \left(1 + \frac{2^{p+2}}{\gamma^p}\right) R \frac{t^{-(1+\nu)}}{1 - t^{1-\nu}} \le C(\nu, t, p, \gamma) = \min\left\{ \left(\frac{t^2}{t + \frac{2^{p+2}}{\gamma^p}} \right)^{\frac{2}{p-2}}, \frac{\gamma}{4}, t\,\gamma(1 - t^\nu) \right\} \tag{7.2}$$

and either

$$\Delta^{12}_{x_0,R} \ge 2\,\gamma\,,\ \Delta^{31}_{x_0,R} \le -2\,\gamma\,, \tag{7.3$_1$}$$

or

$$\Delta^{23}_{x_0,R} \ge 2\,\gamma\,,\ \Delta^{12}_{x_0,R} \le -2\,\gamma\,, \tag{7.3$_2$}$$

or

$$\Delta^{31}_{x_0,R} \ge 2\,\gamma\,,\ \Delta^{23}_{x_0,R} \le -2\,\gamma\,. \tag{7.3$_3$}$$

Then, for any $k \in \mathbb{N}_0$, we get

$$\Psi(x_o, t^k R) \le \min\left\{ \left(\frac{t^2}{t + \frac{2^{p+2}}{\gamma^p}} \right)^{\frac{2}{p-2}}, \frac{\gamma}{4} \right\}, \tag{7.4}$$

$$\Psi(x_o, t^{k+1} R) \le t^{\nu(k+1)} \left[\Psi(x_0, R) + c_1\, t^{-(1+\nu)} \left(1 + \frac{2^{p+2}}{\gamma^p}\right) R \sum_{s=0}^{k} t^{s(1-\nu)} \right] \tag{7.5}$$

and either

$$\begin{cases} \Delta^{12}_{x_o,R} + \gamma \ge \Delta^{12}_{x_0,t^kR} \ge \gamma \\ \Delta^{31}_{x_o,R} - \gamma \le \Delta^{31}_{x_0,t^kR} \le -\gamma, \end{cases} \tag{7.6$_1$}$$

or

$$\begin{cases} \Delta^{23}_{x_o,R} + \gamma \ge \Delta^{23}_{x_0,t^kR} \ge \gamma \\ \Delta^{12}_{x_o,R} - \gamma \le \Delta^{12}_{x_0,t^kR} \le -\gamma, \end{cases} \tag{7.6$_2$}$$

or

$$\begin{cases} \Delta^{31}_{x_o,R} + \gamma \ge \Delta^{31}_{x_0,t^kR} \ge \gamma \\ \Delta^{23}_{x_o,R} - \gamma \le \Delta^{23}_{x_0,t^kR} \le -\gamma, \end{cases} \tag{7.6$_3$}$$

respectively.

Proof. Lemma 7.1 follows by induction from Lemma 6.1 taking into account the inequalities

$$\begin{aligned}\left|\Delta^{ij}_{x_0,t^{s+1}R}-\Delta^{ij}_{x_0,t^sR}\right| &\le |(\sigma)_{x_0,t^{s+1}R}-(\sigma)_{x_0,t^sR}|\\ &\le \fint_{B_{t^{s+1}R}(x_0)} |\sigma-(\sigma)_{x_0,t^sR}|\,dx \le t^{-1}\Psi(x_0,t^sR).\end{aligned}$$

For a detailed computation in a similar setting we refer to [20, 21]. □

In order to prove Theorem 2.4 we first observe that

$$\begin{aligned}a(\sigma)=&\left\{x\in\Omega_0 : \Delta^{12}_x>0,\ \Delta^{31}_x<0\right\}\cup\\ &\left\{x\in\Omega_0 : \Delta^{23}_x>0,\ \Delta^{12}_x<0\right\}\cup\\ &\left\{x\in\Omega_0 : \Delta^{31}_x>0,\ \Delta^{23}_x<0\right\},\end{aligned}\tag{7.7}$$

Ω_0 denoting the Lebesgue set of σ. Next take some $x_0\in a(\sigma)$, w.l.o.g. we may assume that

$$x_0\in\left\{x\in\Omega_0 : \Delta^{12}_x > 0, \Delta^{31}_x < 0\right\},$$

hence we have

$$\lim_{R\downarrow 0}\Delta^{12}_{x_0,R} = \gamma_{12}>0,\quad \lim_{R\downarrow 0}\Delta^{31}_{x_0,R} = \gamma_{31}<0.\tag{7.8}$$

Next we consider a subdomain $\Omega'\subset\subset\Omega$ such that $x_0\in\Omega'$ and define the numbers $p>2$ and $\gamma>0$ according to

$$2\gamma < \min\{\gamma_{12},-\gamma_{31}\}.\tag{7.9}$$

For any $\nu\in(0,1)$ we select a number t with (7.1). By Poincare's inequality (compare (6.13)) we know

$$\Psi(x,R)\underset{R\downarrow 0}{\longrightarrow} 0\tag{7.10}$$

for any x in Ω. According to (7.8)–(7.10) there exists a radius $0<R<\frac{1}{2}R_0=\frac{1}{2}\operatorname{dist}(\partial\Omega,\Omega')$ such that

$$\begin{aligned}&\Psi(x_0,R)+c_1\left(1+\frac{2^{p+2}}{\gamma^p}\right)R\frac{t^{-(1+\nu)}}{1-t^{1-\nu}}<C(\nu,t,p,\gamma),\\ &\Delta^{12}_{x_0,R}>2\gamma,\ \Delta^{31}_{x_0,R}<-2\gamma.\end{aligned}\tag{7.11}$$

Since the functions $x\mapsto\Psi(x,R), x\mapsto\Delta^{12}_{x,R}, x\mapsto\Delta^{31}_{x,R}$ are continuous, we find a neighborhood U of x_0 such that (7.11) holds for any $x\in U$. Then we deduce from Lemma 7.1 that σ is Hölder continuous near x_0 with exponent ν, moreover, (7.6_1) implies that this neighborhood belongs to the set $\{x\in\Omega_0 : \Delta^{12}_x>0,\ \Delta^{31}_x<0\}$. This completes the proof of Theorem 2.4. □

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