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Viability Results for Nonautonomous Differential Inclusions

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We obtain measurable viability results for the differential inclusion $y'(t) \in F(t, y(t))$ via the approximation of F through the Aumann integral means. The admissibility of a preorder is also studied by reducing it to the viability of its sections.

1. Introduction

Our aim here is to use a result of Guseinov, Subbotin and Ushakov [11] (see [1, p.85] for a proof), Proposition 2.1 below, in order to obtain measurable viability results for the differential inclusion

$$y'(s) \in F(s, y(s)) \tag{1.1}$$

in finite dimensional spaces, via the approximation of F through the Aumann integral means

$$F_h(t, x) = \frac{1}{h} \int_t^{t+h} F(s, x) ds.$$

This technique of approximation is not new; see, e.g., [6], [13], [14]. Since a tangency condition involving F and a tangent cone, say $\mathcal{T}$, leads to a tangency condition involving F_h and $\overline{\text{conv}}\, \mathcal{T}$, in the papers cited above the authors consider situations where the tangent cone $\mathcal{T}$ itself is closed and convex. Proposition 2.1 allows us to use tangency conditions involving the Bouligand tangent cone, which is not necessarily convex.

We have to note that Proposition 2.1, which is our main tool, holds true in the finite dimensional setting. Therefore our results and our techniques are related to the finite dimensional versions of the corresponding theorems in [6], [13], [14].

We mention that in all the previous papers concerning this subject the tangency condition is imposed to hold either for every t and for every x or for almost every t and for every x. In other words, the negligible set where the tangency condition may not hold is independent of x. Our conditions (V3) or (V4) below allow that set to depend on x as well. Moreover, our machinery leads to an extension of Proposition 2.1 to the Carathéodory case.

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In Section 2 we develop the discussion above while in Section 3 we apply the results of Section 2 to the problem of monotone solutions associated with a preorder. There we use an idea from [5] that, in the autonomous case, reduces the admissibility of a preorder to the viability of its sections.

2. Viability results

In the sequel we shall denote by X a finite dimensional normed space. For a set A in X we denote $|A| := \sup\{\|x\|; x \in A\}$.

Let $D \subset X$ and let $F : [a, b) \times D \to 2^X$. By a solution to the differential inclusion (1.1) we mean a locally absolutely continuous function $y : I \to D$, where I is an interval in $[a, b)$, which satisfies (1.1) for almost all $s \in I$. We denote by $B(x, r)$ the closed ball of center x and radius r in X. We say that $u \in X$ is *tangent* to D at $x \in D$ if for each $\delta > 0$ and each neighbourhood Ω of 0 there exist $t \in (0, \delta)$ and $p \in \Omega$ such that $x + t(u + p) \in D$. It is easy to see that $u \in X$ is tangent to D at $x \in D$ if and only if there exist a sequence (t_n) decreasing to 0 and a sequence (p_n) convergent to 0 such that $x + t_n(u + p_n) \in D$ for each $n \in \mathbb{N}$. The set of all tangent elements to D at $x \in D$ is denoted by $T_D(x)$ and is called the Bouligand tangent cone to D at x. It is a closed cone (see, e.g., [2] for details). Let us state the result of Guseinov, Subbotin and Ushakov [11] cited above.

Proposition 2.1. *Let K be a closed set in X and let $F : K \to 2^X$ be an upper semicontinuous convex compact valued multifunction. Then the following two conditions are equivalent*

(i) $\forall x \in K, \quad F(x) \cap T_K(x) \neq \emptyset$;

(ii) $\forall x \in K, \quad F(x) \cap \overline{\text{conv}}\, T_K(x) \neq \emptyset$.

We note that the proof of Proposition 2.1 works also in case K is locally closed, i.e., for each $x \in K$ there exists $B(x, r)$ such that $K \cap B(x, r)$ is closed.

Definition 2.2. The set D is *viable* with respect to (1.1) if for every $(\tau, \xi) \in [a, b) \times D$ there exists a solution $y : [\tau, \theta] \to D$ to (1.1) with $y(\tau) = \xi$ and with $\theta > \tau$.

Let us record some conditions which we shall refer to in the sequel.

(A) For every $x \in D$ the multifunction $F(\cdot, x)$ is measurable on $[a, b)$.

(B) For almost every $t \in [a, b)$ the multifunction $F(t, \cdot)$ is upper semicontinuous on D.

(C) For every $t \in [a, b)$ and for every $x \in D$, $F(t, x)$ is nonempty compact and convex.

(D) For every compact $K \subset X$ there exists a locally bounded function $\mu : [a, b) \to \mathbb{R}$ such that $|F(t, x)| \leq \mu(t)$ for almost every $t \in [a, b)$ and for every $x \in D \cap K$.

Before stating our main result let us recall that a solution $y : [\tau, T) \to D$ to the differential inclusion (1.1) is said to be *saturated* if there does not exist any other solution $z : [\tau, \sigma) \to D$ to (1.1) such that both $T \leq \sigma$ and y equals the restriction of z to $[\tau, T)$.

Theorem 2.3. *Let D be a locally closed subset of X. Assume that conditions* (A), (B), (C) *and* (D) *are satisfied. Then the following conditions are equivalent:*

(V1) *there exists a negligible set $Z \subseteq [a, b)$ such that for every $t \in [a, b) \setminus Z$ and for every $x \in D$,*

$$F(t, x) \cap T_D(x) \neq \emptyset;$$

(V2) *there exists a negligible set $Z \subseteq [a,b)$ such that for every $t \in [a,b) \setminus Z$ and for every $x \in D$,*

$$F(t,x) \cap \overline{\text{conv}}\, T_D(x) \neq \emptyset;$$

(V3) *for each $x \in D$ there exists a negligible set $Z_x \subseteq [a,b)$ such that for every $t \in [a,b) \setminus Z_x$,*

$$F(t,x) \cap T_D(x) \neq \emptyset;$$

(V4) *for each $x \in D$ there exists a negligible set $Z_x \subseteq [a,b)$ such that for every $t \in [a,b) \setminus Z_x$,*

$$F(t,x) \cap \overline{\text{conv}}\, T_D(x) \neq \emptyset;$$

(V5) *the set D is viable with respect to* (1.1)*;*

(V6) *for every $(\tau,\xi) \in [a,b) \times D$ there exists a saturated solution $y : [\tau,T) \to D$ to the differential inclusion* (1.1) *with $y(\tau) = \xi$.*

Proof. Since the implications (V3)→(V4), (V1)→(V3), (V1)→(V2), (V2)→(V4) and (V6)→(V5) are obvious, it remains to prove the implications (V4)→(V5), (V5)→(V1) and (V5)→(V6). To prove (V5)→(V6) we can apply the Zorn Theorem or an ordering principle of Brézis - Browder [4] as in [5, Proposition 3].

Suppose now (V4). In order to prove (V5) it is sufficient to work on a subinterval $[a,c]$ of $[a,b)$ and we have to prove that for every $(\tau,\xi) \in [a,c) \times D$ there exists a solution $y : [\tau,\theta] \to D$ to the differential inclusion (1.1) with $y(\tau) = \xi$.

For each $(t,x) \in [a,c] \times D$ and $h > 0$ consider

$$F_h(t,x) = \frac{1}{h} \int_{I_{t,h}} F(s,x)ds$$

where $I_{t,h} := [t,t+h] \cap [a,c]$.

By [6, p.337] or [14, p.259], for each $h > 0$ the multifunction $F_h(\cdot,\cdot)$ is upper semicontinuous in both variables. Furthermore, it has compact and convex values. Since for every $x \in D$, $t \mapsto F(t,x)$ is measurable, it follows that $t \mapsto F(t,x) \cap \overline{\text{conv}}\, T_D(x)$ is measurable too and has closed values. Therefore it has a measurable selection [7, p.65], $g(s) \in F(s,x) \cap \overline{\text{conv}}\, T_D(x)$, a.e. $s \in [a,c]$. By condition (D), the function $g(\cdot)$ is integrable, therefore,

$$\frac{1}{h} \int_{I_{t,h}} g(s)ds \in \overline{\text{conv}}\, T_D(x) \cap F_h(t,x)$$

for all $t \in [a,c]$, $x \in D$ and $h > 0$. We thus have

$$F_h(t,x) \cap \overline{\text{conv}}\, T_D(x) \neq \emptyset$$

for all $t \in [a,c)$, $x \in D$ and $h > 0$. We now apply Proposition 2.1 and conclude that

$$F_h(t,x) \cap T_D(x) \neq \emptyset$$

for all $t \in [a,c)$, $x \in D$ and $h > 0$. This implies that

$$\{1\} \times F_h(t,x) \cap T_{[a,c) \times D}(t,x) \neq \emptyset$$

for all $(t, x) \in [a, c) \times D$ and $h > 0$.

Since $[a, c) \times D$ is locally closed, we can apply Haddad's viability theorem [12], [2], [5], to the differential inclusion

$$z'(t) = 1, \quad y'(t) \in F_h(z, y)$$

to conclude that for every $(\tau, \xi) \in [a, c) \times D$ there exists a solution $y_h : [\tau, \theta_h] \to D$ to the differential inclusion

$$y'(s) \in F_h(s, y(s)) \tag{2.1}$$

with $y_h(\tau) = \xi$.

From the proof of Haddad's viability theorem in [1, p. 95-96] it follows that if we fix $(\tau, \xi) \in [a, c) \times D$ and $r > 0$ such that $[\tau, \tau + r] \subset [a, c)$, $K := D \cap B(\xi, r)$ is compact and

$$|F_h(t, x)| \leq L \tag{2.2}$$

for $(t, x) \in [\tau, \tau + r] \times K$, then we can obtain a solution to (2.1) on a time interval $[\tau, \theta]$ which depends only on r and L. Moreover that solution remains in $B(\xi, r)$. It is easy to see, due to condition (D), that we can obtain (2.2) with L independent of h.

To summarize, for $(\tau, \xi) \in [a, c) \times D$ and r as above, there exists $\theta < c$ such that for every $h > 0$ there exists $y_h : [\tau, \theta] \to D \cap B(\xi, r)$, absolutely continuous and satisfying (2.1) for almost all $s \in [\tau, \theta]$.

By (2.2) we obtain $\|y_h'(s)\| \leq L$ a.e. on $[\tau, \theta]$, so that, by taking a subsequence if necessary, we may suppose that $y_h \to y$ uniformly on $[\tau, \theta]$ and that $y_h' \to y'$ weakly in $L^1[\tau, \theta]$. We have to show that $y'(s) \in F(s, y(s))$ a.e. in $[\tau, \theta]$. This fact is proved in [14, p.260]. See also [6, Lemma 6.5] in case F is measurable in both variables. Hence, the implication (V4)→(V5) is proved.

Let us prove now that (V5)→(V1). As in [10, Lemma 2.6], one can prove via a Scorza Dragoni type theorem that there exists a negligible set $Z \subset [a, b)$ such that for every $\tau \in [a, b) \setminus Z$, for every $\xi \in D$ and $\epsilon > 0$ there exists $\delta > 0$ such that for every solution $y : [\tau, \theta) \to D$ to (1.1) with $y(\tau) = \xi$ we have

$$\frac{y(\tau + h) - y(\tau)}{h} \in F(\tau, \xi) + \epsilon B(0, 1)$$

for $0 < h \leq \delta$. Since $F(\tau, \xi)$ is compact, it readily follows that there exist $w \in F(\tau, \xi)$, $h_n \to 0$, $p_n \to 0$ such that

$$y(\tau + h_n) = y(\tau) + h_n(w + p_n)$$

for every $n \in \mathbb{N}$. Since $y(\tau + h_n) \in D$ we get $w \in T_D(\xi)$ and the proof is completed. □

We end this section with some bibliographical comments. Theorem 2.3 extends [14, Theorems 1 and 2] where Clarke tangent cone is involved. The same situation appears in [6, Theorem 7.2] where D is convex and closed and thus $T_D(x)$ is the Clarke tangent cone too. Theorem 2.3 is also related to [9, Theorem 3.1] where the more general case when D depends on t is considered and where the tangency condition involves the closed convex hull of the Bouligand tangent cone. For a pioneering work on viability in case the right hand side is a Carathéodory function and D is independent of t see [15]. More recent treatments of the subject are to be found in e.g. [1], [3] and [8].

3. Admissibility of preorders

We discuss in this section the problem of monotone trajectories associated with the differential inclusion (1.1) and the preorder $\leq$ (reflexive and transitive relation) on D. It is convenient to identify the preorder $\leq$ on D with the multifunction $P : D \to 2^D$ defined by

$$P(x) = \{y \in D; x \leq y\}$$

for all $x \in D$. In the sequel we characterize *admissibility* of the preorder P with respect to the differential inclusion (1.1).

Definition 3.1. The solution $y : [\tau, \theta) \to D$ to (1.1) is *monotone* with respect to $\leq$ if for every $s \in [\tau, \theta)$ and for every $t \in [s, \theta)$ we have $y(t) \in P(y(s))$.

Definition 3.2. The preorder $P : D \to 2^D$ is *admissible* with respect to (1.1) if for every $(\tau, \xi) \in [a, b) \times D$ there exists a monotone solution with respect to $\leq$, $y : [\tau, \theta) \to D$ to (1.1), such that $y(\tau) = \xi$.

In case F is upper semicontinuous and independent of t and the multifunction P is continuous we have the well known result of Haddad [12]. Cârjă and Ursescu [5] obtained a similar result when P is assumed to have closed graph in $D \times D$. As a matter of fact, in [5] it is presented a completely new approach. It reduces the admissibility of preorder P to the viability of its sections $P(x)$. Usually the admissibility of preorders is obtained by remaking and lengthening the proof of the viability of sets (cf. e.g. [2], [12] and [13]).

Here we show that the approach from [5] does work in case F is of Carathéodory type and thus we obtain a necessary and sufficient condition for the admissibility of P via Theorem 2.3 above. This approach allows us to extend the finite dimensional version of the results of [13] in three directions: (1) the tangency condition involves the Bouligand tangent cone instead of the Clarke tangent cone; (2) the preorder P has closed graph in $D \times D$ instead of its continuity; (3) the tangency condition is satisfied "for almost all t" instead of "for all t".

We first state the version of Proposition 9 in [5] when F is of Carathéodory type, which can be proved in a similar way.

Theorem 3.3. *Let D be locally closed, let $F : [a, b) \times D \to 2^X$ satisfy conditions* (A), (B), (C), (D) *and let P be a preorder on D such that its graph is closed in $D \times D$. Then P is admissible with respect to* (1.1) *if and only if for each $x \in D$ the set $P(x)$ is viable with respect to* (1.1).

We are now prepared to state the main result of this section.

Theorem 3.4. *Let D be locally closed, let $F : [a, b) \times D \to 2^X$ satisfy conditions* (A), (B), (C), (D) *and let P be a preorder on D such that its graph is closed in $D \times D$. Then the following conditions are equivalent:*

(M1) *there exists a negligible set $Z \subseteq [a, b)$ such that for every $t \in [a, b) \setminus Z$ and for every $x \in D$,*

$$F(t, x) \cap T_{P(x)}(x) \neq \emptyset;$$

(M2) *there exists a negligible set $Z \subseteq [a, b)$ such that for every $t \in [a, b) \setminus Z$ and for every*

$x \in D$,

$$F(t,x) \cap \overline{\text{conv}}\, T_{P(x)}(x) \neq \emptyset;$$

(M3) *for each $x \in D$ there exists a negligible set $Z_x \subseteq [a,b)$ such that for every $t \in [a,b) \setminus Z_x$,*

$$F(t,x) \cap T_{P(x)}(x) \neq \emptyset;$$

(M4) *for each $x \in D$ there exists a negligible set $Z_x \subseteq [a,b)$ such that for every $t \in [a,b) \setminus Z_x$,*

$$F(t,x) \cap \overline{\text{conv}}\, T_{P(x)}(x) \neq \emptyset;$$

(M5) *the preorder P is admissible with respect to* (1.1)*;*

(M6) *for every $(\tau, \xi) \in [a,b) \times D$ there exists a saturated solution $y : [\tau, T) \to D$ to the differential inclusion* (1.1) *with $y(\tau) = \xi$ which is monotone with respect to $\leq$.*

Proof. Since the implications (M3)→(M4), (M1)→(M3), (M1)→(M2), (M2)→(M4) and (M6)→(M5) are obvious, it remains to prove the implications (M4)→(M5), (M5)→(M1) and (M5)→(M6). The implication (M5)→(M6) follows by the Zorn Theorem or by the Brézis - Browder result as in [5, Proposition 10].

The implication (M4)→(M5) follows easily by combining Theorems 2.3 and 3.3. Indeed, thanks to Theorem 3.3, in order to prove (M5) we need to show that $P(x)$ is viable for every $x \in D$. By Theorem 2.3 applied to the restriction of F to $[a,b) \times P(x)$ we have to verify (V4), i.e., that for every $y \in P(x)$ and for almost all $t \in [a,b)$ we have

$$F(t,y) \cap \overline{\text{conv}}\, T_{P(x)}(y) \neq \emptyset.$$

This is clearly implied by (M4) because, for $y \in P(x)$ we have $P(y) \subset P(x)$ and thus $T_{P(y)}(y) \subset T_{P(x)}(y)$. Therefore, (M4) implies (M5).

To prove that (M5) implies (M1) we proceed as in the proof of (V5)→(V1) in Theorem 2.3. Indeed, there exists a negligible set $Z \subset [a,b)$ such that for every $\tau \in [a,b) \setminus Z$, for every $\xi \in D$ and $\epsilon > 0$ there exists $\delta > 0$ such that for every solution $y : [\tau, \theta) \to D$ to (1.1) with $y(\tau) = \xi$ we have

$$\frac{y(\tau + h) - y(\tau)}{h} \in F(\tau, \xi) + \epsilon B(0,1)$$

for $0 < h \leq \delta$. Assuming (M5), take a monotone solution above, observe that $y(\tau + h) \in P(\xi)$ for $h > 0$ and the proof is completed. □

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