Journal of Convex Analysis
Volume 7 (2000), No. 2, 445–452

Minimal Pairs Representing Selections of Four Linear Functions in $\mathbb{R}^3$

J. Grzybowski
Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Matejki 48/49, 60769 Poznań, Poland.
e-mail: jgrz@amu.edu.pl

D. Pallaschke
Institut für Statistik und Mathematische Wirtschaftstheorie, Universität Karlsruhe, Kaiserstr. 12, 76128 Karlsruhe, Germany.
e-mail: lh09@rz.uni-karlsruhe.de

R. Urbański
Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Matejki 48/49, 60769 Poznań, Poland.
e-mail: rich@amu.edu.pl

Received July 5, 1999
Revised manuscript received January 24, 2000

In this paper we investigate minimal pairs of continuous selections of four linear functions in $\mathbb{R}^3$. Our purpose is to find minimal pairs of compact convex sets (polytops) which represent all 166 (see [2]) continuous selections in $CS(y_1, y_2, y_3, -\sum_{i=1}^{3} y_i)$ in $\mathbb{R}^3$. We find that these 166 selections are represented by 16 essentialy different minimal pairs which were studied in [5], [9]. Three out of 16 cases are minimal pairs that are not unique minimal representations in their own quotient classes. One of these quotient classes was already studied in [5], [10], [15].

Keywords: Minimal pairs of convex sets, convex analysis

1991 Mathematics Subject Classification: 52A07, 26A27

1. Introduction

Let $U \subset \mathbb{R}^n$ be an open subset and $f, f_1, \ldots, f_m : U \to \mathbb{R}$ continuous functions. If $I(x) = \{i \in \{1, \ldots, m\} \mid f_i(x) = f(x)\}$ is nonempty at every point $x \in U$, then f is called a *continuous selection* of the functions $f_1, \ldots, f_m$. We denote by $CS(f_1, \ldots, f_m)$ the set of all continuous selections of $f_1, \ldots, f_m$. The set $I(x)$ is called the active index set of f at the point x. The functions $f_1, \ldots, f_m$ will be called *generating functions.* Typical examples for continuous selections are the functions

$$f_{\max} = \max(f_1, \ldots, f_m), \quad f_{\min} = \min(f_1, \ldots, f_m)$$

or, more generally, any finite superposition of maximum and minimum operations over subsets of the functions $f_1, \ldots, f_m$.

In [8] the notion of a nondegenerate critical point for a continuous selections of C^2-functions has been defined and it has been shown that a continuous selection f of C^2

ISSN 0944-6532 / \$ 2.50

functions is topologically equivalent to a function of the form

$$y \to f(x_0) + g(y_1, \dots, y_k) - \sum_{i=k+1}^{k+\mu} y_i^2 + \sum_{j=k+\mu+1}^{n} y_j^2$$

in a neighbourhood of a nondegenerate critical point x_0, where $k =\mid I(x_0) \mid -1$, $g \in CS(y_1, \dots, y_k, -\sum_{i=1}^{k} y_i)$, and μ is the quadratic index of f at x_0. For more details see [7], Chapter 7, and [8].

In [2] it has been shown that every continuous selection of linear functions $l_1, \dots, l_m$ on $\mathbb{R}^n$ has a representation of the form

$$l(x) = \min_{i \in \{1,\dots,r\}} \max_{j \in M_i} l_j(x) \tag{1.1}$$

where $M_i \subset \{1, \dots, n+1\}$ and that this representation is unique, provided the linear functions are affinely independent, i.e. $\sum_{i=1}^{m} \lambda_i l_i = 0$, $\sum_{i=1}^{m} \lambda_i = 0$ implies that $\lambda = 0$, and $M_i \subset M_j$ if and only if $i = j$. Note that in particular the functions $l_i(x) = x_i$, $i = 1, \dots, n$, $l_{n+1}(x) = -\sum_{i=1}^{n} x_i$ are affinely independent. The topological structure of a continuous selection of C^2 functions in the vicinity of a nondegenerate critical point is thus completely determined by its quadratic index μ and a unique collection of index sets $M_1, \dots, M_r$. This fact has been used in [1] to extend the classical smooth Morse theory to piecewise smooth functions.

Observe that every function l of the form (1.1) can be represented as a difference of two polyhedral support functions, since

$$l(x) = \min_{i \in \{1,\dots,r\}} \max_{j \in M_i} l_j(x) \;=\; \max_{i \in \{1,\dots,r\}} \{\sum_{\substack{k=1 \\ k \neq i}}^{r} \max_{j \in M_k} -l_j(x)\} - \sum_{k=1}^{r} \max_{j \in M_k} -l_j(x) \tag{1.2}$$

holds. Now, we identify the difference of the support functions $p_A - p_B$ of two compact convex sets A and B with the quotient class $[A, B]$ in the Rådström-Hörmander lattice [6] of equivalence classes of pairs of nonempty compact convex sets.

In other words: As in [9] let us denote for a real topological vector space X the set of all nonempty compact convex subsets by $\mathcal{K}(X)$ and the set of all pairs of nonempty compact convex subsets by $\mathcal{K}^2(X)$, i.e. $\mathcal{K}^2(X) = \mathcal{K}(X) \times \mathcal{K}(X)$. The equivalence relation between pairs of compact convex sets is given by: "$(A, B) \sim (C, D)$ *if and only if* $A + D = B + C$" using the Minkowski-sum, and a partial order is given by the relation: "$(A, B) \leq (C, D)$ *if and only if* $A \subseteq C$ *and* $B \subseteq D$." By $[A, B]$, we denote the equivalence class of (A, B) in $\mathcal{K}^2(X)/_{\sim}$. For two compact convex sets $A, B \in \mathcal{K}(X)$ we will use the notation

$$A \vee B := \mathrm{conv}(A \cup B),$$

where the operation "conv" denotes the convex hull and $\bar{A}$ denotes the closure of a set A.

Let us recall that for a real topological vector space X a pair $(A, B) \in \mathcal{K}^2(X)$ is *minimal* if and only if for every equivalent pair $(C, D) \in \mathcal{K}^2(X)$ the relation $(C, D) \leq (A, B)$ implies $C = A$ and $D = B$.

In the proofs, we will use frequently an easy identity for compact convex sets which was first observed by A. Pinsker [12], namely: For $A, B, C \in \mathcal{K}(X)$ we have:

$$(A \ + \ C) \vee (B \ + \ C) = C \ + \ (A \vee B).$$

Finally let us state explicitely the order cancellation law (see [6], [14]).

Let X be a real topological vector space and $A, B, C \subseteq X$ compact convex subsets. Then the inclusion

$$A \ + \ B \subseteq A \ + \ C \quad \textit{implies} \quad B \subseteq C.$$

2. The Representation Theorem

Now we are able to prove the following result:

Theorem 2.1. *The set*

$$CS(y_1, y_2, y_3, -\sum_{i=1}^{3} y_i)$$

consists of 166 continuous selections which are represented by 16 essentialy different minimal pairs. Three out of these 16 cases are minimal pairs that are not unique minimal representations in their own quotient classes.

Proof. Throughout the proof we will use the following notations. Let $a, b, c, d \in \mathbb{R}^3$ be affinely independent vectors such that $a+b+c+d=0$. For convenience we identify these vectors with linear functions, i.e. $a : \mathbb{R}^3 \to \mathbb{R}$, $a(x) =< a, x >$, where $< \cdot, \cdot >$ denotes the scalar product.

If $a = (1,0,0)$, $b = (0,1,0)$, $c = (0,0,1)$, $d = (-1,-1,-1)$ then

$$CS(y_1, y_2, y_3, \ -y_1, -y_2 - y_3) = CS(a, b, c, d).$$

In ([2]) it has been shown that $CS(y_1, y_2, y_3, \ -y_1, -y_2 - y_3)$ consists of 166 continuous selections. Our purpose is to find minimal pairs of polytops that represent these 166 continuous selections.

Therefore, we identify the difference of the support functions $p_A - p_B$ of two compact convex sets with the quotient class $[A, B]$. Then, the function a is identified with $[\{a\}, \{0\}]$. For convenience we will write $[a, 0]$.

According to all possible max-min combinations of the functions $l_1, ..., l_4$ we have to consider the following 16 cases:

(1) The trivial selections a, b, c and d can be represented by the minimal pairs $(a, 0)$, $(b, 0)$, $(c, 0)$, and $(d, 0)$.

(2) Denote $\max(a, b)$ by $\overline{ab}$ and $\min(a, b)$ by $\underline{ab}$. Applying (1.2) we obtain $\overline{ab} = [a \vee b, 0]$ and $\underline{ab} = [a + b, a \vee b] = [0, -(a \vee b)]$. The pairs $(a \vee b, 0), (0, -(a \vee b))$ are minimal because one of two sets in each pair is a one-point set. In a similar way we find representations of 12 selections in total: $\overline{ab}, \overline{ac}, \overline{ad}, \ldots, \underline{ab}, \underline{ac}, \ldots$.

(3) Notice that $\overline{abc} = \overline{\overline{ab}\,c}$ and $\overline{abc} = [a \vee b \vee c, 0]$. Also $\underline{abc} = [0, -(a \vee b \vee c)]$ and both pairs $(a \vee b \vee c, 0)$ and $(0, -(a \vee b \vee c))$ are minimal and $a \vee b \vee c$ is a *triangle.* In this way we find representations of 8 selections in total.

(4) Take $\underline{\overline{ab}\,c}$ that is $\min(\max(a,b),c)$. Then $\underline{\overline{ab}\,c} = [a \vee b + c, a \vee b \vee c]$. Also

$$\underline{\overline{ac}\ \overline{bc}} = [a{\vee}c{+}b{\vee}c, a{\vee}b{\vee}c] = [(a{+}b){\vee}(a{+}c){\vee}(b{+}c), a{\vee}b] = [-(a{\vee}b{\vee}c), -(a{\vee}b){-}c]$$

The reader can compute these equalities for himself.
The pairs $(a \vee b + c, a \vee b \vee c)$ and $(-(a \vee b \vee c), -(a \vee b) - c)$ consist of a *triangle* and an *interval.* It follows from the criteria proved in [10] that they are minimal. Similar pairs represent 24 selections in total.

(5) Notice that

$$\begin{aligned}
&\underline{\overline{ab}\ \overline{ac}\ \overline{bc}} = \\
&[a \vee b + b \vee c + a \vee c, (a \vee b + a \vee c) \vee (a \vee b + b \vee c) \vee (a \vee c + b \vee c)] = \\
&[a \vee b \vee c + (a+b) \vee (a+c) \vee (b+c), a \vee b \vee c + a \vee b \vee c] = \\
&[(a+b) \vee (a+c) \vee (b+c), a \vee b \vee c].
\end{aligned}$$

Again, the pair of *triangles* $((a+b) \vee (a+c) \vee (b+c), a \vee b \vee c)$ is minimal (see [11]). Similar pairs represent 4 selections in total.

(6) Take $\overline{abcd} = [a \vee b \vee c \vee d, 0]$ and $\underline{abcd} = [0, -(a \vee b \vee c \vee d)]$. The pairs $(a \vee b \vee c \vee d, 0)$ and $(0, -(a \vee b \vee c \vee d))$ are minimal and $a \vee b \vee c \vee d$ is a *tetrahedron.*

(7) Now,

$$\begin{aligned}
&\underline{\overline{ab}\,cd} = [a \vee b + c + d, (a \vee b + c) \vee (a \vee b + d) \vee (c+d)] = \\
&[a \vee b + c + d, (a \vee b + c \vee d) \vee (c+d)].
\end{aligned}$$

Then $a{\vee}b{+}c{+}d$ is an *interval* parallel to two edges of the *pyramid* $(a{\vee}b{+}c{\vee}d){\vee}(c{+}d)$. The pair $(a \vee b + c, (a \vee b + c \vee d) \vee (c+d))$ is minimal, cf. [10]. Also $\underline{\overline{acd}\ \overline{bcd}} = [-((a \vee b + c \vee d) \vee (c+d)), -(a \vee b) - c - d]$. Similar pairs represent 12 selections in total.

(8) Observe that

$$\begin{aligned}
&\underline{\overline{ab}\ \overline{ac}\ \overline{bc}\,d} = \underline{\underline{\overline{ab}\ \overline{ac}\ \overline{bc}}\,d} = \\
&[(a+b) \vee (a+c) \vee (b+c) + d, (a+b) \vee (a+c) \vee (b+c) \vee (d + a \vee b \vee c)] = \\
&[-c \vee b \vee a, (a+b) \vee (a+c) \vee (b+c) \vee (d+a) \vee (d+b) \vee (d+c)].
\end{aligned}$$

This is a pair of *triangle* and *octahedron.* Also $\underline{\overline{abd}\ \overline{acd}\ \overline{bcd}} = [(a+b) \vee (a+c) \vee (b+c) \vee (a+d) \vee (b+d) \vee (c+d), a \vee b \vee c)$. These pairs are minimal, cf. [10] and similar pairs represent 8 selections in total.

(9) Notice that

$$\underline{\overline{ab}\;\overline{ac}\;\overline{ad}\;\overline{bc}\;\overline{bd}\;\overline{cd}} =$$
$$[a \vee b + \ldots + c \vee d, (a \vee b + \ldots + b \vee d) \vee \ldots \vee (a \vee c + \ldots + c \vee d)] =$$
$$[\bigvee\{3x + 2y + z \mid x, y, z \in \{a, b, c, d\}, x \neq y \neq z \neq x\},$$
$$\bigvee\{3x + 2y \mid x, y \in \{a, b, c, d\}, x \neq y\}] =$$
$$[a \vee b \vee c \vee d + (a + b) \vee (a + c)\ (a + d) \vee (b + c) \vee (b + d) \vee (c + d)+$$
$$(a + b + c) \vee (a + b + d) \vee (a + c + d) \vee (b + c + d),$$
$$a \vee b \vee c \vee d + 2((a + b) \vee (a + c) \vee (a + d) \vee (b + c) \vee (b + d) \vee (c + d))] =$$
$$[-(a \vee b \vee c \vee d), (a + b) \vee (a + c) \vee (a + d) \vee (b + c) \vee (b + d) \vee (c + d)].$$

The polytop $a \vee b + \ldots + c \vee d$ is a *"tetrakaidekahedron"* represented in [13] chapter VII. The pair $(-(a \vee b \vee c \vee d), (a + b) \vee (a + c) \vee (a + d) \vee (b + c) \vee (b + d) \vee (c + d))$ consisting of a *tetrahedron* and an *octahedron* is minimal.
Similarly, $\underline{\overline{abc}\;\overline{abd}\;\overline{acd}\;\overline{bcd}} = [(a + b) \vee \ldots \vee (c + d), a \vee b \vee c \vee d]$.

(10) Take $\underline{\overline{ab}\;\overline{ac}\,d} = \underline{\underline{\overline{ab}\;\overline{ac}}\,d} = [a \vee b + a \vee c + d, (a \vee b + a \vee c) \vee (d + a \vee b \vee c)]$. Since $(a \vee b + a \vee c + d, (a + b) \vee (a + c) \vee (b + c) + d) \sim (a \vee b + a \vee c, (a + b) \vee (a + c) \vee (b + c)) \sim (d + a \vee b \vee c, d + b \vee c)$.
Then $\underline{\overline{ab}\;\overline{ac}\,d} = [(a + b) \vee (a + c) \vee (b + c) + d, (a + b) \vee (a + c) \vee (b + c) \vee (d + b \vee c)] = [-(a \vee b \vee c), (b \vee c + d \vee a) \vee (b + c)]$.
Similarly $\underline{\overline{ad}\;\overline{bcd}} = [-((b \vee c + d \vee a) \vee (b + c)), a \vee b \vee c]$. These pairs consisting of a *triangle* and a *pyramid* are minimal. Similar pairs represent 24 selections.

(11) $\underline{\overline{ab}\;\overline{ac}\;\overline{ad}\;\overline{bc}\;\overline{bd}} = [a \vee b + a \vee c + a \vee d + b \vee c + b \vee d, (a \vee b + a \vee c + a \vee d + b \vee c) \vee (a \vee b + a \vee c + a \vee d + b \vee d) \vee (a \vee b + a \vee c + b \vee c + b \vee d) \vee (a \vee b + a \vee d + b \vee c + b \vee d) \vee (a \vee c + a \vee d + b \vee c + b \vee d)] = [(a \vee b + a \vee c + b \vee c + a + b) \vee (d + a \vee b + a \vee b + a \vee c + b \vee c) \vee (2d + a \vee b + a \vee c + b \vee c), (a \vee b + a \vee b + a \vee c + b + c) \vee (d + (a \vee b + a \vee c + b \vee c) \vee 3a \vee 3b) \vee (2d + 2a \vee 2b \vee 2c)] = [-(a \vee b \vee c \vee d), (a \vee b + c \vee d) \vee (c + d)]$.
The reader can verify the last equality by computation.
Also, $\underline{\overline{ab}\;\overline{acd}\;\overline{bcd}} = [-((a \vee b + c \vee d) \vee (c + d), a \vee b \vee c \vee d]$. These pairs consist of a *tetrahedron* and a *pyramid*. Similar pairs represent 12 selections.

(12) Take $\underline{\overline{abc}\,d} = [a \vee b \vee c + d, a \vee b \vee c \vee d]$ and $\underline{\overline{ad}\;\overline{bd}\;\overline{cd}} = [-(a \vee b \vee c \vee d), -(a \vee b \vee c) - d]$. These minimal pairs consist of a *triangle* and a *tetrahedron*. Similar pairs represent 8 selections.

(13) $\underline{\overline{ac}\;\overline{bd}} = [a \vee c + b \vee d, a \vee b \vee c \vee d]$ also $\underline{\overline{ab}\;\overline{bc}\;\overline{cd}\;\overline{da}} = [-(a \vee b \vee c \vee d), a \vee c + b \vee d]$. These minimal pair consist of a *square* and a *tetrahedron*. Similar pairs represent 6 selections.

(14) $\underline{\overline{ab}\;\overline{bc}\;\overline{ca}\;\overline{ad}} = \underline{\underline{\overline{ab}\;\overline{bc}\;\overline{ca}}\;\overline{ad}} = [(a + b) \vee (a + c) \vee (b + c) + a \vee d, (a + b) \vee (a + c) \vee (b + c) \vee (a \vee b \vee c + a \vee d)] = [a \vee d - d - (a \vee b \vee c), 2a \vee (a + b) \vee (a + c) \vee (b + c) \vee (a + d) \vee (b + d) \vee (c + d)]$. (See the pair (A, B) of Figure 2.1)
Moreover $\underline{\overline{ab}\;\overline{bc}\;\overline{ca}\;\overline{ad}} = \underline{\underline{\overline{ab}\;\overline{ac}\;\overline{ad}}\;\overline{bc}} = [b \vee c - (a \vee b \vee c \vee d), (b \vee c - (b \vee c \vee d) - a) \vee (-(a \vee b \vee c \vee d)] = [b \vee c - (a \vee b \vee c \vee d), (b - c - a) \vee (b - d - a) \vee (c - b - a) \vee (c - d - a) \vee (-b) \vee (-c) \vee (-d)]$. (See the pair (C, D) of Figure 2.2)
The minimal pairs represented in Figures 2.1 and 2.2 are equivalent and not translations of each other.

Similarly $\underline{\overline{ab}\ \overline{ac}\ \overline{bcd}} = [(a+b)\vee(a+c)\vee(b+c)\vee(a+d)\vee(b+d)\vee(c+d)\vee(-2a), a\vee b\vee c+d-(a\vee d)] = [(c+a-b)\vee(d+a-b)\vee(b+a-c)\vee(d+a-c)\vee b\vee c\vee d, a\vee b\vee c\vee d-(b\vee c)]$. Similar pairs represent 24 selections.

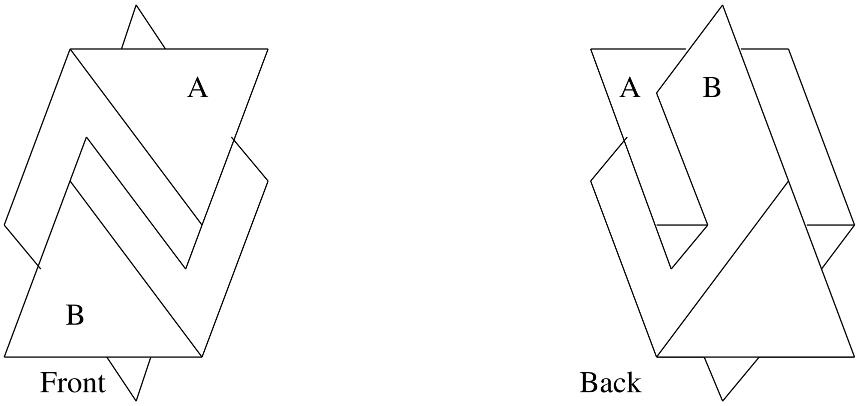

Figure 2.1

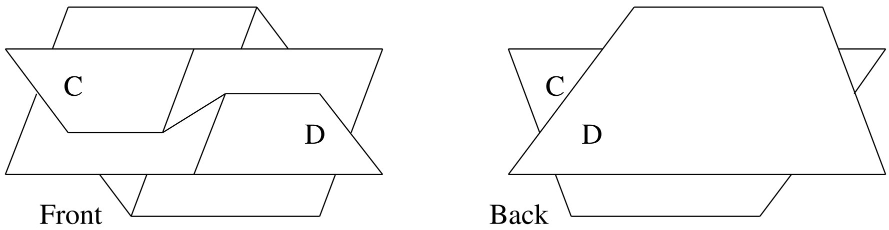

Figure 2.2

(15) Take $\underline{\overline{ab}\ \overline{bc}\ \overline{cd}} = \underline{\underline{\overline{ab}\ \overline{bc}}\ \overline{cd}} = [(a+b)\vee(b+c)\vee(a+c)+c\vee d, (a+b)\vee(b+c)\vee(c+a)\vee 2c\vee(d+c)\vee(d+a)] = [(d\vee c)-d-(a\vee b\vee c), a\vee c+b\vee c\vee d]$. (See the pair (A,B) of Figure 2.3)
On the other hand $\underline{\overline{ab}\ \overline{bc}\ \overline{cd}} = \underline{\overline{dc}\ \overline{cb}\ \overline{ba}} = [a\vee b-a-(b\vee c\vee d), d\vee b+a\vee b\vee c]$. (See the pair (C,D) of Figure 2.4)
The two pairs represented in the Figures 2.3 and 2.4 are equivalent and minimal and they are not translations of each other.
Similar pairs represent 12 selections.

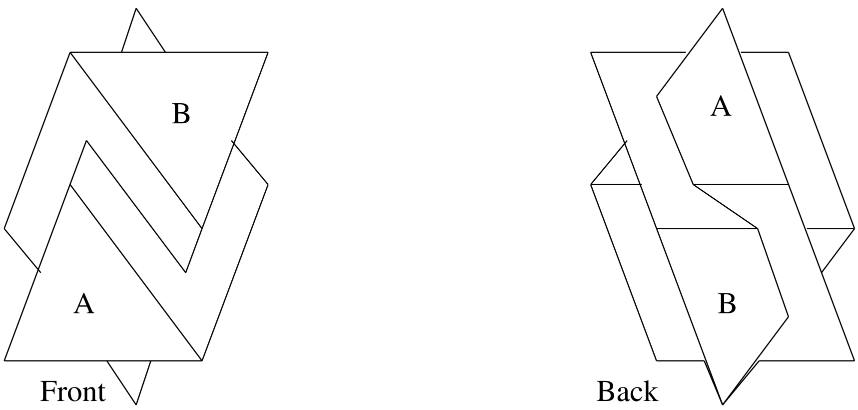

Figure 2.3

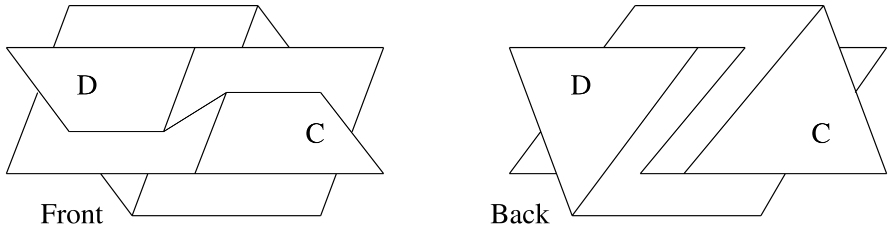

Figure 2.4

(16) $\overline{abc}\ \overline{ad}\ \overline{bd}\ \overline{cd} = [a \vee b \vee c + a \vee d + b \vee d + c \vee d, (a \vee b \vee c + (a+b) \vee (a+c) \vee (b+c)) \vee (d + a \vee b \vee c + a \vee b \vee c) \vee (2d + a \vee b \vee c) \vee 3d] = [(a \vee b \vee c + a + b + c) \vee (d + a \vee b \vee c + (a+b) \vee (a+c) \vee (b+c)) \vee (2d + a \vee b \vee c + a \vee b \vee c) \vee (3d + a \vee b \vee c), (a \vee b \vee c + (a+b) \vee (a+c) \vee (b+c)) \vee (d + a \vee b \vee c + a \vee b \vee c) \vee (2d + a \vee b \vee c) \vee 3d] = [(a \vee b \vee c + a + b + c) \vee (d + a \vee b \vee c + (a+b) \vee (a+c) \vee (b+c)) \vee (2d + a \vee b \vee c + a \vee b \vee c), (a \vee b \vee c + (a+b) \vee (a+c) \vee (b+c)) \vee (d + a \vee b \vee c + a \vee b \vee c) \vee (2d + a \vee b \vee c)] = [(a+b+c) \vee ((a+b) \vee (a+c) \vee (b+c) + d) \vee (2d + a \vee b \vee c), (a+b) \vee (a+c) \vee (b+c) \vee (d + a \vee b \vee c) \vee 2d]$.
A similar pair as in Figure 2.5 was studied in [5] and [11]. The minimal pair (C, D) of Figure 2.6 is an example of a minimal pair which is equivalent to the pair (A, B) of Figure 2.5 and which is not a translation of (A, B). Similar pairs as in Figure 2.6 represent 4 selections.

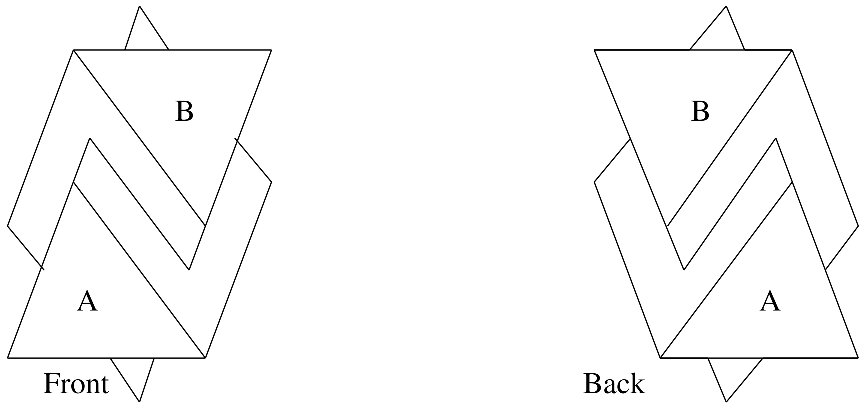

Figure 2.5

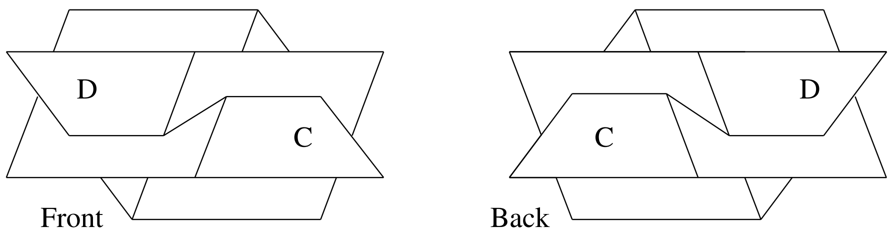

Figure 2.6

□

Remark 2.2. (1) In the cases (1)–(10) and in case (12) of the proof the minimal pairs are uniquely determined except for translations

(2) In the cases (14)–(16) of the proof the minimal pairs are not uniquely determined except for translations

(3) In the remaining cases (11) and (13) of the proof we do not know whether the minimal pairs are unique determined except for translations.

References

[1] A. A. Agrachev, D. Pallaschke, S. Scholtes: On Morse theory for piecewise smooth functions, Journal of Dynamical and Control Systems 3 (1997) 449–469.

[2] S. G. Bartels, L. Kuntz, S. Scholtes: Continuous selections of linear functions and nonsmooth critical point theory, Nonlinear Analysis, Theory, Meth. & Appl. 24 (1995) 385–407.

[3] V. F. Dem'yanov: Quasidifferential Calculus, New York, 1986.

[4] G. Ewald: Combinatorial Convexity and Algebraic Geometry, Springer-Verlag, Berlin et al., 1996.

[5] J. Grzybowski: Minimal pairs of convex compact sets, Arch. Math. 63 (1994) 173–181.

[6] L. Hörmander: Sur la fonction d'appui des ensembles convexes dans un espace localement convexe, Arkiv för Matematik 3 (1955) 181–186.

[7] H. Th. Jongen, P. Jonker, F. Twilt: Nonlinear Optimization in $\mathbb{R}^n$, I. Morse Theory, Chebyshev Approximation, Peter Lang Verlag, Frankfurt et al., 1983.

[8] H. Th. Jongen, D. Pallaschke: On linearization and continuous selections of functions, Optimization 19 (1998) 343–353.

[9] D. Pallaschke, S. Scholtes, R. Urbański: On minimal pairs of convex compact sets, Bull. Polish. Acad. Sci. Math. 39 (1991) 105–109.

[10] D. Pallaschke, R. Urbański: Some criteria for the minimality of pairs of compact convex sets, Zeitschrift für Operations Research 37 (1993) 129–150.

[11] D. Pallaschke, R. Urbański: A continuum of minimal pairs of compact convex sets which are not connected by translation, Journal of Convex Analysis 1 (1996) 83–95.

[12] A. G. Pinsker: The space of convex sets of a locally convex space, Trudy Leningrad Engineering-Economic Institute, 63 (1966) 13–17.

[13] H. Steinhaus: Kalejdoskop Matematyczny, Warszawa, 1989.

[14] R. Urbański: A generalization of the Minkowski-Rådström-Hörmander Theorem, Bull. Acad. Polon. Sér. Sci. Math. 24 (1976) 709–715.

[15] M. Wiernowolski: Minimality in asymmetry classes, Studia Math. 124(2) (1997) 149–154.