The Splendors and Miseries of Martingales

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Over the past sixty years, martingales have become central in the mathematics of randomness. They appear in the general theory of stochastic processes, in the algorithmic theory of randomness, and in some branches of mathematical statistics. Yet little has been written about the history of this evolution. This special issue explores some of the territory that a history of martingales would need to traverse.

The historian of martingales faces an immense task. We can find traces of martingale thinking at the very beginning of probability theory, because this theory began as a study of gambling, and the evolution of a gambler's holdings as a result of following a particular strategy can always be understood as a martingale. More recently, in the second half of the twentieth century, martingales became important in the theory of stochastic processes at the very same time that stochastic processes were becoming increasingly important in statistics and in science.

Moreover, a history of martingales, like a history of any other branch of mathematics, must go far beyond an account of mathematical ideas and techniques. It must explore the context in which the evolution of ideas took place: the broader intellectual milieux of the actors, the networks that already existed or were created by the research, even the social and political

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conditions that favored or hampered the circulation and adoption of certain ideas.

So we do not pretend to treat the history of martingales exhaustively. We present only a stroll through this history, in part a guided tour, in part a random walk. We first present historical studies on the period from 1920 to 1950, when martingales emerged as a distinct mathematical concept. Then we offer insights on the period from 1950 into the 1980s, when the concept showed its value in stochastic processes, mathematical statistics, and algorithmic randomness.

We offer eleven articles. Most were written for this occasion. A few are revisions of articles important for our topic, translated from the French for the first time. These eleven articles vary greatly in tone and purpose, for they treat different aspects of the history of martingales and have authors with a variety of perspectives. As we come closer to the present, and more concerned with ideas that have not yet been fully digested by all who might use them, history is sometimes mixed with exposition. But each of the contributions opens to our view another important landscape within the vast territory traversed by the history of martingales.

The first article, by Roger MANSUY, takes us on a colorful journey into the past of the word *martingale*. As MANSUY shows us, the word's ultimate etymology may remain uncertain, but we can find enjoyment, and surprises, in the different meanings it has taken over the course of time.

In the second article, Bernard BRU, Marie France BRU, and the recently deceased Kai Lai CHUNG recount Emile Borel's encounter with martingales and place it in the context of the life's work of this powerful mathematician, who launched modern probability theory in some respects, and whose vision of its practical importance drove much of its development between the two world wars. Borel first encountered a mathematical definition of martingales when he was nearly 70, in the thesis work of his student and research assistant Jean Ville, and he then used the idea to revisit the St. Petersburg paradox and to continue a debate about the value of probability theory that he had begun nearly forty years earlier with the biologist Félix Le Dantec.

In another article, Bernard BRU and Salah EID introduce us to Børge Jessen and Paul Lévy, whose work in the 1930s anticipated what we now know as Joseph Doob's martingale convergence theorem. Lévy is sometimes considered an inventor of martingales because of his work on dependent random variables. His zero-one law – Lévy's lemma, as BRU and EID call it – contains the central idea of martingale convergence and is also closely related to Jessen's construction of infinite dimensional integrals as the limit of finite dimensional integrals. BRU and EID base their study on a correspondence between the two mathematicians, initiated by Lévy when he discovered the kinship of his work with Jessen's. Their study is particularly important as an introduction to Jessen's early work, too little known because it was in Danish. The appendices to their article include passages from Jessen's theses, translated from the Danish, and Jessen's later correspondence with Doob

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and with Jean Dieudonné.

In our fourth article, Laurent MAZLIAK surveys Lévy's work related to martingales in the 1930s and studies Lévy's relationship with his much younger colleague Ville, who invented martingales in a different way in the same period. It was Ville, not Lévy, who inspired Doob's work and thus launched martingales in probability theory, but Lévy never understood Ville's purposes and never rated his talent highly.

Jean André Ville is a central figure in the history of martingales. His doctoral thesis, where the concept of a martingale was first explicitly formulated, is mentioned in many of the articles in this issue, but both the details of his mathematical accomplishment and the story of his life are relatively little known. How did he come upon the concept of a martingale, and why did he abandon mathematical research on martingales? Glenn SHAFER, who is writing a personal and scientific biography of Ville, offers here an account of Ville's early life and work, from his birth in Marseille in 1910 to his defense of his thesis in 1939.

Our sixth article, by Laurent BIENVENU, Glenn SHAFER, and Alexander SHEN, places martingales in the context where Ville first discovered them: the study of randomness. Ville was inspired by the work of Richard von Mises and Abraham Wald, who wanted to base probability theory on a definition of random sequences, or *collectives*. Von Mises and Wald's collectives were characterized by limiting frequencies that do not change when the sequence is replaced by a subsequence; Ville's collectives were characterized by their resistance to gambling strategies that try to multiply the capital risked by an infinite factor. BIENVENU, SHAFER, and SHEN review both the work of Wald and Ville in the 1930s and the revival of the study of randomness after the discovery, by Andrei Kolmogorov and others in the 1960s, of the concept of algorithmic complexity and its ability to characterize randomness. Here martingales again found their place, especially in the work of Claus-Peter Schnorr and Leonid Levin.

Bernard LOCKER focuses on a specific event: Joseph Leo Doob's presentation of his work on martingales at Lyon in 1948, at a conference organized by Maurice Fréchet. Here Doob demonstrated the power of his results on martingale convergence by showing how easily they deal with the strong law of large numbers and certain problems of statistical estimation. LOCKER reviews the text of Doob's presentation and places it in the context of previous contributions by Lévy, Ville, and Doob himself.

The four remaining articles are written by mathematicians who have pioneered the use of martingales in various arenas. Two of these articles are written by probabilists, and two are written by statisticians.

First, we present an article by the late Paul-André MEYER (1934–2003). Here MEYER reviews the history of the theory of stochastic processes from 1950 into the 1990s. Although he emphasizes the aspects of this development that interested him personally as a working mathematician, he surveys a vast territory, and this survey amply demonstrates the importance of mar-

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tingales. The first step of their development, after Doob had exhibited their power, was the generalization, in the 1950s, of Itô's stochastic integration for Brownian motion. The martingale property, generalized in several directions, especially in the concepts of local martingale and semimartingale, emerged as the fundamental property required for stochastic integration and stochastic differential equations. The second step, in which MEYER and his seminar at Strasbourg were central, was the *general theory of processes* that emerged in the 1960s. It demonstrated both the generality of many ideas that had first been developed for Markov processes and the role of martingales in very general representations of stochastic processes.

Another distinguished probabilist, Shinzo WATANABE, provides some details about how Japanese mathematicians became involved in research in martngales. He begins of course with Kiyosi Itô's early work, but he emphasizes work by the many Japanese researchers who took inspiration from Itô's collaboration with Henry McKean in the 1950s and from their lectures at Kyoto after Itô's return from the United States in 1956. While emphasizing the Japanese contribution, he helps us understand the diffusion of martingale techniques across mathematics as well as across continents.

In the first of our articles by statisticians, Tze Leung LAI reviews the role of martingales in sequential analysis and time series, beginning with Wald's 1945 article on the sequential probability ratio test. LAI worked on martingales and sequential analysis at Columbia University, with Yuan Shih Chow, Herbert Robbins, and David Siegmund. He sees the emergence of martingales in sequential analysis as a process that stretched from 1945 to 1975, with their role in time series analysis emerging later, from 1975 to 1985.

The second article is by a team of statisticians, Odd AALEN, Per Kragh ANDERSEN, Niels KEIDING, Richard GILL and Ørnulf BORGAN, whose work on the applications of martingales to survival analysis began with Aalen's doctoral thesis at Berkeley in 1975. They see the subsequent fifteen years as a period during which the general theory of processes developed by the Strasbourg school allowed the clarification and implementation of intuitive ideas brought into survival analysis by the Mantel-Haenszel test (1959) and David Cox's concept of partial likelihood (1975).

To complete the package, we present a number of significant documents. Some are original texts, some documents from archives, public or personal. Again, some were previously published, but not in English.

First comes a biography of Børge Jessen by Christian Berg, based on the obituary Berg published in Danish right after Jessen's death in 1993. Then a short memoir by Klaus Krickeberg, an important contributor to the theory of martingales after World War II.

Then six documents concerning Jean Ville. They include a partly autobiographical article by Ville and an obituary of Ville by his old friend and colleague from student days, Bernard d'Oregval. Others are translations from archival documents. We call special attention to the presentation of Pierre Crépel's interview and correspondence with Ville in 1984–85, a unique source

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of information about this often elusive individual.

We include two letters, one from Joseph Doob to Pierre Crépel, and one from Paul Lévy to Takeyuki Hida. We conclude with a very brief film, generously provided by Hida. Made by Hida in March 1968, it provides a glimpse of Paul and Suzanne Lévy on the balcony of their apartment in Paris.

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