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Planar embeddability of the vertices of a graph using a fixed point set is NP-hard

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Abstract

Let $G = (V, E)$ be a graph with n vertices and let P be a set of n points in the plane. We show that deciding whether there is a planar straight-line embedding of G such that the vertices V are embedded onto the points P is NP-complete, even when G is 2-connected and 2-outerplanar. This settles an open problem posed in [2, 4, 13].

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1 Introduction

A *geometric graph* H is a graph $G(H)$ together with an injective mapping of its vertices into the plane. An edge of the graph is drawn as a straight-line segment joining its vertices. We use $V(H)$ for the set of points where the vertices of $G(H)$ are mapped to, and we do not make a distinction between the edges of $G(H)$ and H. A *planar* geometric graph is a geometric graph such that its edges intersect only at common vertices. In this case, we say that H is a *geometric planar embedding* of $G(H)$. See [14] for a survey on geometric graphs.

Let P be a set of n points in the plane, and let G be a graph with n vertices. What is the complexity of deciding if there is a straight-line planar embedding of G such that the vertices of G are mapped onto P? This question has been posed as open problem in [2, 4, 13], and here we show that this decision problem is NP-complete. Let us rephrase the result in terms of geometric graphs.

Theorem 1 *Let P be a set of n points, and let G be a graph on n vertices. Deciding if there exists a geometric planar embedding H of G such that $V(H) = P$ is an NP-complete problem.*

The reduction is from 3-partition, a strongly NP-hard problem to be described below, and it constructs a 2-connected graph G. The proof is given in Section 2, and we use that the maximal 3-connected blocks of a 2-connected planar graph can be embedded in different faces. In a 3-connected planar graph, all planar embeddings are topologically equivalent due to Whitney's theorem [9, Chapter 6]. Therefore, it does not seem possible to extend our technique to show the hardness for 3-connected planar graphs.

Related work A few variations of the problem of embedding a planar graph into a fixed point set have been considered. The problem of characterizing what class of graphs can be embedded into any point set in general position (no three points being collinear) was posed in [11]. They showed that the answer is the class of *outerplanar* graphs, that is, graphs that admit a straight-line planar embedding with all vertices in the outer face. This result was rediscovered in [5], and efficient algorithms for constructing such an embedding for a given graph and a given point set are described in [2]. The currently best algorithm runs in $O(n \log^3 n)$ time, although the best known lower bound is $\Omega(n \log n)$.

A tree is a special case of outerplanar graph. In this case, we also can choose to which point the root should be mapped. See [3, 12, 15] for the evolution on this problem, also from the algorithmical point of view. For this setting, there are algorithms running in $O(n \log n)$ time, which is worst case optimal. Bipartite embeddings of trees were considered in [1].

If we allow each edge to be represented by a polygonal path with at most two bends, then it is always possible to get a planar embedding of a planar graph that maps the vertices to a fixed point set [13]. If a bijection between the vertices and the point set is fixed, then we need $O(n^2)$ bends in total to get a planar embedding of the graph, which is also asymptotically tight in the worst case [16].

We finish by mentioning a related problem, which was the initial motivation for this research. A universal set for graphs with n vertices is a set of points S_n such that any planar graph with n vertices has a straight-line planar embedding whose vertices are a subset of S_n. Asymptotically, the smallest universal set is known to have size at least $1.098n$ [6], and it is bounded by $O(n^2)$ [7, 17]. Characterizing the asymptotic size of the smallest universal set is an interesting open problem [8, Problem 45].

2 Planar embeddability is NP-complete

It is clear that the problem belongs to NP: a geometric graph H with $V(H) = P$ and $G(H) \equiv G$ can be described by the bijection between $V(G)$ and P, and for a given bijection we can test in polynomial time whether it actually is a planar geometric graph; therefore, we can take as certificate the bijection between $V(G)$ and P.

For showing the NP-hardness, the reduction is from 3-partition.

> *Problem:* 3-partition
> *Input:* A natural number B, and $3n$ natural numbers $a_1, \ldots, a_{3n}$ with $\frac{B}{4} < a_i < \frac{B}{2}$.
> *Output:* n disjoint sets $S_1, \ldots, S_n \subset \{a_1, \ldots, a_{3n}\}$ with $|S_j| = 3$ and $\sum_{a \in S_j} a = B$ for all S_j.

We will use that 3-partition is a strongly NP-hard problem, that is, it is NP-hard even if B is bounded by a polynomial in n [10]. Observe that because $\frac{B}{4} < a_i < \frac{B}{2}$, it does not make sense to have sets S_j with fewer or more than 3 elements. That is, it is equivalent to ask for subdividing all the numbers into disjoint sets that sum to B. Of course, we can assume that $\sum_{i=1}^{3n} a_i = Bn$, as otherwise it is impossible that a solution exists.

Given a 3-partition instance, we construct the following graph G (see Figure 1):

- Start with a 4-cycle with vertices $v_0, \ldots, v_3$, and edges $(v_{i-1}, v_{i \bmod 4})$. The vertices v_0 and v_2 will play a special role.
- For each a_i in the input, make a path B_i consisting of a_i vertices, and put an edge between each of those vertices and the vertices v_0, v_2.
- Construct $n-1$ triangles $T_1, \ldots, T_{n-1}$. For each triangle T_i, put edges between each of its vertices and v_2, and edges between two of its vertices and v_0. We call each of these structures a separator (the reason for this will become clear later).
- Make a path C of $(B+3)n$ vertices, and put edges between each of the vertices in C and v_0. Furthermore, put an edge between one end of the path and v_1, and another edge between the other end and v_3.

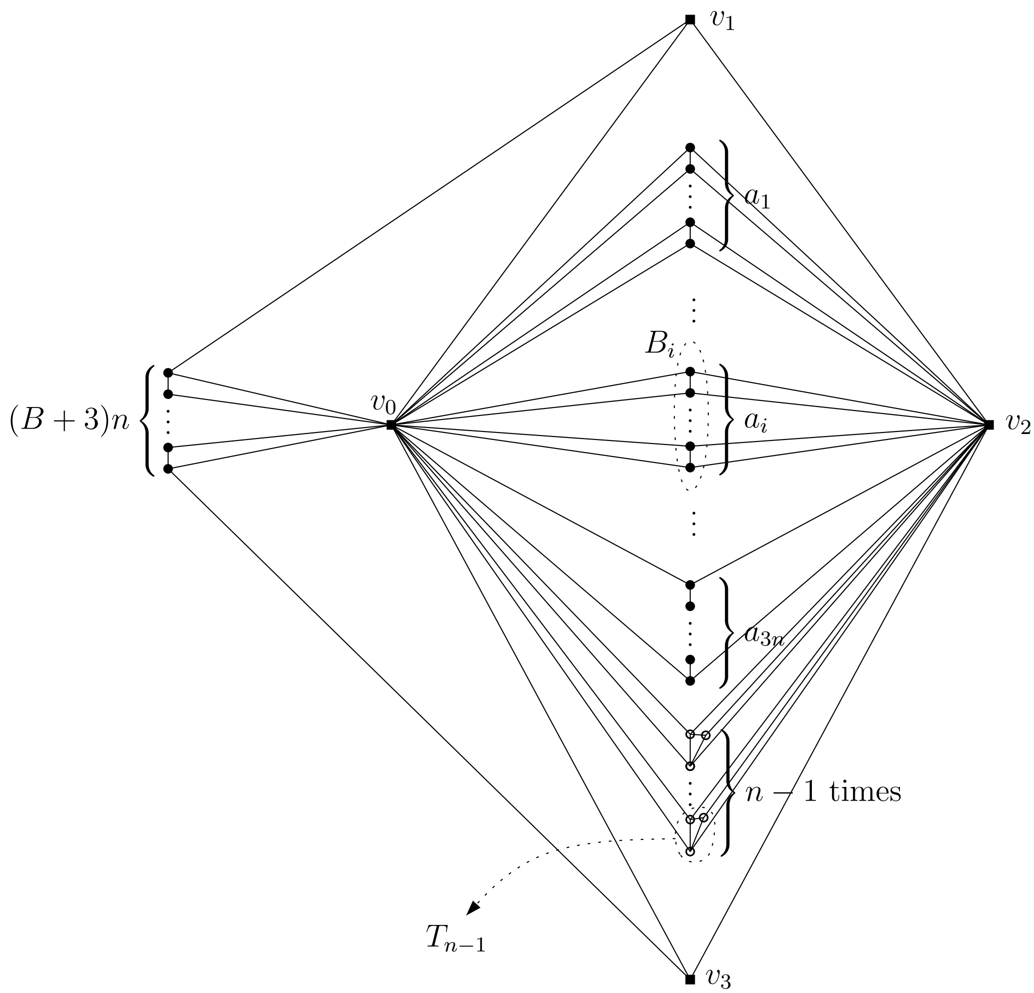

Figure 1: Graph G for the NP-hardness reduction.

It is easy to see that G is planar; in fact, we are giving a planar embedding of it in Figure 1. The idea is to design a point set P such that $G \setminus (B_1 \cup \cdots \cup B_{3n})$ can be embedded onto P in essentially one way. Furthermore, the embedding of $G \setminus (B_1 \cup \cdots \cup B_{3n})$ will decompose the rest of the points into n groups, each of B vertices and lying in a different face. The embedding of the remaining vertices $B_1 \cup \cdots \cup B_{3n}$ will be possible in a planar way if and only if the paths B_i can be decomposed into groups of exactly B vertices, which is equivalent to the original 3-partition instance. The following point set P will do the work (see Figure 2):

- Let $K := (B+2)n$.
- Place $(B+3)n$ points at coordinates $(0,-n),(0,-(n-1)),\dots,(0,-1)$ and at coordinates $(0,1),(0,2),\dots,(0,K)$.
- Place points $p_0 := (1,0), p_1 := (K,K), p_2 := (2K,0), p_3 := (K,-n)$. In the figure, these points are shown as boxes and are labeled.
- For each $i \in \{0,\dots,n-1\}$, place the group of B points $(K,-2+(B+2)i+1),(K,-2+(B+2)i+2),\dots,(K,-2+(B+2)i+B)$.
- For each $i \in \{1,\dots,n-1\}$, place the group of three points $(K,(B+2)i-3),(K,(B+2)i-2),\dots,(K+1,(B+2)i-3)$. In the figure, these points are shown as empty circles.

Let $CH(P)$ be the points in the convex hull of P. Notice that $CH(P)$ consists of the points with x-coordinate equal to 0, and the points p_1, p_2, p_3.

The rest of the proof goes in two steps. Firstly, we will show that, in any geometric planar graph H such that $G(H)$ is isomorphic to G and $V(H) = P$, the vertices of $C \cup \{v_1, v_2, v_3\}$ are mapped to $CH(P)$, and the vertices v_0, v_1, v_2, v_3 are mapped either to p_0, p_1, p_2, p_3, respectively, or to p_0, p_3, p_2, p_1, respectively. In particular, v_0, v_2 are always mapped to p_0, p_2, respectively. Secondly, we will discuss why a mapping of the rest of the vertices, namely vertices in $G \setminus (C \cup \{v_0, v_1, v_2, v_3\})$, onto the rest of the points, namely $P \setminus (CH(P) \cup \{p_0\})$, provides a geometric planar graph if and only if the 3-partition instance has a solution.

For the first step, consider the subgraph $\tilde{G}$ of G induced by the vertices of C and $v_0, \dots, v_3$; see Figure 3. The graph $\tilde{G}$ is a subdivision of a 3-connected graph; consider replacing the path v_1, v_2, v_3 by the edge v_1, v_3. Therefore, because of Whitney's theorem [9, Chapter 6], in any topological planar embedding of $\tilde{G}$, the faces are always induced by the same cycles. In particular, in any planar embedding of $\tilde{G}$ we will get the same faces as shown in Figure 3. Furthermore, in any planar embedding of G, all the vertices of $G \setminus \tilde{G}$ have to be drawn inside the cycle v_0, v_1, v_2, v_3 (notice that this cycle may be the outer face). This means that in any planar embedding of G we have the faces induced by $\tilde{G}$ plus the faces induced by $G \setminus \tilde{G}$ inside the face v_0, v_1, v_2, v_3.

Any planar embedding of G has $Bn + 3(n-1)$ vertices placed inside the cycle v_0, v_1, v_2, v_3, and so any face inside the cycle v_0, v_1, v_2, v_3 has fewer than

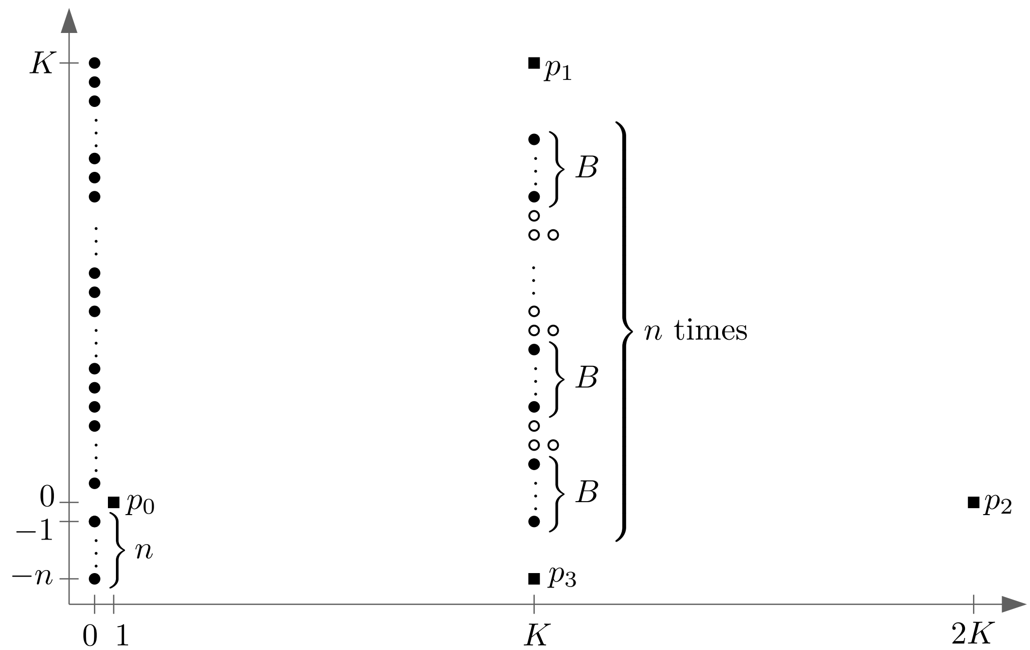

Figure 2: Point set P for the NP-hardness reduction. $K = (B + 2)n$.

$Bn + 3(n - 1) + 4 < (B + 3)n + 3$ vertices. However, we always have a face consisting of $(B + 3)n + 3$ vertices, namely the outer face of $\tilde{G}$ in Figure 3. We conclude that, in any planar embedding of G, there is always a unique face with $(B + 3)n + 3$ vertices, and it is induced by the vertices in $C \cup \{v_1, v_2, v_3\}$.

Recall that $CH(P)$ denotes the points in the convex hull of P, and observe that it consists of exactly $(B + 3)n + 3$ points. In any geometric planar graph, all the points in the convex hull have to be part of the outer face, and therefore, the vertices in $C \cup \{v_1, v_2, v_3\}$ have to be mapped onto the points $CH(P)$.

Next, observe that the vertex v_0 has an edge with each point in $C \cup \{v_1, v_2\}$, and the rest of the vertices have to lie inside one single face, namely v_0, v_1, v_2, v_3. Among the points $P \setminus CH(P)$, only p_0 has this property, and we conclude that the vertex v_0 has to be mapped to the point p_0. Furthermore, the vertices v_0, v_1, v_2, v_3 have to be mapped to either p_0, p_1, p_2, p_3, respectively, or to p_0, p_3, p_2, p_1, respectively. In any case, v_0 is always mapped to the point p_0, and v_2 is mapped to the point p_2.

This concludes the first step of the proof. Figure 4 shows with solid straight segments the part of G which has been already embedded onto P.

For the second step, the only points of P that do not have vertices assigned to them are $\tilde{P} := P \setminus (CH(P) \cup \{p_0\})$. Consider one of the triangles $T_i \subset G$. To realize it, we need to find three points $p, q, r \in \tilde{P}$ such that r is the only point contained in the triangle p_2pq, and the triangle p_0pq contains no points. Only the groups of three adjacent points that are marked with empty circles

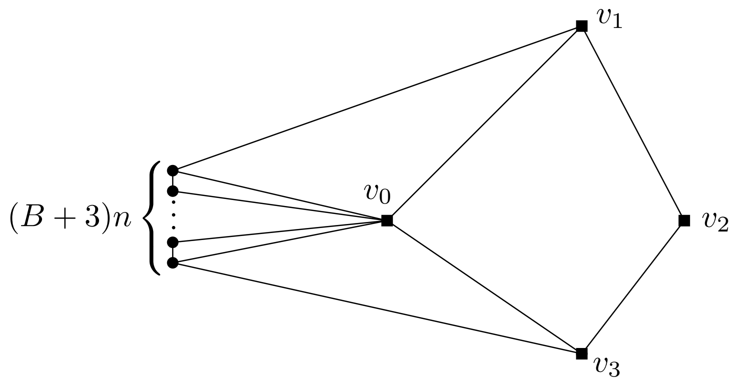

Figure 3: Subgraph $\tilde{G}$ of G induced by C and $v_0, \ldots, v_3$. It is a subdivision of a 3-connected graph; we obtain it by inserting the vertex v_2 in the middle of the edge v_1, v_3.

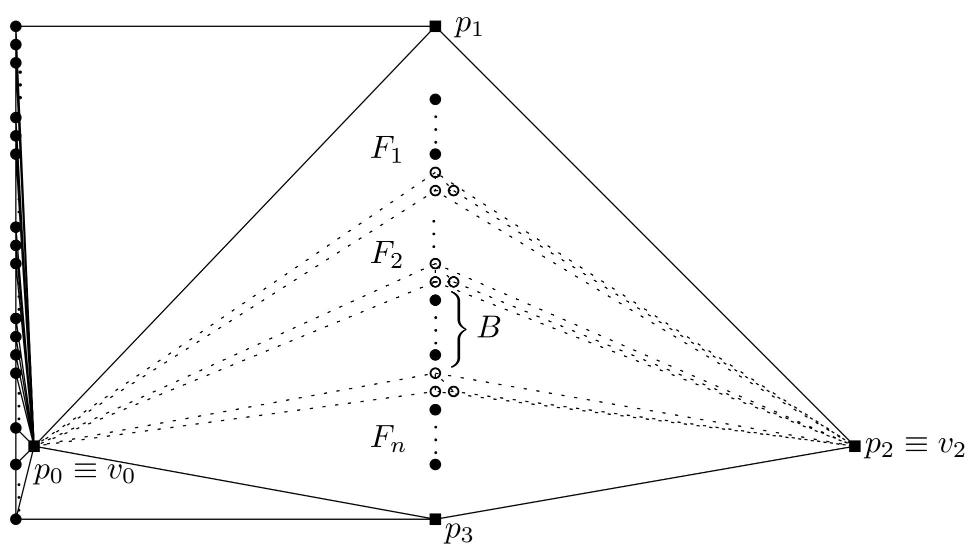

Figure 4: Part of G embedded onto P. The edges with solid segments are embedded at the end of the first step. We can have either $p_1 \equiv v_1$ and $p_3 \equiv v_3$, or $p_1 \equiv v_3$ and $p_3 \equiv v_1$. The edges with dotted segments are the separators placed during the second step. They induce the faces $F_1, \ldots F_n$.

have this property, that is, the separators that were added in the last item when generating the point set P. There are $n-1$ triangles T_i, and $n-1$ separators, so each triangle gets assigned to one separator. The induced edges are drawn with dotted segments in Figure 4.

We are left with n faces $F_1, \dots, F_n$, each containing exactly B points; see Figure 4. For a geometric planar embedding H of G having $V(H) = P$, each of the paths B_i has to lie completely inside some face F_{j_i} . Therefore, a geometric planar embedding of G is possible if and only if $B_1, \dots, B_{3n}$ can be arranged in groups such that each of the groups has exactly B vertices. However, because $|V(B_i)| = a_i$, such a geometric planar embedding of G is possible if and only if the 3-partition instance has a solution.

Because 3-partition is NP-hard even when B is bounded by a polynomial in n, the graph G has a polynomial number of vertices, and the point set P also has a polynomial number of points. Furthermore, the coordinates of the points in P are bounded by polynomials and the whole reduction can be done in polynomial time. This finishes the proof of Theorem 1. □

3 Concluding remarks

The point set P that we have constructed has many collinear points. However, in the proof we have not used this fact, and so it is easy to modify the reduction in such a way that no three points of P are collinear. Probably, the easiest way for keeping integer coordinates is replacing each of the points lying in a vertical line by points lying in a parabola, and adjusting the value K accordingly. Therefore, the result remains valid even if P is in general position, meaning that no 3 points are collinear.

In the proof, the graph G that we constructed is 2-outerplanar, as shown in Figure 1. k-outerplanarity is a generalization of outerplanarity that is defined inductively. A planar embedding of a graph is *k-outerplanar* if removing the vertices of the outer face produces a $(k-1)$-outerplanar embedding, where 1-outerplanar stands for an outerplanar embedding. A graph is k-outerplanar if it admits a k-outerplanar embedding. For (1-)outerplanar graphs, the embedding problem is polynomially solvable [2], but for 2-outerplanar we showed that it is NP-complete. Therefore, regarding outerplanarity, our result is tight.

The graph G that we constructed in the proof is 2-connected; removing the vertices v_1, v_3 disconnects the graph. As mentioned in the introduction, we have strongly used this fact in the proof because in a 2-connected graph the maximal 3-connected blocks can flip from one face to another one. Therefore, it would be interesting to find out the complexity of the problem when the graph G is 3-connected, and more generally, the complexity when the topology of the embedding is specified beforehand.

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