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On Alliance Partitions and Bisection Width for Planar Graphs

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Abstract

An alliance in a graph is a set of vertices (allies) such that each vertex in the alliance has at least as many allies (counting the vertex itself) as non-allies in its neighborhood of the graph. We show how to construct infinitely many non-trivial examples of graphs that can not be partitioned into alliances and we show that any planar graph with minimum degree at least 4 can be split into two alliances in polynomial time. We base this on a proof of an upper bound of n on the bisection width for 4-connected planar graphs with an odd number of vertices. This improves a recently published $n + 1$ upper bound on the bisection width of planar graphs without separating triangles and supports the folklore conjecture that a general upper bound of n exists for the bisection width of planar graphs.

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1 Introduction

An *alliance* is a set of vertices (allies) such that any vertex in the alliance has at least as many allies (including the vertex itself) as non-allies in its neighborhood of the graph. The alliance is said to be *strong* if this holds even without including the vertex itself among the allies. Alliances of vertices in graphs [13] are used to model among other things alliances of individuals or nations and alliances appear many places in the literature under various names: Flake et al. [9] refer to a strong alliance as a *community* and base their work on the assumption that web pages related to each other form communities in the web graph. Gerber and Kobler [11] look at what they refer to as the *Satisfactory Graph Partition Problem* where the objective is to partition a graph into two strong alliances. A partition of a graph into strong alliances can also be viewed as a so called *Nash stable partition* of an *Additive Hedonic Game* [15]. As mentioned above, alliances have been used to model scenarios that might be planar in nature, so in this paper we focus on the problem of partitioning a *planar* graph into two alliances. In Section 2 we show how to compute such a partition in polynomial time for any planar graph with minimum degree at least 4. To prove this, we need an upper bound of n on the bisection width of 4-connected planar graphs with an odd number of vertices. We prove this upper bound in Section 3. This tight upper bound is an improvement over the recently published [8] $n+1$ upper bound for planar graphs without separating triangles, and it supports the folklore conjecture [8], that a general upper bound of n exists for the bisection width of planar graphs.

1.1 Preliminaries

Consider the connected graph G with vertex set V and edge set E where $|V| = n$ and $|E| = m$. The degree $d(v)$ of a vertex v in G is the number of edges incident to v in G. Similarly, for a subset $X \subseteq V$ we define the degree $d_X(v)$ of a vertex v in the subgraph of G induced by $X \cup \{v\}$ as $d_X(v) = |\{u \in X : \{v,u\} \in E\}|$. We denote the minimum degree of the vertices in G as δ and the maximum degree as Δ. A graph G is *k-connected* when at least k vertices are required to be removed in order to disconnect G. A d-regular graph is a graph where all vertices have degree d. A *clique* is a fully connected graph and a *maximal planar graph* is a planar graph with the property that the addition of any new edge destroys planarity. An *alliance* in G is a non empty set $A \subseteq V$ such that $\forall u \in A : d_A(u) + 1 \geq d_{V-A}(u)$. This definition can be tightened giving the notion of a *strong alliance* which is a non empty set $A \subseteq V$ such that $\forall u \in A : d_A(u) \geq d_{V-A}(u)$. A *partition* of G is a collection of non-empty disjoint subsets $V_1 \dots V_l$ of V such that $\bigcup_{i=1}^{l} V_i = V$. For a partition of G into two subsets V_1 and V_2 we will denote the set of edges crossing this partition as $e(V_1, V_2) = \{\{u,v\} \in E : u \in V_1 \wedge v \in V_2\}$. A *bisection* of G is a partition of G into V_1 and V_2 such that $||V_1| - |V_2|| \leq 1$ and the *bisection width* of G is defined as the minimum $|e(V_1, V_2)|$ over all bisections. Throughout this paper when considering a planar graph, we will implicitly consider an embedding of the graph. A *separating triangle* in a planar graph is a triangle where both the interior and the exterior are non-empty.

1.2 Related Work

The problem of partitioning a graph into two strong alliances is NP-hard if we put no restrictions on the graph [2]. There are however classes of graphs for which we can decide whether a partition into strong alliances exists and compute it in polynomial time. Examples of such classes are graphs with maximum degree at most 4 and graphs with girth at least 5 and minimum degree at least 3 [2, 3]. Shafique [12] presents among other things results telling precisely when line graphs can be partitioned into alliances.

For a general graph G, the computational complexity of partitioning G into two alliances is an open problem [4]. Fricke et al. [10] show that any graph G contains an efficiently computable alliance with no more than $\lceil \frac{n}{2} \rceil$ vertices, while the problem of deciding whether an alliance with less than k members exists in G is NP-complete if k is part of the input. This even holds if G is planar [7].

Fan et al. [8] prove an upper bound of $n + 1$ for the bisection width for planar graphs without a separating triangle and an upper bound on $n - 2$ for the bisection width for any triangle-free planar graph. The latter upper bound has subsequently been improved by Li et al. [14] for triangle-free planar graphs. Diks et al. [6] show how to obtain a partition V_1, V_2 with $\min(|V_1|, |V_2|) \geq \frac{n}{3}$ and no more than $2\sqrt{2\Delta n}$ crossing edges in linear time for any planar graph.

2 Alliances in Planar Graphs

Our main aim of this section is to show that a partition into two alliances exists for any planar graph with minimum degree at least 4 and that this partition can be computed in polynomial time. Before doing this in Section 2.2 we present more generic results on alliance partitions where we among other things describe non-trivial infinite families of graphs that can *not* be partitioned into alliances.

2.1 Generic Observations on Alliance Partitions

If all the vertices in a graph have an odd degree then the graph can be efficiently partitioned into two alliances by using the decomposition technique by Bazgan et al. [3]. On the other hand, it is easy to see that any clique with an odd number of vertices can not be partitioned into alliances. Inspired by this fact we now present an observation containing a recipe for constructing infinitely many less trivial examples of graphs that cannot be partitioned into alliances:

Observation 1 *Let $G(V, E)$ be a graph and let B_1, B_2, …, B_l be a partition of V for some $l \geq 2$. The graph G can* not *be partitioned into alliances if the following conditions hold:*

$$|B_1 \cup B_2| \text{ is odd} \tag{1}$$

$$\forall u \in B_1 : d(u) = d_{B_1 \cup B_2}(u) = |B_1 \cup B_2| - 1 \tag{2}$$

$$\forall i \geq 2 \forall u \in B_i : d_{B_{i-1}}(u) > d_{V \setminus B_{i-1}}(u) + 1 \tag{3}$$

Proof: Now assume that V_1, V_2 is a partition of V into alliances. The conditions (1) and (2) imply that B_1 must be fully contained in V_1 or in V_2 – otherwise there would be at least one member of B_1 with too few allies since $|B_1 \cup B_2|$ is odd. The condition (3) implies that all vertices in B_i are members of the alliance containing B_{i-1} for $i \geq 2$ and we arrive at a contradiction since this leaves us with an empty alliance. □

Figure (1a) shows an example of a graph satisfying the conditions of Observation 1 for $l = 3$ where the B-sets are indicated by the different colors. Bazgan et al. [4] note that $K_{3,3,3}$ shown in Figure (1b) can not be partitioned into alliances. The following observation shows among other things how to construct a d-regular graph that is not a clique and impossible to partition into alliances for any even $d \geq 6$. As an example of such a graph we mention the 8-regular graph with 11 vertices defined by having a complement graph consisting of two triangles and a 5-cycle.

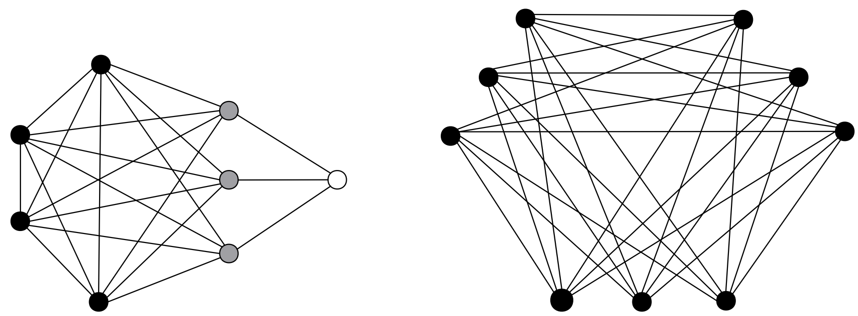
Observation 2 *Let $G(V, E)$ be a regular graph with an odd number of vertices and $d(u) = n - 3$ for all $u \in V$. The graph G can be partitioned into two alliances if and only if G contains a clique with $\lfloor \frac{n}{2} \rfloor$ vertices. Checking whether such a clique exists can be done in polynomial time.*

Proof: If G contains a clique $C \subset V$ with $\lfloor \frac{n}{2} \rfloor$ vertices then this clique is an alliance. The complement graph $\bar{G}$ is 2-regular so there are $2 \lfloor \frac{n}{2} \rfloor = n - 1$ edges in $\bar{G}$ with exactly one endpoint in C and consequently one edge in $\bar{G}$ with both endpoints in $V \setminus C$. We therefore conclude that $V \setminus C$ is an alliance as well.

Now let us on the other hand assume that G does not contain a clique with $\lfloor \frac{n}{2} \rfloor$ vertices. In this case it is not hard to see that we can not have an alliance with $\lfloor \frac{n}{2} \rfloor$ or fewer vertices.

A clique is an independent set in the complement graph and it is easy to find the size of the biggest independent set in a 2-regular graph in polynomial time. □

We now turn our attention to graphs for which alliance partitions exist. In Lemma 1 we characterize a group of general graph partitions that can be refined into an alliance partition using a polynomial time algorithm. In Section 2.2 we will use the lemma to obtain a partition into alliances for planar graphs with minimum degree 4 in polynomial time. Lemma 1 is a precise formulation of the well known principle [4, 10] that a partition into two sets of vertices forms a good starting point for obtaining a partition into alliances if the number of crossing edges is relatively small compared to the cardinality of the smallest set of vertices.



(a) A graph satisfying the conditions of Observation 1 for $l = 3$ where the B-sets are indicated by the different colors.

(b) The graph $K_{3,3,3}$.

Figure 1: Examples of graphs that can not be partitioned into alliances.

Lemma 1 *A graph G can be partitioned into two alliances if there exists a partition V_1, V_2 of G such that*

$$|e(V_1, V_2)| - 2\min(|V_1|, |V_2|) < \delta - 2 \ . \tag{4}$$

The alliances can be computed in polynomial time if V_1 and V_2 can be obtained in polynomial time.

Proof: Let V_1, V_2 be a partition of G satisfying (4). We now run the following simple algorithm:

1. Let $A_1 = V_1$ and $A_2 = V_2$.
2. If A_1 and A_2 both are alliances or if one of them is empty we stop. Otherwise we go to step 3.
3. Assume that A_1 is not an alliance (otherwise we process A_2 similarly). There must be a $u \in A_1$ with $d_{A_1}(u) + 1 < d_{A_2}(u)$. We now move u from A_1 to A_2 and go to step 2.

The number of crossing edges $|e(A_1, A_2)|$ decreases with 2 or more every time step 3 is executed so the algorithm must stop after no more than $\frac{m}{2}$ steps. Assume that the algorithm stops because A_1 is empty and let u be the last vertex to leave A_1. We now consider the point in time where $A_1 = \{u\}$:

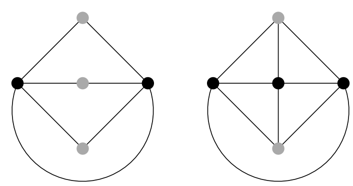
$$d_V(u) = |e(A_1, A_2)| \leq |e(V_1, V_2)| - 2(\min(|V_1|, |V_2|) - 1) \ .$$

We obtain a contradiction since (4) implies that the right hand side is less than δ. We conclude that the algorithm cannot stop with A_1 or A_2 being empty. It has to stop with A_1 and A_2 being alliances. □

For one application of this lemma we consider some sparse graphs: The expected number of $|e(V_1, V_2)|$ is slightly bigger than $\frac{m}{2}$ for a random bisection if n is big. So m just has to be slightly less than $2n$ to ensure a partition into alliances computable in polynomial time using derandomization. If a big graph has average degree a little below 4 it can thus be partitioned into alliances. We omit the technical details on this observation in this paper.

2.2 Alliance Partitions of Planar Graphs

We now prove that all planar graphs with minimum degree at least 4 allow a partition of the vertices into two alliances and that this partition can be computed in polynomial time. This is trivially also true for planar graphs with minimum degree 1 (let one alliance consist of a single vertex with degree 1), while for planar graphs with minimum degree 2 and 3 we can use Observation 1 and construct examples of graphs that can not be partitioned into alliances. Figure 2 shows two examples of such graphs where the black vertices and the gray vertices form the sets B_1 and B_2 from Observation 1 respectively. It is worth noting that we do not guarantee the alliance partition to be a bisection.



(a) (b)

Figure 2: Examples of planar graphs that can not be partitioned into alliances. The black vertices and the gray vertices form the sets B_1 and B_2 from Observation 1 respectively.

We are now ready to state and prove the main result of this section.

Theorem 1 *Any planar graph with $\delta \geq 4$ can be partitioned into two alliances in polynomial time.*

Proof: We start by expanding the graph by adding edges until it is a maximal planar graph which can be done in polynomial time. We now consider two cases:

The expanded graph has a separating triangle: A separating triangle has vertices both inside and outside of the triangle. Let V_1 be the vertices on the side of the triangle containing the fewest vertices and let $V_2 = V \setminus V_1$. There can be no more than one vertex in V_1 having edges to all three vertices in the separating triangle so $|e(V_1, V_2)| \leq 2|V_1| + 1$. This inequality also holds in the original graph so we can now use Lemma 1. The detection and processing of the separating triangle case is easily done in polynomial time.

The expanded graph does not have a separating triangle: In this case the graph is 4-connected since all maximal planar graphs without a separating triangle are 4-connected [5] and thus contains a hamiltonian cycle computable in linear time [1]. Fan et al. [8] show how to efficiently compute a bisection V_1, V_2 of V with $|e(V_1, V_2)| \leq n + 1$ for such a graph. This makes it possible for us to apply Lemma 1 in the case where *n is even* but for *n odd* an upper bound on n for the bisection width is needed to make inequality (4) hold. In Section 3 we prove Theorem 3 stating the existence of an efficiently computable bisection V_1, V_2 with $|e(V_1, V_2)| \leq n$ for any 4-connected planar graph $G(V, E)$ with an odd number of vertices. We now use Lemma 1 in the case where n is odd. □

As mentioned above, Fricke et al. [10] have shown that any graph contains an alliance with no more than $\lceil \frac{n}{2} \rceil$ members. We can now improve this upper bound for planar graphs with $\delta \geq 4$ since such graphs can be partitioned into two alliances that cannot both have more than $\lfloor \frac{n}{2} \rfloor$ members:

Corollary 1 *Any planar graph with $\delta \geq 4$ contains an alliance with no more than $\lfloor \frac{n}{2} \rfloor$ members.*

A planar graph with $\Delta < \frac{1}{18}n$ can be partitioned into two alliances in polynomial time.

Theorem 2 *A planar graph can be partitioned into two alliances in polynomial time if $\Delta < \frac{1}{18}n$.*

Proof: For $\delta = 1$ the case is clear. For $\delta > 1$ we use the work of Diks et al. [6] and obtain a partition V_1, V_2 of G with $|e(V_1, V_2)| \leq 2\sqrt{2\Delta n}$ and $\min(|V_1|, |V_2|) \geq \frac{n}{3}$ in linear time. Finally, we use Lemma 1. □

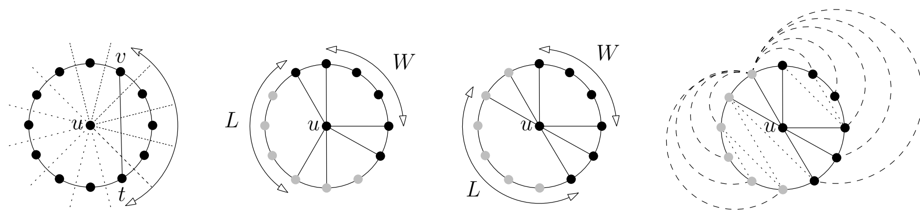
A similar result holds even for *strong* defensive alliances.

3 An Upper Bound for the Bisection Width

We now show that a bisection V_1, V_2 with $|e(V_1, V_2)| \leq n$ can be computed in polynomial time for any 4-connected planar graph with an odd number of vertices. Some of the techniques used are similar to the techniques used by Fan et al. [8] but we also use other techniques and the analysis is considerably more complicated compared to the analysis of Fan et al. Since the bisection width never increases when removing edges from a graph, it is sufficient to only consider maximal 4-connected planar graphs with an odd number of vertices.

Lemma 2 *A maximal 4-connected planar graph with an odd number of vertices has a vertex u with $d(u) \geq 5$ such that $G - u$ is Hamiltonian. The vertex u and the hamiltonian cycle of $G - u$ can be found in polynomial time.*

Proof: Consider a maximal 4-connected planar graph G with an odd number of vertices. There is at least one vertex u in G with $d(u) \geq 5$ since otherwise we would have $\sum_{v \in V} d(v) = 2m = 2(3n-6) \leq 4n$ that could only happen if $n \leq 5$ which would contradict 4-connectedness. The graph G is 4-connected so the graph $G-u$ has a Hamiltonian cycle computable in polynomial time as showed by Thomas and Yu [16]. □



(a) (b) (c) (d)

Figure 3: Illustrations of a configuration. Figure (a) shows the hamiltonian cycle with its k hamiltonian bisections (the dotted lines) and the cycle length of edge $\{v,t\}$. Figure (b) shows a single hamiltonian bisection where the vertices are colored according to which side of the bisection they belong to. Also, it shows L and W for the configuration. Figure (c) shows a compacted neighbor configuration with L and W. Figure (d) shows a heavy compacted neighbor configuration where the dotted edges are the inner edges of the configuration and the dashed edges are the outer edges of the configuration. We point out that the graph in figure (d) is not 4-connected but meant as an illustration.

Let G be a maximal 4-connected graph with an odd number of vertices and let u be a vertex in G with $d(u) \geq 5$ and C a Hamiltonian cycle in $G-u$. We will say that the tuple (G,u,C) represents a *configuration* of G. For any such configuration, there are exactly $k = \lfloor \frac{n}{2} \rfloor$ different ways to split C into two connected and equally sized parts. From any such split into two parts, we construct a *hamiltonian bisection* V_1, V_2 of G by adding u to the part where it has the most neighbors i.e. the part that minimizes $|e(V_1,V_2)|$ (ties are broken arbitrarily). Refer to Figure 3(a) and 3(b). In the following we let $T(G,u,C)$ denote the sum of $|e(V_1,V_2)|$ over the k possible hamiltonian bisections of (G,u,C). We will sometimes omit the arguments if they are clear from the context. The *cycle length* of an *edge* $\{v,t\}$ in $G-u$ is the minimum distance between v and t in the graph induced by the cycle. The contribution to $T(G,u,C)$ of an edge in $G-u$ is precisely the cycle length of the edge. Refer to Figure 3(a). We let L denote the length of the longest path along C starting and ending at a neighbor from u but visiting no other neighbors of u and let W denote the length of the second longest such path. Refer to Figure 3(b).

We refer to the configuration (G,u,C) as a *compacted neighbor configuration* if the neighbors of u can be divided into two subsets N_1 and N_2 of size $\left\lfloor \frac{d(u)}{2} \right\rfloor$ and $\left\lceil \frac{d(u)}{2} \right\rceil$ respectively such that each subset occupies a connected subpath of the hamiltonian cycle C. Refer to Figure 3(c). The *inner edges* are the edges on the same side of C as u. The inner edges that are not incident to u are naturally grouped into (at most) two groups in a compacted neighbor configuration. A compacted neighbor configuration is called *heavy* if the edges from both these groups have cycle lengths $2, 3, 4, \ldots, k, k-1, k-2, k-3, \ldots$ (for both groups we start the sequence from the left) and if the set of *outer edges* has two edges of length 2, two edges of length 3, ... , two edges of length $k-1$ and one edge of length k. Refer to Figure 3(d).

In what follows, we will show that $T(G, u, C) < k(n+1)$ for any configuration (G, u, C) of a maximal 4-connected planar graph with an odd number of vertices. Since $T(G, u, C)$ is the sum of bisection sizes for the k hamiltonian bisections this implies that there exists at least one hamiltonian bisection V_1, V_2 such that $|e(V_1, V_2)| \leq n$ which then gives us the upper bound on the bisection width. Fan et al. [8] use the same strategy to prove the $n+1$ upper bound for the bisection width for planar graphs without a separating triangle but Fan et al. consider a Hamiltonian cycle in G where we consider a Hamiltonian cycle in $G-u$ making the analysis considerably more complicated.

To prove $T(G, u, C) < k(n+1)$ we will first show that the heavy compacted neighbor configurations can be considered as a set of worst case configurations such that for any configuration (G, u, C) there exists a heavy compacted neighbor configuration (G', u', C') where $T(G, u, C) \leq T(G', u', C')$. We then exploit that the heavy compacted neighbor configurations are simple enough that $T(G', u', C') < k(n+1)$ can be shown for this set of configurations.

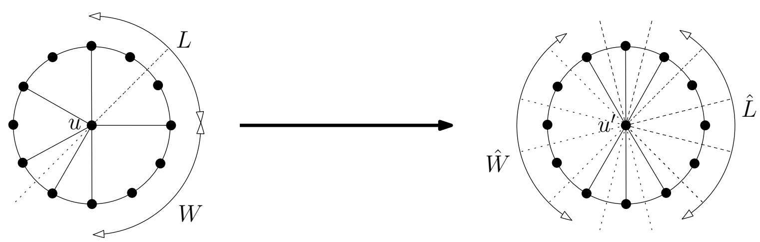


Figure 4: In the configuration (G, u, C) to the left, the hamiltonian bisections where edges incident to u contribute with $\lfloor d(u)/2 \rfloor$ are shown with dotted lines. Similarly, in the configuration $(\hat{G}, u', C)$ to the right, the hamitonian bisections where edges incident to u' contribute with $\lfloor d(u')/2 \rfloor$ are shown with dotted lines.

Lemma 3 *If (G, u, C) is a configuration then it is possible to construct a heavy compacted neighbor configuration (G', u', C) where G and G' have the same number of vertices and $d(u) = d(u')$ such that $T(G, u, C) \leq T(G', u', C)$.*

Proof: Let (G, u, C) represent an arbitrary configuration. We now remove those edges in G that are not on C and not incident to u. We then replace u (and the edges incident to u) with a vertex u' with $d(u) = d(u')$ such that the resulting configuration $(\hat{G}, u', C)$ is a compacted neighbor configuration. Finally, we put in edges to create the graph G' such that (G', u', C) is a heavy

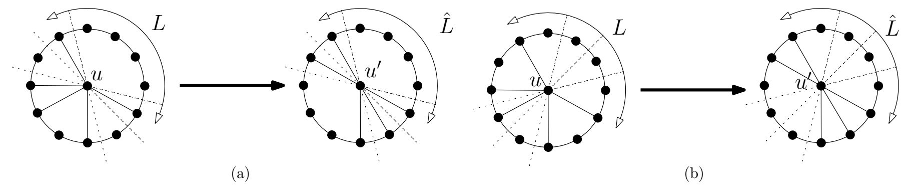


Figure 5: In (a) we show those hamiltonian bisections which fully contain the vertices on the cycle path corresponding to L in (G, u, C) (to the left) and in $(\hat{G}, u', C)$ (to the right). In (b) we show those hamiltonian bisections which does not fully contain the vertices in (G, u, C) (to the left) and in $(\hat{G}, u', C)$ (to the right).

compacted neighbor configuration. Below, we first argue that the contribution to T of edges incident to u in G is not higher than the contribution to T of edges incident to u' in G'. Secondly, we argue that the contribution to T of edges in $G - u$ is not higher than the contribution to T of edges in $G' - u'$.

Edges incident to u': We separate our analysis into a case analysis based on the value of L in G. The values of L and W in $\hat{G}$ are denoted by $\hat{L}$ and $\hat{W}$ respectively.

Case 1: $L \leq k$: We consider the following subcases:

- If $2L + d(u) - 2 \leq 2k$ we build the compacted neighbor configuration $(\hat{G}, u', C)$ such that $\hat{L} - \hat{W}$ is minimized (0 or 1). Refer to Figure 4. The contribution to T of edges incident to u' is $k \left\lfloor \frac{d(u)}{2} \right\rfloor$ which is the maximum obtainable value since u' always chooses to join the partition which contributes the least to T. Refer to Figure 4. The condition $2L + d(u) - 2 \leq 2k$ makes it possible for us to obtain the $k \left\lfloor \frac{d(u)}{2} \right\rfloor$ contribution to T from edges incident to u' and at the same time obtain $\hat{L} \geq L$ and $\hat{W} \geq W$ that is important when we consider the contribution from the other edges.
- If $2L + d(u) - 2 > 2k$ we build the compacted neighbor configuration $(\hat{G}, u', C)$ such that $L = \hat{L}$ and such that the vertices forming the long paths in G and $\hat{G}$ with length L along C with no neighbors of u or u' respectively are the same. Refer to Figure 5. For each of the k hamiltonian bisections in $(\hat{G}, u', C)$ we now show that the number of crossing edges incident to u' has not decreased compared to the corresponding (same partition of C) hamiltonian bisection in (G, u, C).
  - We first consider a bisection V_1, V_2 where the vertices on the path along C of length $L = \hat{L}$ is fully contained within either V_1 or V_2 – say V_1. In this case, u' must choose to join V_2. The

number of neighbors of u' in V_1 is at least as high as the number of neighbors of u in V_1 in G so the number of crossing edges for such a bisection has not decreased. Refer to Figure 5(a).

- We now consider a bisection where the vertices on the path along C of length $L = \hat{L}$ are not fully contained within either side of the bisection. When u' has chosen a side of the bisection u' has only crossing edges to members of either N_1 or N_2 (the two groups of neighbors of u'). If u' has $\left\lfloor \frac{d(u')}{2} \right\rfloor$ crossing edges the case is clear. Otherwise, the number crossing edges has not dropped since every vertex on the other side of the cut and not on the long path is a neighbor to u'. Refer to Figure 5(b).

Case 2: $L > k$: We build the compacted neighbor configuration $(\hat{G}, u', C)$ with $L = \hat{L}$. Consider a bisection V_1, V_2 of $(\hat{G}, u', C)$. When u' chooses side of the bisection u' can not have crossing edges to both N_1 and N_2. If there are no crossing edges the same would be the case for the corresponding bisection of the original configuration. Refer to Figure 6(a). The neighbors of u' are packed around the path with $\hat{L} - 1$ consecutive vertices on the cycle that are not neighbors of u' so if there are crossing edges then the number of neighbors on the other side cannot have decreased. Refer to Figure 6(b).

Edges not incident to u': Since C is in both G and G' the edges on C obviously contribute with the same to T. We now consider the edges in G not incident to u and not on C and the edges of G' not incident to u' and not on C. We will refer to these sets of edges as the G-set and the G'-set respectively. We will now argue that a one-to-one correspondence between the two sets of edges exists such that any edge in the G-set is matched with an edge in the G'-set with the same cycle length or a bigger cycle length.

The first thing we do is to consider the inner edges in the G-set that go between $L+1$ consecutive vertices on the cycle with only the first and last vertex being a neighbor of u. Fan et al. [8] show how to eliminate any triangle of such edges and obtain a new set of edges with higher cycle lengths by replacing some of the edges and Fan et al. also argue that repeated elimination of triangles will produce a heavy configuration – we refer to [8] for more details. In this way we are able to match any edge in the considered subset of the G-set with an edge in the G'-set with the same or a bigger cycle length but with a cycle length not exceeding L. We now repeat this argument where we consider the inner edges in the G-set that go between $W + 1$ other consecutive vertices on the cycle with only the first and last vertex being a neighbor of u. By doing this we are also able to match these edges with edges in the G'-set with the same or a bigger cycle length but with a cycle length not exceeding W. The remaining unmatched edges in the G'-set all have cycle length at least W which makes it possible to match the remaining inner edges in the G-set with an edge in the G'-set with the same cycle length or a bigger cycle length. We repeat the procedure for the outer edges as well. The contribution to T of the edges not incident to u' can consequently not decrease during the transformation since all the edges in the G-set have been matched with an edge in the G'-set with the same cycle length or a bigger cycle length. □

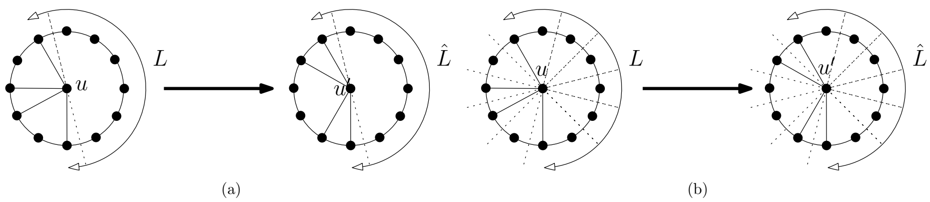


Figure 6: In (a) we illustrate the case where there are no crossing edges for a hamiltonian bisection in G and the corresponding bisection of $\hat{G}$. In (b) we show the bisections where there are crossing edges in which case the number of crossing edges can not have decreased in $\hat{G}$.

Lemma 4 *Let (G, u, C) be a heavy compacted neighbor configuration with $d(u)$ even. The contribution to $T(G, u, C)$ of the edges incident to u is*

$$\frac{d(u)^2}{4} + W\frac{d(u)}{2} - \frac{d(u)}{2} \ .$$

Proof: We group the edges incident to u into pairs such that a pair of edges cuts C into two pieces with the same number of neighbors of u. For a given hamiltonian bisection the contribution to $T(G, u, C)$ of a pair is 1 if the endpoints of the edges are separated and 0 otherwise. The shortest route along the cycle between two vertices in a pair contains $(W-1) + \frac{d(u)-2}{2}$ vertices between the two vertices. There are consequently $W + \frac{d(u)}{2} - 1$ bisections that separate each pair so it is now easy to compute the contribution to $T(G, u, C)$ of the edges incident to u:

$$\frac{d(u)}{2}(W + \frac{d(u)}{2} - 1) \ .$$

□

Lemma 5 *If (G, u, C) is a heavy compacted neighbor configuration then we have the following:*

$$T(G, u, C) < k(n+1) \ .$$

Proof:

We divide the proof into three cases.

Assume that $L \geq k-1$ and that $d(u)$ is even: We compute T in the following way:

$$T = 2k + \left(k + 2\sum_{i=2}^{k-1} i\right) + \left(k + 2\sum_{i=2}^{k-1} i - \sum_{i=1}^{d(u)-3} (W+i)\right) + \left(\frac{d(u)^2}{4} + W\frac{d(u)}{2} - \frac{d(u)}{2}\right) .$$

The first term is the sum of cycle lengths from the edges on the cycle, the second term is the sum of cycle lengths for the outer edges, the third term is the sum of cycle lengths for the inner edges not incident to u, and the fourth term is the contribution from edges incident to u given by Lemma 4. We now use $\sum_{i=2}^{k-1} i = \left(\frac{(k-1)k}{2} - 1\right)$ and $n = 2k+1$:

$$T - k(n+1) = \left(\frac{d(u)^2}{4} + W\frac{d(u)}{2} - \frac{d(u)}{2}\right) - \sum_{i=1}^{d(u)-3} (W+i) - 4 \ .$$

We now work on a part of this sum multiplied by 4 in order to exclusively have integers in the computation:

$$4\left(\left(\frac{d(u)^2}{4} + W\frac{d(u)}{2} - \frac{d(u)}{2}\right) - \sum_{i=1}^{d(u)-3} (W+i)\right)$$

$$= (d(u)-2)d(u) + 2Wd(u) - 4(d(u)-3)W - 2(d(u)-3)(d(u)-2)$$

$$= -d(u)^2 + 8d(u) - 12 + 12W - 2Wd(u) = -(d(u)-6)(d(u)-2) + 12W - 2Wd(u)$$

$$= -d(u)^2 + 8d(u) - 12 + 12W - 2Wd(u) = -(d(u)-6)(d(u)-2+2W) \ ,$$

and finally we get

$$T - k(n+1) = -\frac{(d(u)-6)(d(u)-2+2W)}{4} - 4 < 0 \ , \qquad (5)$$

where we have used that the degree of u is at least 6.

Now assume that $L \leq k-2$ and that $d(u)$ is even: In this case we get

$$T = \sum_{i=2}^{L} i + \sum_{i=2}^{W} i + \frac{d(u)^2}{4} + W\frac{d(u)}{2} - \frac{d(u)}{2} + 2k + k^2 - 2$$

$$= \frac{L(L+1)}{2} - 1 + \frac{W(W+1)}{2} - 1 + \frac{d(u)^2}{4} + W\frac{d(u)}{2} - \frac{d(u)}{2} + 2k + k^2 - 2$$

implying

$$4T - 4k(2k+2)$$

$$= 2L(L+1) - 8 + 2W(W+1) + d(u)^2 + 2Wd(u) - 2d(u) + 8k + 4k^2 - 8 - 4k(2k+2)$$

$$= 2L(L+1) + 2W(W+1) + d(u)(d(u)+2W-2) - 4k^2 - 16 \ .$$

We now use $W + L - 2 + d(u) = 2k$:

$$4T - 4k(2k+2) = 2L(L+1) + 2W(W+1) + (2k+2-W-L)(2k+W-L) - 4k^2 - 16$$

$$= 3L^2 + W^2 - 4kL + 4W + 4k - 16. \tag{6}$$

We now use $L \geq W$ in (6):

$$4T - 4k(2k+2) \leq 4L^2 + (4-4k)L + 4k - 16$$
$$= 4((L-1)(L-k+2) - 2)\ .$$

implying

$$T - k(n+1) \leq (L-1)(L-k+2) - 2 < 0 \text{ for } L \in \{1, 2, \ldots, k-2\}\ .$$

Now assume that $d(u)$ is odd: We remove the edge of u from the group with $\left\lceil \frac{d(u)}{2} \right\rceil$ edges that is closest to the path along the cycle corresponding to W. It is not hard to see that the contribution to T of the edges of u is unchanged after the removal of this edge. For $d(u) > 5$ there is consequently a heavy compacted neighbor configuration considered above with a higher value of T compared to the original graph. Now consider the case $d(u) = 5$. We now remove an edge of u as described above implying no change in the contribution to T of the edges of u and *insert* an edge with cycle length $W + 1$ and obtain a heavy compacted neighbor configuration with $d(u) = 4$. We can now use (5) with $d(u) = 4$ and W replaced by $W + 1$ to compute T for this configuration. Since we have inserted an edge with cycle length $W + 1$ we have to subtract $W + 1$ to obtain the value for T for the original configuration with $d(u) = 5$:

$$T - k(n+1) = -\frac{(4-6)(4-2+2(W+1))}{4} - 4 - (W+1) = -3 < 0\ .$$

□

We are now ready to present the main theorem of this section:

Theorem 3 *A bisection V_1, V_2 exists with $|e(V_1, V_2)| \leq n$ for any 4-connected planar graph $G(V, E)$ with an odd number of vertices and such a bisection can be obtained in polynomial time.*

Proof: Let $G(V, E)$ be a 4-connected planar graph with an odd number of vertices. As noted earlier, we can assume that G is a maximal planar graph without loss of generality. Lemma 2 assures that we can efficiently obtain a configuration (G, u, c). We now examine all the k hamiltonian bisections of the configuration. By using Lemma 3 and Lemma 5 we know that at least one of the hamiltonian bisections satisfies $|e(V_1, V_2)| \leq n$. □

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