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Drawing Graphs on Few Circles and Few Spheres

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Abstract

Given a drawing of a graph, its *visual complexity* is defined as the number of geometrical entities in the drawing, for example, the number of segments in a straight-line drawing or the number of arcs in a circular-arc drawing (in 2D). Recently, Chaplick et al. [GD 2016] introduced a different measure for the visual complexity, the *affine cover number*, which is the minimum number of lines (or planes) that together cover a crossing-free straight-line drawing of a graph G in 2D (3D). In this paper, we introduce the *spherical cover number*, which is the minimum number of circles (or spheres) that together cover a crossing-free circular-arc drawing in 2D (or 3D). It turns out that spherical covers are sometimes significantly smaller than affine covers. For complete, complete bipartite, and platonic graphs, we analyze their spherical cover numbers and compare them to their affine cover numbers as well as their segment and arc numbers. We also link the spherical cover number to other graph parameters such as treewidth and linear arboricity.

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1 Introduction

A drawing of a given graph can be evaluated by many different quality measures depending on the concrete purpose of the drawing. Classical examples are the number of crossings, the ratio between the lengths of the shortest and the longest edge, or the angular resolution. Clearly, different layouts (and layout algorithms) optimize different measures. Hoffmann et al. [15] studied ratios between optimal values of quality measures implied by different graph drawing styles. For example, there is a circular-arc drawing of the icosahedron with perfect angular resolution (that is, the edges are equiangularly spaced around each vertex), whereas the best straight-line drawing has an angular resolution of at most 15°, which yields a ratio of $72°/15° = 4.8$. Hoffmann et al. also constructed a family of graphs whose straight-line drawings have unbounded edge–length ratio, whereas there are circular-arc drawings with edge–length ratios arbitrarily close to 3 [15, Figures 4 and 6].

A few years ago, a new type of quality measure was introduced: the number of geometric objects that are needed to draw a graph given a certain style. Schulz [23] coined this measure the *visual complexity* of a drawing. More concretely, Dujmović et al. [7] defined the *segment number* $\mathrm{seg}(G)$ of a graph G to be the minimum number of straight-line segments over all straight-line drawings of G. Similarly, Schulz [23] defined the *arc number* $\mathrm{arc}(G)$ with respect to circular-arc drawings of G and showed that circular-arc drawings are an improvement over straight-line drawings not only in terms of visual complexity but also in terms of area consumption; see Schulz [23, Theorem 1]. Mondal et al. [20] showed how to minimize the number of segments in convex drawings of 3-connected planar graphs both on and off the grid. Igamberdiev et al. [17] fixed a bug in the algorithm of Mondal et al. and compared the resulting algorithm to two other algorithms in terms of angular resolution, edge length, and face aspect ratio. Hültenschmidt et al. [16] studied the visual complexity of drawings of planar graphs. For example, they showed upper bounds for the number of segments and arcs in drawings of trees, triangulations, and general planar graphs. Recently, Kindermann et al. [18] presented a user study showing that people without mathematical or computer science background prefer drawings that consist of few line segments, that is, drawings of low visual complexity. (Users with such a background had a slight tendency to prefer drawings that are more symmetric.) The study, however, was done for trees only.

Durocher et al. [9] investigated the complexity of computing minimum-segment drawings (and related problems). Among others, they showed that it is NP-hard to compute the segment number of plane graphs (that is, planar graphs with fixed embedding), even if the graphs have maximum degree 4. As an open problem, the authors suggested to study *minimum-line drawings*, which they define to be minimum-segment drawings whose edges lie in the union of the smallest number of straight lines (among all minimum-segment drawings).

Chaplick et al. [4] defined a similar quality measure, which they call the *affine cover number*. Given a graph G and two integers l and d with $0 < l < d$, they defined $\rho^l_d(G)$ to be the minimum number of l-dimensional affine subspaces that together cover a crossing-free straight-line drawing of G in d-dimensional space. It turned out that it suffices to consider $l \leq 2$ because otherwise $\rho^l_d(G) = 1$. In [4] the authors also show that every graph can be drawn in 3-space as effectively as in high dimensional spaces, i.e., for any integers $1 \leq l \leq 3 \leq d$ and for any graph G, it holds that $\rho^l_d(G) = \rho^l_3(G)$. Note that, in general, the minimum-line drawings mentioned above are different from ρ^1_2-optimal drawings since there are graphs that do not have a ρ^1_2-optimal cover with the minimum number of segments; see Example 1 in Section 7.

Among others, Chaplick et al. showed that the affine cover number can be asymptotically smaller than the segment number, constructing an infinite family of triangulations $(T_n)_{n>1}$ such that T_n has n vertices and $\rho^1_2(T_n) = O(\sqrt{n})$, but $\text{seg}(T_n) = \Omega(n)$. On the other hand, they showed that $\text{seg}(G) = O(\rho^1_2(G)^2)$ for any connected planar graph G. In a companion paper [5], Chaplick et al. show that most variants of the affine cover number are NP-hard to compute.

Our contribution. Combining the approaches of Schulz and Chaplick et al., we introduce the *spherical cover number* $\sigma^l_d(G)$ of a graph G to be the minimum number of l-dimensional spheres in $\mathbb{R}^d$ such that G has a crossing-free circular-arc drawing that is contained in the union of these spheres. Note that $\sigma^1_2(G)$ is defined for planar graphs only.

Firstly, we provide some basic observations and preliminary results that our work heavily relies on.

We obtain bounds for the spherical cover number σ^2_3 of the complete and complete bipartite graphs which show that spherical covers can be asymptotically smaller than affine covers; see Table 1 and Section 3.

Then we turn to platonic graphs, that is, to 1-skeletons of platonic solids; see Section 4. These graphs possess several nice properties: they are regular, planar and Hamiltonian. We use them as indicators to compare the above-mentioned measures of visual complexity; we provide bounds for their segment and arc numbers (see Table 2) as well as for their affine and spherical cover numbers (see Table 3). For the upper bounds, we present straight-line drawings with (near-) optimal affine cover number ρ^1_2 and circular-arc drawings with optimal spherical cover number σ^1_2; see Figures 4–6. We note that sometimes optimal spherical covers are more symmetric than optimal affine covers. For example, it seems that there is no symmetric drawing of the cube that is ρ^1_2-optimal, whereas there are symmetric σ^1_2-optimal drawings; see Fig. 4.

For general graphs, we present lower bounds for the spherical cover numbers by means of many combinatorial graph characteristics, in particular, by the edge-chromatic number, treewidth, balanced separator size, linear arboricity, and bisection width; see Section 5.

We decided to start with our more concrete (and partially stronger) results and postpone the structural observations to Section 5, although this means that we'll sometimes have to use forward references to Theorem 3, our main result in Section 5. Finally, we formulate an integer linear program (MIP) that yields lower bounds for the segment number of embedded planar graphs; see Section 6.

For the platonic solids, the lower bounds (see Table 4) that we computed using the MIP turned out to be tight. We conclude with a few open problems.

2 Preliminary Results

In this section we state some preliminary results. Firstly, we note that any drawing with straight-line segments and circular arcs can be transformed into a drawing that uses circular arcs only.

Proposition 1 *Given a graph G and a drawing Γ of G that represents edges as straight-line segments or circular arcs on r l-dimensional planes or spheres in $\mathbb{R}^d$, there is a circular-arc drawing Γ' of G on r l-dimensional spheres in $\mathbb{R}^d$. In particular, $\sigma^l_d(G) \le \rho^l_d(G)$ for any graph G and $1 \le l < d$.*

Proof: Take an arbitrary sphere $S \subset \mathbb{R}^d$ that does not intersect any of the r spheres or planes that support the given drawing Γ of G. Without loss of generality, assume that S is centered at the origin. This implies that none of the spheres supporting Γ goes through the origin. Let ρ be the radius of S. Invert the drawing with respect to S by the map $x \mapsto \rho x/\|x\|$. The resulting drawing is a circular-arc drawing of G on r l-dimensional spheres in $\mathbb{R}^d$. Indeed, using basic properties of the inversion (see, for instance, [10] or [3, Chapter 5.1]), it can be proved that this inversion transforms planes into spheres of the same dimension and preserves spheres, in other words, the set of images of points on a sphere forms another sphere of the same dimension. □

Therefore, we may consider any line a "circle of infinite radius", any plane a "sphere of infinite radius", and any affine cover a spherical cover. By "line" we always mean a straight line.

Trivial bounds on $\sigma^1_3(G)$ follow from the fact that every circle is contained in a plane and that we have more flexibility when drawing in 3D than in 2D. Note again that $\sigma^1_2(G)$ and $\sigma^1_3(G)$ are only defined when G is planar.

Proposition 2 *For any graph G, it holds that $\rho^2_3(G) \le \sigma^1_3(G)$. If G is planar, we additionally have $\sigma^1_3(G) \le \sigma^1_2(G)$.*

The spherical cover number $\sigma^2_3(G)$ can be considered a characteristic of a graph G that lies between its *thickness* $\theta(G)$, which is the smallest number of planar graphs whose union is G, and its *book thickness* $\mathrm{bt}(G)$, also called page number, which is the minimum number of pages (halfplanes) needed to draw the edges of G when the vertices lie on the *spine* of the book (the line that bounds all halfplanes).

Proposition 3 *For every graph G, it holds that $\theta(G) \le \sigma^2_3(G) \le \lceil \mathrm{bt}(G)/2 \rceil$.*

Proof: Each sphere covers a planar subgraph of G, so $\sigma^2_3(G)$ is bounded from below by $\theta(G)$. On the other hand, given a book embedding of a graph G with the minimum number of pages (equal to $\mathrm{bt}(G)$), we put the vertices from the spine along a circle which is the common intersection of $\lceil \text{bt}(G)/2 \rceil$ spheres; see Fig. 1a. Then, for each page, we draw all its edges as arcs onto a hemisphere. Thus, we obtain a drawing witnessing $\sigma_3^2(G) \leq \lceil \text{bt}(G)/2 \rceil$. □

To bound $\sigma_2^1(G)$ and $\sigma_3^1(G)$ for the platonic solids in Section 4 from below we use a combinatorial argument similar to that in Lemma 7(a) and Lemma 7(b) in [4] which is based on the fact that each vertex of degree at least 3 must be covered by at least two lines and two lines can cross at most once, therefore, providing a lower bound on the number of lines given the number of vertices. We use a similar argument together with the fact that two circles can cross at most twice.

Proposition 4 *For any integer $d \geq 1$ and any graph G with n vertices and m edges, the following bounds hold:*

(a) $\sigma_d^1(G) \geq \frac{1}{2}\left(1 + \sqrt{1 + 2\sum_{v \in V(G)} \left\lceil \frac{\deg v}{2} \right\rceil \left(\left\lceil \frac{\deg v}{2} \right\rceil - 1 \right)}\right)$;

(b) $\sigma_d^1(G) \geq \frac{1}{2}\left(1 + \sqrt{2m^2/n - 2m + 1}\right)$ *for any graph G with $m \geq n \geq 1$;*

3 Complete and Complete Bipartite Graphs

In this section we investigate the spherical cover numbers of complete graphs and complete bipartite graphs. We first cover these graphs by spheres then by circular arcs, in 3D (and higher dimensions).

Theorem 1

(a) For any $n \geq 3$, it holds that $\lfloor (n+7)/6 \rfloor \leq \sigma_3^2(K_n) \leq \lceil n/4 \rceil$.

(b) For any $1 \leq p \leq q$, it holds that $pq/(2p+2q-4) \leq \sigma_3^2(K_{p,q}) \leq p$ and, if additionally $q > p(p-1)$, it holds that $\sigma_3^2(K_{p,q}) = \lceil p/2 \rceil$.

Proof: (a) By Proposition 3, $\theta(K_n) \leq \sigma_3^2(K_n) \leq \lceil \text{bt}(K_n)/2 \rceil$. It remains to note that, e.g., Duncan [8] showed that $\theta(K_n) \geq \lfloor (n+7)/6 \rfloor$ and Bernhart and Kainen [2] showed that $\text{bt}(K_n) = \lceil n/2 \rceil$.

(b) Again, it suffices to bound the values of the graph's thickness and book thickness. It can be easily shown that $\text{bt}(K_{p,q}) \leq \min\{p, q\}$. On the other hand, Harary [14, Section 7, Theorem 8] showed that $\theta(K_{p,q}) \geq pq/(2p+2q-4)$. Due to Proposition 3, $\theta(K_{p,q}) \leq \sigma_3^2(K_{p,q}) \leq \min\{p, q\} \leq p$. In particular, if $q > p(p-1)$ then $\text{bt}(K_{p,q}) = p$, due to Bernhart and Kainen [2, Theorem. 3.5], and $\lceil pq/(2p+2q-4) \rceil = \lceil p/2 \rceil$, so in this case $\sigma_3^2(K_{p,q}) = \lceil p/2 \rceil$. □

Theorem 1 implies that any n-vertex graph G has $\sigma_3^2(G) \leq \lceil n/4 \rceil$.

On the other hand, given a graph G, we can bound $\sigma_3^1(G)$ from below in terms of the *bisection width* $\text{bw}(G)$ of G, that is, the minimum number of edges between the two sets (W_1, W_2) of a *bisection* of G that is, a partition of the vertex set $V(G)$ of G into two sets W_1 and W_2 with $|W_1| = \lceil n/2 \rceil$ and $|W_2| = \lfloor n/2 \rfloor$.

Proposition 5 *For any graph G and $d \geq 2$, it holds that $\sigma_d^1(G) \geq \mathrm{bw}(G)/2$.*

Proof: The proof is similar to the proof in Theorem 9(a) in [4]. It is based on the fact that for any finite set of points in $\mathbb{R}^d$, there is a hyperplane that bisects the point set into two almost equal subsets (that is, one subset may have at most one point more than the other). Given a drawing of G with σ arcs, a hyperplane bisecting $V(G)$ can cross at most 2σ edges since a hyperplane can cross an arc at most twice. □

Next we analyze the bisection width of the complete (bipartite) graphs.

Proposition 6 *For any n, p, and q, $\mathrm{bw}(K_n) = \lfloor n^2/4 \rfloor$ and $\mathrm{bw}(K_{p,q}) = \lceil pq/2 \rceil$.*

Proof: Let (W, W') be a bisection of K_n such that $|W| = \lfloor n/2 \rfloor$. Then the width of this bisection is $\lfloor n^2/4 \rfloor$.

Now let $P \cup Q = V(K_{p,q})$ be the bipartition of $K_{p,q}$, and let (W, W') be a bisection of $K_{p,q}$ that contains r vertices from P and s vertices from Q (with $r + s = \lfloor (p+q)/2 \rfloor$). Then the width of this bisection is $r(q-s) + s(p-r)$. The minimum of this value can be found by a routine calculation of the minimum of a quadratic polynomial on the grid over the possible values of r and s. □

Theorem 2 *For any positive integers n, p, and q, it holds that*

(a) $\lfloor n^2/8 \rfloor \leq \sigma_3^1(K_n) \leq (n^2 + 5n + 6)/6$ *and*

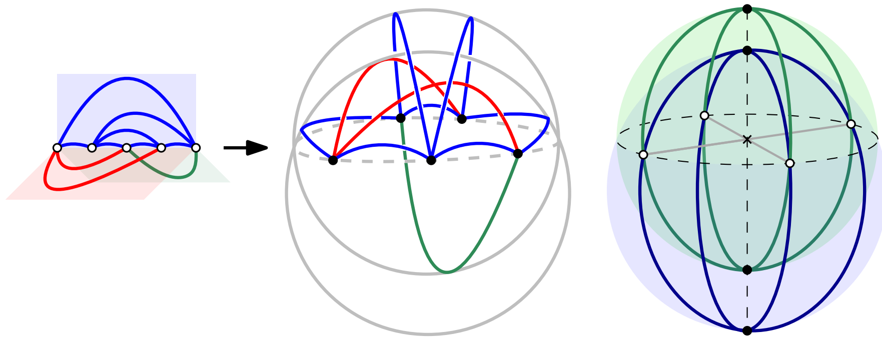
(b) $\lceil pq/4 \rceil \leq \sigma_3^1(K_{p,q}) \leq \lceil p/2 \rceil \lceil q/2 \rceil$.

Proof: The lower bounds follow from Proposition 5 and 6.

To show the upper bound for $\sigma_3^1(K_n)$, we use a partition of K_n into (mutually edge-disjoint) subgraphs of K_3 (that is, copies of K_3, paths of length 2, and single edges). Using Steiner triple systems, one can show that $(n^2 + 5n + 6)/6$ subgraphs suffice [4, Theorem 12]. For distinct points a, b, and c, let $L(a, b)$ be the line through a and b and let $C(a, b, c)$ be the (unique) circle through a, b, and c. For $n \leq 3$, it is clear how to draw K_n. For $n \geq 4$, we iteratively construct a set P of n points in $\mathbb{R}^3$ satisfying the following conditions:

- no four distinct points of P are coplanar,
- for any five distinct points $p_1, \ldots, p_5 \in P$, it holds that $C(p_1, p_2, p_5) \cap C(p_3, p_4, p_5) = \{p_5\}$ and $L(p_1, p_2) \cap C(p_3, p_4, p_5) = \emptyset$.
- for any six distinct points $p_1, \ldots, p_6 \in P$, it holds that $C(p_1, p_2, p_3) \cap C(p_4, p_5, p_6) = \emptyset$.

It can be checked that these conditions forbid only a so-called *nowhere dense set* of $\mathbb{R}^3$ to place the next point of P, so we can always continue. Finally, we map the vertices of K_n to the distinct points of the set P. Consider our partition of K_n into subgraphs of K_3. Each subgraph of K_3 with at least two edges uniquely determines a circle or a circular arc, which we draw. For each subgraph that consists of a single edge, we draw the line segment that connects



(a) $\sigma_3^2(K_5) \le \lceil \theta(K_5)/2 \rceil = 2$ (b) $\sigma_3^2(K_{4,4}) \le 2$

Figure 1: Upper bounds for the spherical cover number of complete (bipartite) graphs.

the two vertices. The above conditions ensure that the drawings of no two subgraphs have a crossing.

The upper bound for $\sigma_3^1(K_{p,q})$ can be seen as follows. Let $p' = \lceil p/2 \rceil \ge p/2$ and $q' = \lceil q/2 \rceil \ge q/2$. Draw a bipartite graph $K_{2p',2q'} \supset K_{p,q}$ in 3D as follows; see Fig. 1b. Let $V(K_{2p',2q'}) = P \cup Q$ be the natural bipartition of its vertices. Fix any family of p' distinct spheres with a common intersection circle. Place the $2q'$ vertices of Q on q' distinct pairs of antipodal points on the circle. Consider a line going through the center of the circle and orthogonal to its plane. Place the $2p'$ vertices of P into p' pairs of distinct intersection points of the line with the circles of the family, the points from each pair belonging to the same sphere. Now each pair of antipodal points in Q together with each pair of cospheric points in P determine a unique circle that contains all these points and provides a drawing of the four edges between them. The union of all these circles is the desired drawing of $K_{2p',2q'}$ onto $p'q'$ circles. □

We remark that Proposition 3 and all the bounds for 3D in this section also hold for higher dimensions.

Table 1 summarizes the known bounds for the affine cover numbers [4] and the new bounds for the spherical cover numbers of complete (bipartite) graphs in 3D.

4 Platonic Graphs

In this section we analyze the segment numbers, arc numbers, affine cover numbers, and spherical cover numbers of platonic graphs. We provide upper bounds via the corresponding drawings; see Figures 2–6.

To bound the spherical cover numbers σ_2^1 and σ_3^1 of the platonic graphs from below, we use a single combinatorial argument—Proposition 4(a); see Section 5. For the affine cover number ρ_2^1, a similar combinatorial argument fails [4,

G	K_n	$K_{p,q}$	references
$\rho^1_3(G)$	$\binom{n}{2}$	$pq - \lfloor \frac{p}{2} \rfloor - \lfloor \frac{q}{2} \rfloor$	[4, Expl. 10 & 25(c)]
$\rho^2_3(G)$	$\frac{n^2-n}{12} \sim \frac{n^2+5n+6}{6}$	$\left\lceil \frac{\min\{p,q\}}{2} \right\rceil$	[4, Thm. 12, Expl. 11]
$\sigma^1_3(G)$	$\left\lfloor \frac{n^2}{8} \right\rfloor \sim \frac{n^2+5n+6}{6}$	$\lceil \frac{pq}{4} \rceil \sim \lceil \frac{p}{2} \rceil \lceil \frac{q}{2} \rceil$	Theorem 2
$\sigma^2_3(G)$	$\lfloor \frac{n+7}{6} \rfloor \sim \lceil \frac{n}{4} \rceil$	$\left\lceil \frac{pq}{2(p+q-2)} \right\rceil \sim \left\lceil \frac{\min\{p,q\}}{2} \right\rceil$	Theorem 1

Table 1: Lower and upper bounds on the three-dimensional line, plane, circle, and sphere cover numbers of K_n for any $n \geq 1$ and of $K_{p,q}$ for any $p, q \geq 3$. The cells with only one entry contain tight bounds.

Lemma 7(a)]. Therefore, we bound ρ^1_3 (and, hence, also ρ^1_2) from below for each platonic graph individually; see Proposition 7. For an overview of our results, see Tables 2 and 3. We abbreviate every platonic graph by its capitalized initial; for example, C for the cube.

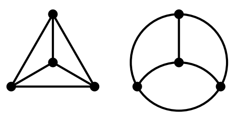
Proposition 7 *(a)* $\rho^1_3(T) \geq 6$*; (b)* $\rho^1_3(O) \geq 9$*; (c)* $\rho^1_3(C) \geq 7$*; (d)* $\rho^1_3(D) \geq 9$*; (e)* $\rho^1_3(I) \geq 13$.

Proof. (a) Follows from [4, Ex. 10].

(b) Consider a straight-line drawing of the octahedron O covered by a family $\mathcal{L}$ of ρ lines. Observe that every vertex of the octahedron is adjacent to every other except the opposite vertex. Therefore, no line in $\mathcal{L}$ can cover more than three vertices, otherwise the edges on the line would overlap. Hence, every line covers at most two edges, and these must be adjacent. Moreover, the two end vertices of these length-2 paths cannot be adjacent. Since there are only three pairs of such vertices, at most three lines cover two edges each. Since the octahedron has twelve edges, $\rho \geq 9$.

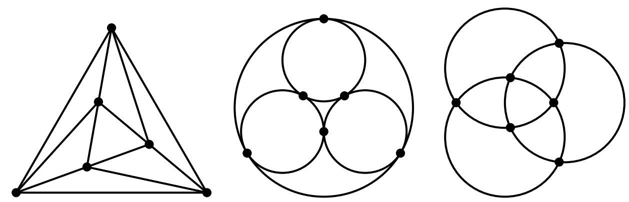
(c) Now consider a straight-line drawing of the cube C covered by a family $\mathcal{L}$ of ρ lines. We distinguish two cases.

Assume first that the drawing of the cube lies in a single plane. Each embedding of the cube contains two nested cycles, namely, the boundary of the outer face and the innermost face. We consider three cases depending on the shape of the outer face. (i) If the outer cycle is drawn as the boundary of a (strictly) convex quadrilateral, then none of the lines covering its sides can be used to cover the edges of the innermost cycle, therefore, it needs three additional lines. (ii) If the outer cycle is drawn as the boundary of a non-convex quadrilateral, then we need three additional lines to cover the three edges going from its three convex angles to the innermost cycle. (iii) Now assume that the outer cycle is drawn as a triangle. Then none of the lines covering its sides can be used to cover the edges of the innermost cycle. If this cycle is drawn as a quadrilateral, then we need four additional lines to cover its sides. If the innermost cycle is drawn as a triangle, then we need three lines for the triangle and an additional



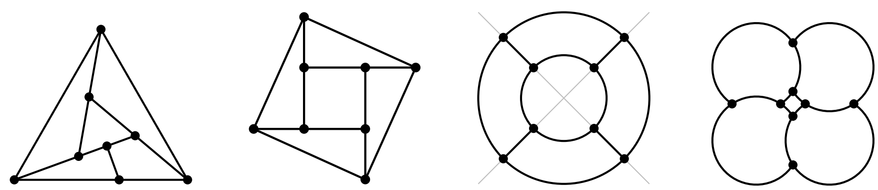
(a) 6 segm. (b) 3 arcs

Figure 2: Drawings of the tetrahedron



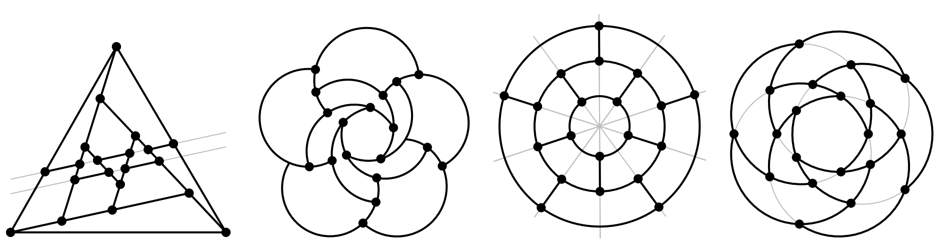
(a) 9 segm. / lines (b) 4 arcs / circles (c) 3 arcs / circles

Figure 3: Drawings of the octahedron



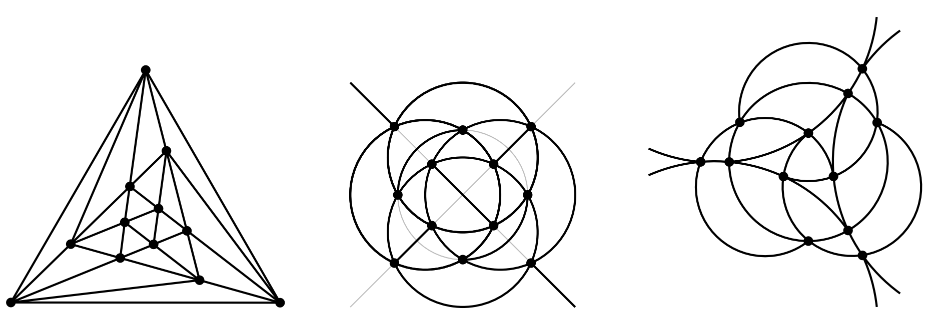
(a) 7 segm. / lines (b) 8 segm. / lines (c) 6 arcs / 4 circles (d) 4 arcs

Figure 4: Drawings of the cube



(a) 13 segm. / 10 lines [22] (b) 10 arcs / 10 circ. [23] (c) 13 arcs / 8 circ. [22] (d) 10 arcs / 5 circles

Figure 5: Drawings of the dodecahedron



(a) 15 segments / lines (b) 10 arcs / 7 circles (c) 7 arcs / 7 circles

Figure 6: Drawings of the icosahedron

$G = (V, E)$	$\lvert V\rvert$	$\lvert E\rvert$	$\lvert F\rvert$	seg	upp. bd.	arc	lower bd.	upp. bd.
tetrahedron	4	6	4	6	Fig. 2a	3	Prop. 4(a)	Prop. 8(a)
octahedron	6	12	8	9	Fig. 3a	3	Prop. 4(a)	Prop. 8(b)
cube	8	12	6	7	Fig. 4a	4	[7, Lem. 5]	Prop. 8(c)
dodecahedron	20	30	12	13	Fig. 5a	10	[7, Lem. 5]	Prop. 8(d)
icosahedron	12	30	20	15	Fig. 6a	7	Prop. 4(a)	Prop. 8(e)

Table 2: Bounds on the segment and arc numbers of the platonic graphs. We obtained the lower bounds on the segment number with the help of an integer linear program; see Table 4 in Section 6. The upper bounds for the segment numbers of the dodecahedron and icosahedron have been established by Schulz [23] and Scherm [22, Fig. 2.1(c)].

graph	ρ^1_2	ρ^1_3	lower bd.	upp. bd.	σ^1_2	σ^1_3	upp. bd.
tetrahedron	6	6	[4, Expl. 10]	Fig. 2a	3	3	Fig. 2b
octahedron	9	9	Prop. 7(b)	Fig. 3a	3	3	Fig. 3c
cube	7	7	Prop. 7(c)	Fig. 4a	4	4	Fig. 4d
dodecahedron	9 . . . 10	9 . . . 10	Prop. 7(d)	Fig. 5a	5	5	Fig. 5d
icosahedron	13 . . . 15	13 . . . 15	Prop. 7(e)	Fig. 6a	7	7	Fig. 6c

Table 3: Bounds on the affine cover numbers ρ^l_d and the spherical cover numbers σ^l_d for platonic graphs. The lower bounds on σ^1_2 and σ^1_3 stem from Proposition 4(a).

line to cover the edge incident to the vertex of the innermost cycle which is not a vertex of the triangle. In each of the three cases (i)–(iii), we need at least seven lines to cover the cube.

Now assume that the drawing of the cube is not contained in a single plane. Then its convex hull has (at least) four extreme points. In order to cover the cube, we need at least one pair of intersecting lines of $\mathcal{L}$ for each vertex of the cube and at least three such pairs for each extreme point, that is, at least $4+4\cdot 3 = 16$ pairs of intersecting straight lines in total. So, $\binom{\rho}{2} \geq 16$ and $\rho \geq 7$.

(d) Consider a straight-line drawing of the dodecahedron D covered by a family $\mathcal{L}$ of ρ lines. Again, we distinguish two cases.

Assume first that the drawing of the dodecahedron lies in a single plane. Again we make a case distinction depending on the shape of the outer cycle. (i) If the outer cycle is drawn as the boundary of a convex polygon, let $\mathcal{L}_0 \subseteq L$ be the family of lines that support the edges on the outer cycle. This family consists of at least three lines. None of them covers any of the at most 15 vertices remaining in the interior of the convex polygon. Thus each of these vertices is an intersection point of two lines of $\mathcal{L} \setminus \mathcal{L}_0$. Since $\mathcal{L} \setminus \mathcal{L}_0 \leq \rho - 3$, this family of lines can generate at most $\binom{\rho-3}{2}$ intersection points. Therefore, $\binom{\rho-3}{2} \geq 15$ and, hence, $\rho \geq 9$. (ii) Assume that the outer cycle is drawn as a non-convex quadrilateral. Then the drawing is contained in a convex angle opposite to the reflex angle. To cover the angle sides, we need a family $\mathcal{L}_0$ consisting of at least two lines. None of them covers any of the at least 15+1 vertices remaining in the interior of the angle. Similarly to the previous paragraph, we obtain $\binom{\rho-2}{2} \geq 16$ and, hence, $\rho \geq 9$. (iii) Assume that the outer cycle is drawn as a pentagon P. Since the angle sum of a pentagon is 3π, P has at most two reflex angles, and therefore, at least three convex angles. Each vertex of D drawn as a vertex of a convex angle is an intersection point of (at least) three covering lines, because it has degree 3. There exists an edge e of P such that P is contained in one of the half-planes created by the line ℓ spanned by e (see, for instance, [21]). It is easy to check that ℓ can cover only edge e of the outer face of D. Then the family $\mathcal{L} \setminus \{\ell\}$ covers all edges of G but e. The angles of P incident to e are convex. Let v be a vertex of D drawn as a vertex of a convex angle not incident to e. In order to cover D, we need at least one pair of intersecting lines from $\mathcal{L} \setminus \{\ell\}$ for each vertex of D different from v and at least three such pairs for v, that is, at least $19 + 3 = 22$ pairs of intersecting lines in total. Therefore, $\binom{\rho-1}{2} \geq 22$ and, hence, $\rho \geq 9$. Note that, in each of the three cases (i)–(iii), we have $\rho \geq 9$.

Now assume that the drawing of D is not contained in a single plane. Then its convex hull has (at least) four extreme points. In order to cover D, we need at least one pair of intersecting lines of $\mathcal{L}$ for each vertex of D and at least three such pairs for each extreme point, that is, at least $16 + 4 \cdot 3 = 28$ pairs of intersecting lines in total. Therefore, $\binom{\rho}{2} \geq 28$. But if we have equality then any two lines of $\mathcal{L}$ intersect. So all of them share a common plane or a common point. In the first case the drawing is contained in a single plane; in the second case the family $\mathcal{L}$ cannot cover the drawing. Thus $\binom{\rho}{2} > 28$, and, hence, $\rho \geq 9$.

(e) If the drawing of the icosahedron I is not contained in a single plane, then we can pick four extreme points of the convex hull of the drawing. Each of these points represents a vertex of degree 5, so we need five lines to cover edges incident to this vertex, that is, 20 lines in total, but we have double-counted the lines that go through pairs of the extreme points that we picked. Of these, there are at most $\binom{4}{2} = 6$. Thus we need at least $20 - 6 = 14$ lines to cover the drawing.

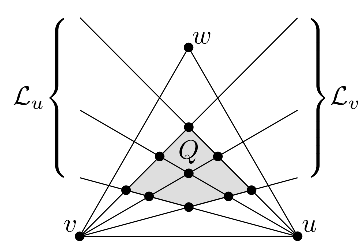


Figure 7: The families of lines $\mathcal{L}_u$ and $\mathcal{L}_v$

Now assume that there exists a straight-line drawing of the icosahedron in a single plane covered by a family $\mathcal{L}$ of twelve lines. Let u, v, w be the vertices of the outer face of I. Clearly, three distinct lines in $\mathcal{L}$ form the triangle uvw. For

$s \in \{u, v, w\}$, we denote by $\mathcal{L}_s$ the lines in $\mathcal{L}$ that go through s and do not cover edges of the outer face. Since I is 5-regular, $|\mathcal{L}_s| = \deg(s) - 2 = 3$. Consider the set P of intersection points between the line families $\mathcal{L}_u$ and $\mathcal{L}_v$. The set P lies in the triangle uvw and is bounded by the quadrilateral Q formed by the outer pairs of lines in $\mathcal{L}_v$ and $\mathcal{L}_u$; see Fig. 7.

The quadrilateral Q is convex and eight of the nine points in P lie on the boundary of Q, hence, for any line ℓ in $\mathcal{L}_w$, we have $|\ell \cap P| \leq 3$. Observe that $|\ell \cap P| = 3$ implies that ℓ goes through the only point of P that lies in the interior of Q. Thus the lines in $\mathcal{L}_w$ can create at most seven triple intersection points with the lines in $\mathcal{L}_u$ and $\mathcal{L}_v$.

The icosahedron is 5-regular, so all vertices must be placed at the intersection of at least three lines. We need at least nine triple intersection points in order to place all $12 - 3$ inner vertices of the icosahedron—a contradiction. □

Proposition 8 *(a)* $\operatorname{arc}(T) \leq 3$*; (b)* $\operatorname{arc}(O) \leq 3$*; (c)* $\operatorname{arc}(C) \leq 4$*; (d)* $\operatorname{arc}(D) \leq 10$*; (e)* $\operatorname{arc}(I) \leq 7$*.*

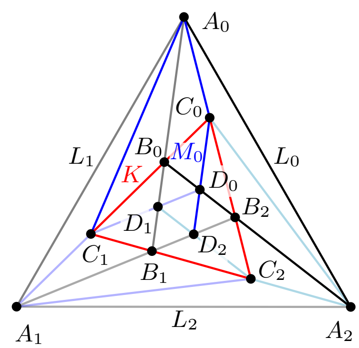
Proof: For the upper bounds for (a)–(d) see the drawings of the graphs in Figures 2b, 3c, 4d, and 5d respectively. While it is easy to see that these drawings are valid, we argue more carefully that the icosahedron does indeed admit a drawing with seven arcs. To construct the drawing in Fig. 6c (for details see Fig. 8), we first cover the edges of the icosahedron by seven objects, grouped into a single cycle K and two sets $L = \{L_0, L_1, L_2\}$ and $M = \{M_0, M_1, M_2\}$, where K is a cycle of length 6 and all elements of L and M are simple paths of length 4; see Fig. 8a. We identify the paths and cycles with their drawings as arcs and circles. For a set $S \in \{\{K\}, L, M\}$ and a number $i \in \{0, 1, 2\}$, let (d_S, α_{S_i}) be the polar coordinates of the center $c(S_i)$ of the circle of radius r_S that covers arc $S_i \in S$ (see Fig. 8b). We set the coordinates and radii as follows:

$$\begin{aligned}
&\alpha_K = 0 && d_K = 0 && r_K = 1 \\
&\alpha_{L_i} = i \cdot 2\pi/3 && d_L = (3 + \sqrt{3})/2 && r_L = \sqrt{5/2 + \sqrt{3}} \\
&\alpha_{M_i} = \pi/2 + i \cdot 2\pi/3 && d_M = (3 - \sqrt{3})/2 && r_M = \sqrt{5/2 - \sqrt{3}}
\end{aligned}$$

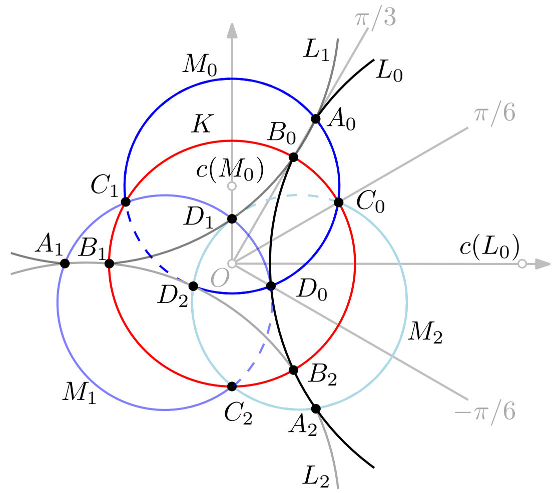
Using the law of cosines, it is easy to compute the intersection points:

$$\begin{aligned}
\{A_i\} &:= L_i \cap L_{i+1} \cap M_i && \Rightarrow\ A_i = \big(i \cdot 2\pi/3, (1 + \sqrt{3})/2\big); \\
\{B_i\} &:= L_i \cap L_{i+1} \cap K && \Rightarrow\ B_i = (i \cdot 2\pi/3, 1); \\
\{C_i\} &:= M_i \cap M_{i+2} \cap K && \Rightarrow\ C_i = (\pi/3 + i \cdot 2\pi/3, 1); \\
\{D_i\} &:= L_i \cap M_i \cap M_{i+1} && \Rightarrow\ D_i = \big(\pi/2 + i \cdot 2\pi/3, (\sqrt{3} - 1)/2\big).
\end{aligned}$$

For $i = 0, 1, 2$, let L_i be the larger arc of the covering circle between the points A_i and B_i, let M_i be the larger arc of the covering circle between the points C_{i+1} and D_{i+2} (with indices modulo 3), and let K be the whole unit circle. □



(a) symmetric path partition: the black and gray arcs are in L, the light and dark blue arcs in M, and the red cycle is K



(b) illustration of the proof of Proposition 8(e)

Figure 8: Bounding the arc number of the icosahedron

5 Lower Bounds for σ_d^1

Given a graph G, we obtain lower bounds for $\sigma_d^1(G)$ via standard combinatorial characteristics of G in the same way as for the bounds for $\rho_d^1(G)$ [4]. In particular, we prove a general lower bound for $\sigma_d^1(G)$ in terms of the treewidth $\mathrm{tw}(G)$ of G, which follows from the fact that graphs with low parameter $\sigma_d^1(G)$ have small separators. This fact is interesting by itself and has yet another consequence: graphs with bounded vertex degree can have a linearly large value of $\sigma_d^1(G)$ (hence, the factor of n in the trivial bound $\sigma_d^1(G) \le m \le n \cdot \Delta(G)/2$ is best possible).

We need the following definitions. The *linear arboricity* $\mathrm{la}(G)$ of a graph G is the minimum number of linear forests that partition the edge set of G [13]. Let $W \subseteq V(G)$. A set of vertices $S \subset V(G)$ is a *balanced W-separator* of the graph G if $|W \cap C| \le |W|/2$ for every connected component C of $G - S$. Moreover, S is a *strongly balanced W-separator* if there is a partition $W \setminus S = W_1 \cup W_2$ such that $|W_i| \le |W|/2$ for both $i = 1, 2$ and there is no path between W_1 and W_2 that avoids S. Let $\mathrm{sep}_W(G)$ ($\mathrm{sep}^*_W(G)$) denote the minimum k such that G has a (strongly) balanced W-separator S with $|S| = k$. Furthermore, let $\mathrm{sep}(G) = \mathrm{sep}_{V(G)}(G)$ and $\mathrm{sep}^*(G) = \mathrm{sep}^*_{V(G)}(G)$. Note that $\mathrm{sep}_W(G) \le \mathrm{sep}^*_W(G)$ for any $W \subseteq V(G)$ and, in particular, $\mathrm{sep}(G) \le \mathrm{sep}^*(G)$.

It is known [11, Theorem 11.17] that $\mathrm{sep}_W(G) \le \mathrm{tw}(G) + 1$ for any $W \subseteq V(G)$. On the other hand, $\mathrm{tw}(G) \le 3k$ if $\mathrm{sep}_W(G) \le k$ for every W with $|W| = 2k + 1$.

Recall that the bisection width $\mathrm{bw}(G)$ of a graph $G = (V, E)$ is the minimum number of edges between two sets of vertices W_1 and W_2 with $|W_1| = \lceil n/2 \rceil$

and $|W_2| = \lfloor n/2 \rfloor$ partitioning V. Note that $\mathrm{sep}^*(G) \leq \mathrm{bw}(G) + 1$.

Now we use these graph parameters to bound the spherical cover number from below. The proofs are similar to those regarding the affine cover number [4]. We restate Proposition 5 as item (a) to make the following theorem more self-contained.

Theorem 3 *For any integer $d \geq 1$ and any graph G with n vertices and m edges, the following bounds hold:*

(a) $\sigma^1_d(G) \geq \mathrm{bw}(G)/2$.

(b) $\sigma^1_d(G) > n/10$ *for almost all cubic graphs with n vertices;*

(c) $\lceil \frac{3}{2}\sigma^1_d(G) \rceil \geq \mathrm{la}(G)$*;*

(d) $\sigma^1_d(G) \geq \mathrm{sep}^*_W(G)/2$ *for every $W \subseteq V(G)$;*

(e) $\sigma^1_d(G) \geq \mathrm{tw}(G)/6$.

Proof: For the proof of (a) see Proposition 5.

(b) The claim follows from (a) and from the fact that a random cubic graph on n vertices has bisection width at least $n/4.95$ with probability $1 - o(1)$ [19].

(c) Given the drawing of the graph G on $r = \sigma^1_d(G)$ circles, we remove an edge from each of the circles (provided such an edge exists), obtaining at (most) r linear forests. The removed edges we group into (possible, degenerated) pairs, obtaining at most $\lceil r/2 \rceil$ additional linear forests. So, $\mathrm{la}(G) \leq r + \lceil r/2 \rceil$.

(d) The proof is similar to Theorem 9(c) in [4]. The difference of a factor of 1/2 is due to the fact that a straight line pierces the plane at most once whereas and a circle pierces the hyperplane at most twice.

(e) follows from (d) by the above mentioned relationship between treewidth and balanced separators. □

Corollary 1 *$\sigma^1_d(G)$ cannot be bounded from above by a function of* $\mathrm{la}(G)$ *or* $v_{\geq 3}(G)$ *or* $\mathrm{tw}(G)$*, where $v_{\geq 3}(G)$ is the number of vertices with degree at least 3.*

Proof: $\mathrm{la}(G)$: Akiyama et al. [1] showed that, for any cubic graph G, $\mathrm{la}(G) = 2$. On the other hand, $v_{\geq 3}(G) = n$, so $\sigma^1_3(G) > \sqrt{n}$ by Proposition 4(a). Theorem 3(b) yields an even larger gap.

$v_{\geq 3}(G)$: Let G be the disjoint union of k cycles. Then $v_{\geq 3}(G) = 0$. Clearly, an arrangement A of ℓ circles has at most ℓ^2 vertices. Each cycle of G "consumes" at least two vertices of A or a whole circle, so $\sigma^1_d(G) = \Omega(\sqrt{k})$.

$\mathrm{tw}(G)$: Let G be a caterpillar with linearly many vertices of degree 3. Then, $\mathrm{tw}(G) = 1$. On the other hand, by Proposition 4(a), we have $\sigma^1_d(G) = \Omega(\sqrt{n})$. □

Lemma 1 *A circular-arc drawing $\Gamma \subset \mathbb{R}$ of a graph G that contains k nested cycles cannot be covered by fewer than k circles.*

Proof: Fix any point inside the closed Jordan curve in Γ that corresponds to the innermost cycle of G. Let ℓ be an arbitrary line through this point. Then ℓ crosses at least twice each of the Jordan curves that correspond to the nested cycles in G. Hence, there are at least $2k$ points where ℓ crosses Γ.

On the other hand, consider any set of r circles whose union covers Γ. Then it is clear that ℓ crosses each of these r circles in at most two points, so there are at most $2r$ points where ℓ crosses Γ. Putting together the two inequalities, we get $r \geq k$ as desired. □

At last we remark that there are graphs whose σ_3^1-value is a lot smaller than their σ_2^1-value.

Theorem 4 *For infinitely many n there is a planar graph G on n vertices with $\sigma_2^1(G) = \Omega(n)$ and $\sigma_3^1(G) = O(n^{2/3})$.*

Proof: We use the same family $(G_i)_{i\geq 1}$ of graphs as Chaplick et al. [4, Theorem 24(b)] with $G_i = C_3 \times P_i$ and P_i a path with i vertices. Then G_i has $n_i = 3i$ vertices, $\rho_2^1(G_i) = \Omega(n_i)$, and $\rho_3^1(G_i) = O(n_i^{2/3})$. The lower bound on $\sigma_2^1(G_i)$ follows from Lemma 1. The upper bound on $\sigma_3^1(G_i)$ follows from Proposition 1 for $l = 1$ and $d = 3$, which states that, for any graph G, $\sigma_3^1(G) \leq \rho_3^1(G)$. □

6 An MIP Formulation for Estimating the Segment Number

In this section, we exploit an integer programming formulation for *locally consistent angle assignments* [6], which we define below, to obtain lower bounds on the segment numbers of planar graphs. Our MIP determines a locally consistent angle assignment with the maximum number of π-angles between incident edges. Note that such angle assignments are not necessarily realizable with straight-line edges in the plane. This is why the MIP yields only an upper bound for the number of π-angles—and a lower bound for the segment number. For the platonic graphs, however, it turns out that the bounds are tight; see Tables 2 and 4.

Let $G = (V, E)$ be a 3-connected graph with fixed embedding given by a set $\mathcal{F}$ of faces and an outer face f_0. For any vertex $v \in V$ and any face $f \in \mathcal{F}$, we introduce a fractional variable $x_{v,f} \in (0, 2)$ whose value is intended to express the size of the angle at v in f, divided by π. Thus, $(\pi \cdot x_{v,f})_{v\in V, f\in\mathcal{F}}$ is an angle assignment for G. The following constraints guarantee that the assignment is locally consistent. (For a vertex v and a face f, we write $v \sim f$ to express that

v is incident to f.)

$$\sum_{f \sim v} x_{v,f} = 2 \qquad \text{for each } v \in V;$$
$$\sum_{v \sim f} x_{v,f} = \deg(f) - 2 \qquad \text{for each } f \in \mathcal{F} \setminus \{f_0\};$$
$$\sum_{v \sim f_0} x_{v,f_0} = \deg(f_0) + 2.$$

For any vertex v, let $L_v = \langle v_1, \dots, v_k \rangle$ be the list of vertices adjacent to v, in clockwise order as they appear in the embedding. Due to the 3-connectivity of G, any two vertices v_t and v_{t+1} that are consecutive in L_v (and adjacent to v) uniquely define a face $f(v,t)$ incident to v, v_t, and v_{t+1}. For two vertices v_i and v_j with $i < j$, we express the angle $\angle(v_i v v_j)$ as the sum of the angles at v in the faces between v_i and v_j. As shorthand, we use $y_{v,i,j} = \angle(v_i v v_j)/\pi \in (0,2)$:

$$y_{v,i,j} = \sum_{t=i}^{j-1} x_{v,f(v,t)} \qquad \text{for each } v \in V,\ 1 \le i < j \le \deg(v).$$

We want to maximize the number of π-angles between any two edges incident to the same vertex. To this end, we introduce a 0–1 variable $s_{v,i,j}$ for any vertex v and $1 \le i < j \le \deg(v)$. The intended meaning of $s_{v,i,j} = 1$ is that $\angle(v_i v v_j) = \pi$. We add the following constraints to the MIP:

$$\left.\begin{aligned} s_{v,i,j} &\in \{0,1\} \\ s_{v,i,j} &\le y_{v,i,j} \\ s_{v,i,j} &\le 2 - y_{v,i,j} \end{aligned}\right\} \qquad \text{for each } v \in V,\ 1 \le i < j \le \deg(v).$$

If $y_{v,i,j} < 1$, the second constraint will force $s_{v,i,j}$ to be 0 and the third constraint will not be effective. If $y_{v,i,j} > 1$, the third constraint will force $s_{v,i,j}$ to be 0, and the second constraint will not be effective. Only if $y_{v,i,j} = 1$ (and $\angle(v_i v v_j) = \pi$), both constraints will allow $s_{v,i,j}$ to be 1. This works because we want to maximize the total number of π-angles between incident edges in a locally consistent angle assignment. To this end, we use the following objective:

$$\text{Maximize} \quad \sum_{v \in V} \sum_{1 \le i < j \le \deg(v)} s_{v,i,j}.$$

Every π-angle between incident edges saves a segment; hence, in any straight-line drawing of G, the number of segments equals the number of edges minus the number of π-angles. In particular, this holds for a drawing that minimizes the number of segments (and simultaneously maximizes the number of π-angles). Thus,

$$\text{seg}(G) = |E| - \text{ang}_\pi(G).$$

Since

$$\text{ang}_\pi(G) \le \sum_{v \in V} \sum_{1 \le i < j \le \deg(v)} s_{v,i,j},$$

graph G	octahedron	cube	dodecahedron	icosahedron
$\text{ang}_\pi(G) \leq$	3	5	17	15
$\text{seg}(G) \quad \geq$	9	7	13	15
variables	60	48	120	180
constraints	137	114	277	395
runtime [s]	0.011	0.009	0.015	0.066

Table 4: Upper bounds on the number of π-angles and corresponding lower bounds on the segment numbers of the platonic graphs (except for the tetrahedron) obtained by the MIP and sizes of the MIP formulation for these instances. Running times were measured on a 64-bit machine with 7.7 GB main memory and four Intel i5 cores with 1.90 GHz, using the MIP solver IBM ILOG CPLEX Optimization Studio 12.6.2.

the above relationship provides the lower bound

$$\text{seg}(G) \geq |E| - \sum_{v \in V} \sum_{1 \leq i < j \leq \deg(v)} s_{v,i,j}$$

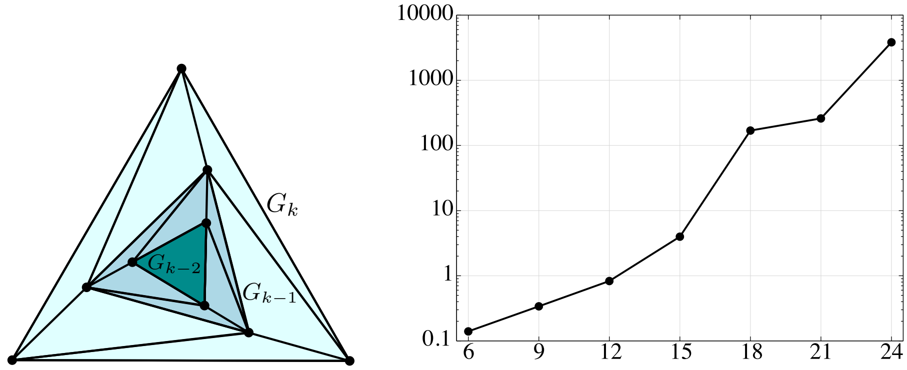
for the segment number, which can be computed by solving the MIP.

Our MIP has $O(n^3)$ variables and constraints. The experimental results for the platonic graphs (which are 3-connected and thus have a unique planar embedding) are displayed in Table 4.

In addition, to check the capabilities of the MIP, we have tested it on a family of triangulations $(G_k)_{k \geq 2}$ constructed by Dujmović et al. [7, Lemma 17]; a variant of the nested-triangles graph (see Fig. 9a). The graph G_2 is the octahedron. For $k > 2$, the triangulation G_k is created by recursively nesting a triangle into the innermost triangle of G_{k-1} and connecting its vertices to the vertices of the triangle it was nested into. Note that G_k has $n_k = 3k$ vertices. Dujmović et al. showed a lower bound of $2n_k - 6$ on the segment number of G_k and a tight lower bound of $2n_k - 3$ (see the proof of [7, Lemma 17] and Fig. 9a) on the number of segments given the fixed embedding. Figure 9b shows the runtime of the MIP in logarithmic scale for the triangulations $G_2, G_3, \ldots, G_8$. As expected, the runtime is (at least) exponential. Interestingly, for each of these (embedded) graphs, our MIP finds a solution with $2n_k - 3$ "segments", thus matching the tight lower bound of Dujmović et al. for the fixed-embedding case.

7 Discussion and Open Problems

As mentioned in the introduction, we now show that minimum-line drawings are indeed different from ρ^1_2-optimal drawings. Then we state some open problems regarding affine and spherical cover numbers.



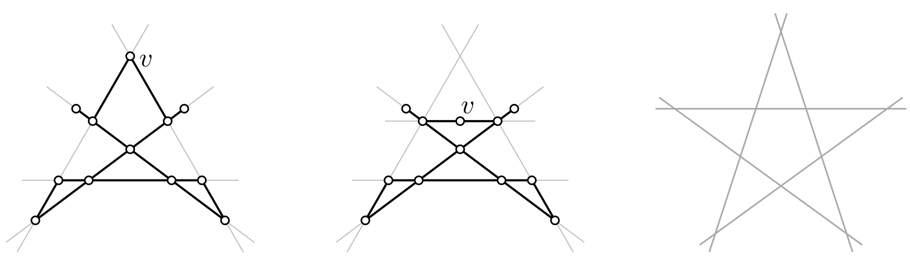
(a) optimal drawing of the triangulation G_k (with $n_k = 3k$ vertices) of Dujmović et al. [7] using $2n_k - 3$ segments

(b) runtime of the MIP applied to the graphs $G_2, G_3, \ldots, G_8$ of Dujmović et al. [7]. The numbers of vertices of the graphs are on the x-axis; the runtime in seconds is on the y-axis. Note the log-scale at the y-axis.

Figure 9: Testing the MIP: instances and runtime

Example 1 *Minimum-line drawings are different from ρ^1_2-optimal drawings.*

Proof: We provide a graph G with $\rho^1_2(G) = 5$ and $\text{seg}(G) \leq 6$; see Fig. 10a. Then we show that every embedding of G on any arrangement of five straight lines consists of at least seven segments.



(a) a ρ^1_2-optimal drawing of G on 5 lines and 7 segments

(b) a minimum-line drawing of G on 6 lines and 6 segments

(c) a star-shaped arrangement of 5 straight lines

Figure 10: A graph G that shows that ρ^1_2-optimal drawing and minimum-line drawings are indeed different.

Chaplick et al. [4] defined a vertex of a planar graph to be *essential* if it has degree at least 3 or belongs to a cycle of length 3. They observe that in any drawing of a graph any essential vertex is shared by two edges not lying on the same line. Observe that G has nine essential vertices. Hence, any arrangement of straight lines that cover a drawing of G consists of at least five straight lines (with potentially ten intersection points). Moreover, for the same reason, an arrangement of five straight lines covering a drawing of G must be *simple*, that is, every two straight lines intersect and no three straight lines have a point in common. There is only one such arrangement of five straight lines in the projective plane [12]. This combinatorially unique arrangement is star-shaped; see Fig. 10c.

The graph G has three triangles that are attached via one vertex in a chain-like fashion. These triangles can only be embedded into faces of the arrangement; otherwise there would be a triangle that consumes two additional intersection points of the arrangement. Therefore, there is only one way to embed the three triangles on the arrangement, namely on some three consecutive spikes of the star. This forces the degree-2 vertex v (see Fig. 10a) to be on a bend (incident to two segments in the drawing) and makes the embedding combinatorially unique. In this embedding of G we have seven segments, but $\mathrm{seg}(G) \leq 6$; see Fig. 10b.

Finally, if a graph does not have a drawing with six segments covered by five straight lines in the projective plane, it also does not have one in the Euclidean plane, because we can embed a line arrangement in the Euclidean plane into one in the projective plane preserving the number of segments. So we need at least six straight lines for a drawing with six segments. □

We close with some open problems.

We conjecture that our drawings in Figures 5a and 6a are optimal. This would mean that $\rho_3^1(D) = 10$ and $\rho_3^1(I) = 15$, but we have no proof for this.

Is there a family $\mathcal{G}$ of graphs such that the affine cover number $\rho_d^l(G)$ of every graph $G \in \mathcal{G}$ can be bounded by a function of the spherical cover number $\sigma_d^l(G)$? For example in the plane (recall that we only consider planar graphs there), $\rho_2^1(G) \in O(n)$, since we can use a different line for each single edge, moreover, according to Proposition 4(a) $\sigma_2^1(G) \in \Omega(\sqrt{n})$, therefore, we have that $\rho_2^1(G) \in O(\sigma_2^1(G)^2)$. For the given family of graphs can this relation be tightened? For example, Chaplick et al. [4, Example 22] showed that there are triangulations for which $O(\sqrt{n})$ lines suffice. It would be even more interesting to find families of graphs where there is an asymptotic difference between the two cover numbers.

We have already seen that $\sigma_3^2(K_n)$ grows asymptotically more slowly than $\rho_3^2(K_n)$. Is there a family of planar graphs where σ_2^1 grows asymptotically more slowly than ρ_2^1?

Chaplick et al. [4] showed that the hierarchy of affine cover numbers collapses in the following sense: For every graph G, for every integer $d > 3$, and for every integer l with $1 \leq l \leq d$, it holds that $\rho_d^l(G) = \rho_3^l(G)$. The proof of this fact is based on affine maps, which transform planes into planes, but not spheres into spheres, so we don't know whether the hierarchy of spherical cover numbers collapses, too.

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