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EXTREMALS OF THE GENERALIZED EULER-BERNOULLI ENERGY AND APPLICATIONS

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Abstract. In this survey we describe a general method to deal with the variational problem associated to the generalized elastic curves, paying special attention to closed critical curves in real space forms due to its special geometric significance. We illustrate the method by studying particular choices of this energy in some more detail. Finally, we will review also some interesting applications of generalized elasticae to other higher dimensional variational problems in Physics, Biophysics and the Theory of Submanifolds.

1. Introduction

In 1691 J. Bernoulli posed the problem of the elastic beam and three years later he published his own solution. However, Huygens criticized his work for not showing all the possible solutions. In 1742, D. Bernoulli proposed to minimize the squared radius of curvature in order to determine the shape of an elastic rod subject to pressure at both ends. Thus, following the D. Bernoulli's simple geometric model, an *elastic curve* (also, *elastica*) is a minimizer of the *bending energy*

$$\mathcal{F}_\lambda(\gamma) = \int_\gamma (\kappa^2 + \lambda)\, \mathrm{d}s \tag{1}$$

where κ represent the curvature of the curve γ and λ corresponds to a constraint on its length. When $\lambda = 0$, critical points of bending energy are called *free elastic* curves or, simply, *elasticae*. In 1743, L. Euler determined the plane elastic curves [23] (explicit expressions for them can be found in [22]). Elastic curves in $\mathbb{R}^3$ were first considered by J. Radon 1910 and R. Irrgang 1933. More recently, J. Langer and D. Singer [38], [40], [41], [59], have made significant contributions to the subject. The Hamiltonian theory has been developed by V. Jurdjevic in [30] and a different approach has also been proposed by R. Bryant and Ph. Griffiths in [18]. This work deals with the study of curves which are essentially one-dimensional