## BOOK REVIEW

*Algebraic Models in Geometry*, by Yves Félix, John Oprea and Daniel Tanré, Oxford University Press, Oxford, 2008, xxi + 460pp., ISBN 978-0-19-920652-0.

In his landmark paper [*Infinitesimal Computations in Topology*, Publ. I. H. E. S. **47** (1977) 269-331], D. Sullivan demonstrated the power of differential forms for homotopy theory. Sullivan constructed a version of the de Rahm algebra of forms but with rational coefficients and without any smoothness hypotheses. Remarkably, this construction leads to a complete algebraic model for the homotopy type of a simply connected (or, more generally, a nilpotent) space modulo torsion. Precisely, given any differential graded (DG) algebra $(A, d)$ there is an associated minimal DG algebra $(\wedge V, d)$ (a free algebra with decomposable differential) and a DG algebra map $\varphi \colon \wedge V \to A$ inducing an isomorphism on cohomology. The minimal DG algebra is uniquely determined by $(A, d)$ up to isomorphism. Applying this procedure to Sullivan's de Rahm algebra yields the Sullivan minimal model of a space.

The Sullivan minimal model is a faithful invariant of rational homotopy type: two nilpotent spaces have homotopy equivalent rationalizations if and only if their minimal models are isomorphic. Consequently, all the rational homotopy invariants of a nilpotent space can, in principle, be recovered from its minimal model. While Sullivan's minimal models thus provide an exceptionally powerful tool for homotopy theory, arguably the most striking applications of the theory were originally in geometry. The famous paper of Deligne-Griffiths-Morgan-Sullivan applied Hodge theory and Sullivan's minimal models to prove that the real homotopy type of a Kähler manifold is a formal consequence of its real cohomology. (Sullivan later improved this result in the above mentioned paper, replacing $\mathbb{R}$ with $\mathbb{Q}$.) In another direction, Sullivan models proved decisive in settling an important case of the "closed geodesic problem" for Riemannian manifolds. In the late 1960s, D. Gromoll and W. Meyer proved that a Riemannian manifold $M$ admits infinitely many distinct, periodic geodesics if the free loop space $LM$ has unbounded Betti numbers. Using this result and an explicit Sullivan model for the free loop space,

M. Vigué-Poirrier and Sullivan proved that $M$ admits infinitely many closed geodesics provided the algebra $H^*(M; \mathbb{Q})$ requires at least two generators.

In the thirty years since the appearance of Sullivan's paper, the promise of minimal models as a tool for geometry has been richly fulfilled. Minimal models and related techniques have found significant applications in diverse areas including blow-ups in symplectic geometry, curvature of Riemannian manifolds, configuration spaces, toral actions on manifolds and, most recently, the Chas-Sullivan loop product. The book under review provides an impressively comprehensive account of these and many other developments. We discuss the specific contents now.

Chapter 1 gives a nice survey of Lie groups and related objects: classifying spaces, bundles and homogenous spaces. Two important antecedents of Sullivan models are discussed in this context. First, given a left $G$-manifold $M$, the left invariant forms $\Omega_L(M)$ are defined and the inclusion $\Omega_L(M) \hookrightarrow A_{\mathrm{DR}}(M)$ is shown to induce an isomorphism in cohomology where the latter space is the usual de Rahm algebra of forms on $M$. Applied to a compact Lie group $G$, this result gives perhaps the simplest example of a Sullivan minimal model. A more significant predecessor of Sullivan's theory is the Cartan-Weil model for a homogeneous space. An explicit version of this model is described providing a prototype for the constructions to come. Also included here are the classical Hopf Theorem on $H^*(G; \mathbb{Q})$ and two separate proofs of Borel's calculation of $H^*(BG; \mathbb{Q})$ for $G$ a compact Lie group.

Chapter 2 gives a condensed treatment of Sullivan's rational homotopy theory. The homotopy theory of DG algebras, minimal models, the construction of Sullivan's rational polynomial forms, relative minimal models (the Sullivan model of a map), the celebrated elliptic-hyperbolic dichotomy theorem and semi-free modules are all discussed with many examples but limited proofs. Readers without background in this area may wish to consult an alternate source, e.g., the text by Félix-Halperin-Thomas, for a more leisurely exposition of these topics. Included here as well is a discussion of formal spaces which figure prominently in the sequel.

A basic theme of this text is to understand how the particular geometric structures of a space are reflected in its algebraic model. The first examples in this connection are given in Chapter 3. Compact manifolds satisfy Poincaré duality and so it is natural to look for a version of this structure at the level of models. Results of Sullivan, J. Barge, J. Stasheff and, most recently, P. Lambrechts on Poincaré duality models are described. While compact Lie groups, their classifying spaces and many homogenous spaces are formal spaces, the generic manifold is not. A theorem of T. Miller proves that highly-connected manifolds are formal while a

theorem of Stasheff gives a useful criterion for determining formality of Poincaré duality spaces. The minimal models of nilmanifolds, biquotients and Riemannian manifolds are also presented here. In the last case, the Hodge decomposition of harmonic forms allows for an explicit description.

Chapter 4 continues the theme of geometric structure reflected in algebraic models with a thorough and technical study of complex manifolds. The de Rahm algebra of a complex manifold $M$ admits a natural bigrading stemming from the complex structure on its tangent spaces. This leads to a decomposition of the de Rahm differential $d = \partial + \overline{\partial}$ into parts of bidegree $(1, 0)$ and $(0, 1)$, respectively. Using the differential $\overline{\partial}$ gives the bigraded Dolbeault complex of the manifold. When $M$ is a compact Kähler manifold, the $\partial\overline{\partial}$-lemma explains the interaction between these two differentials and is the key to the proof that compact Kähler manifolds are formal. Two extensions of this important result are given. First, it is shown that compact Kähler manifolds are actually "strictly formal" which means that their Dolbeault complex, de Rahm complex and Dolbeault cohomology are all linked by maps inducing cohomology isomorphisms. Second, compact Kähler manifolds are shown to have no non-vanishing negative degree derivations of their rational cohomology and thus satisfy the conclusion of a famous conjecture concerning certain formal spaces in rational homotopy theory due to S. Halperin.

Computing the Dolbeault cohomology for general complex manifolds is a challenging problem. Using the Borel spectral sequence, this calculations is made for the space $S^1 \times S^{2n+1}$. The Frölicher spectral sequence, arising from the filtration of the Dolbeault complex by the first degree, converges from the Dolbeault cohomology to the de Rahm cohomology of $M$. For compact Kähler manifolds, the Frölicher spectral sequence collapses at the $E_1$-term while examples due to H. Pittie and the third author show the spectral sequence need not collapse for even dimensional Lie groups. Chapter 4 concludes with a discussion of symplectic manifolds. In contrast to the Kähler case, these are not generally formal. Some conditions which do ensure formality for (cohomologically)-symplectic manifolds are given.

Chapter 5 takes up the study of geodesics on Riemannian manifolds. Following Vigué-Poirrier and Sullivan, a model for the free loop space $LX$ for any simply connected space $X$ is constructed. It is then a purely algebraic exercise to show $LX$ has unbounded Betti numbers when $H^*(X; \mathbb{Q})$ requires at least two generators. By the result of Gromoll-Meyer mentioned above, this implies the existence of infinitely many closed geodesics when $X = M$ is a closed Riemannian manifold. This basic result is generalized to the context of $A$-invariant geodesics where $A$ is a given isometry of $M$. Extending the Gromoll-Meyer Theorem, a

result of M. Tanaka implies $M$ admits infinitely many $A$-invariant geodesics provided the Betti numbers of $M_A^I$ are unbounded where $M_A^I$ denotes the space of paths $c\colon I \to M$ with $c(1) = A(c(0))$. A Sullivan model for $M_A^I$ is constructed and this construction, in turn, yields a theorem of K. Grove and Halperin giving the existence of infinitely many $A$-invariant geodesics for simply connected compact Riemannian manifolds with unbounded rational homotopy groups. A third question considered in this area concerns the asymptotic behavior, with respect to a length $T$, of the number $n_T$ of distinct geodesics of length $\leq T$. By a result of Gromov, this problem is again linked to the rational cohomology of the free loop space. Gromov's conjecture suggests that $n_T$ grows exponentially in the generic case.

Chapter 6 describes some recent results where algebraic models apply to problems related to curvature in differential geometry. Grove conjectured that the Riemannian $n$-manifolds admitting a metric with sectional curvature in a given range and diameter bounded by a fixed bound should represent only finitely many rational homotopy types. Counterexamples to this conjecture are described due to F. Fang and X. Rong and, independently, B. Totaro. Another problem in this area is the question of whether the total space $E$ of a real vector bundle $E \to M$ admits a metric with non-negative sectional curvature when $M$ does. Negative examples are given due to M. Özaydin and G. Walschap. A result of I. Belegradek and V. Kapovitch giving necessary conditions for the existence of such a metric in certain cases is framed in terms of the vanishing of derivations of rational cohomology. This result then establishes a surprising link between the question of the existence of metrics of non-negative sectional curvature and Halperin's conjecture, mentioned above.

Historically, one of the first examples of the applicability of rational homotopy theory to problems in geometry concerned toral actions on manifolds. These developments are described in Chapter 7. Given any space $X$, the rational toral rank of $X$, $\mathrm{rk}_0(X)$, is defined to be the maximal $r$ such that an $r$-dimensional torus acts almost freely on some finite complex $Y$ with the same rational homotopy type as $X$. The toral rank conjecture suggests an inequality of the form $\dim H^*(M;\mathbb{Q}) \geq 2^{\mathrm{rk}_0(M)}$ for all nilpotent, compact manifolds $M$. Basic rational homotopy theory techniques are applied to calculate the rational toral rank in special cases. A characterization in terms of relative Sullivan models of the inequality $\mathrm{rk}_0(M) \geq r$ is due to C. Allday and Halperin with further confirmed cases of the toral rank conjecture as consequences. A related problem is to describe the rational homotopy of the fixed point set corresponding to a smooth action of a torus on a manifold. Results of Allday and V. Puppe allow for an explicit description of

the rational homotopy Lie algebra of a component of this space provided $M$ has vanishing odd rational homotopy.

Chapters 8 describes some areas of current research. The work of P. Lambrechts and D. Stanley gives a Sullivan model for the complement of a submanifold $N \subseteq M$. Sullivan models for complex and symplectic blow-ups are used to prove the existence of non-formal, simply connected, symplectic manifolds following I. Babenko and I. Taimanov. Given $N$, a closed, oriented manifold, M. Chas and Sullivan recently defined a product, called the loop product, on the desuspension of the homology of the free loop space $LN$ with field coefficients. A rational model for the loop product is presented when $N$ is 2-connected. The rational loop algebra is identified with the rational Hochschild cohomology algebra following work of R. Cohen and J.D.S. Jones.

Chapter 9 surveys a variety of algebraic models for geometric constructions. Models for configuration spaces, hyperplane arrangements, algebraic varieties, function and section spaces are all presented with a number of examples and applications. K. Chen's theory of iterated integrals offers an alternate approach to rational homotopy theory and has proved useful in applications. The book concludes with a number of conjectures and open questions. A text of this scope has a large number of prerequisites. The authors address some of these background requirements with three appendices covering differential forms, spectral sequences and basic homotopy theory.

In summary, the book under review is an invaluable addition to the literature, surveying decades of vital research and taking the reader to the frontiers with countless open questions and paths into unexplored terrain. The book serves as an excellent companion volume to the recent text on rational homotopy theory by Félix-Halperin-Thomas.

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