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# BOOK REVIEW

*NIST Handbook of Mathematical Functions*, by Frank W. J. Olver, Daniel W. Lozier, Ronald F. Boisvert and Charles W. Clark (Eds), Cambridge University Press, Cambridge 2010, xv + 951pp., ISBN 978-0-521-19225-5 Hardback, ISBN 978-0-521-14063-8 Paperback.

The National Institute of Standard and Technology (NIST) Handbook of Mathematical Functions, together with its Web counterpart, the NIST Digital Library of Mathematical Functions (DLMF), is a heavily extended authoritative replacement for the well known and highly successful Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, published in 1964 by the National Bureau of Standards, edited by M. Abramowitz and I. A. Stegun [1] (referred further as A&S Handbook). Included with every copy of the book is a CD with a searchable PDF's of each chapter.

In this book the major ingredients of the classic A&S Handbook are retained while taking advantage of new technological achievements. The online version DLMF (at http://dlmf.nist.gov) presents the same information along with extensions and innovative interactive features consistent with the new medium. Among these features there is the interactive capability permitting generation of tables and graphs on demand.

The new Handbook and DLMF are products of many people: editors, associate editors, authors, and validators. A summary of the responsibilities of these groups of outstanding experts is given in the Preface.

The material of the book is organized in 36 chapters. Compared to the 29 chapters of A&S book in the present edition all numerical tables, including the tables of mathematical and physical constants (collected in A&S chapters 1 and 2) are omitted. The material of all other A&S chapters is extended, updated, reorganized and displayed with appropriate internal links and references. The list of References is updated to include hundreds of the research papers and books, published since the year 1964, in particular the five Handbooks of Integrals and Series by Prudnikov, Brychkov and Marichev. As a result more information is provided about the special functions and several new special functions are included (in chapters 16, 17, 21, 27, 29, 33-36).

The first four chapters may be viewed as introductory and methodological. They provide a detailed, but not overladen coverage of the most important mathematical topics for the theory and applications of special functions: algebraic, analytic and numerical methods, asymptotic approximations, and elementary transcendental functions. The highly mathematical style is avoided, while the mathematical correctness is maintained.

Chapter 26 addresses combinatorial analysis (lattice paths (binomial and multinomial coefficients), set partitions (Bell numbers, Stirling numbers), integer partitions, permutations). In Chapter 34 the definitions and the basic properties of $3j$, $6j$, and $9j$ Symbols (the angular momentum coupling coefficients) are described. These symbols (which are missing in the A&S) are essential in the study of symmetry problems of molecular, atomic, nuclear and particle systems.

The core material of the NIST handbook, the special functions and their applications, is exposed in the remaining 30 sections. Each chapter treats a separate class of special functions, but the material in almost all sections is structured in a similar manner in sections and paragraphs: Special Notations, Basic Definitions, Properties, Applications, Computation, References. The notations used in the current chapter are described and their relation to notations in some other books are given. The subsequent subsections present definitions, differential equations, integral representations, recurrence relations, functional relations, series expansions, asymptotic expansions, polynomial approximations, special values, derivative formulas, integrals, zeros, graphical representations, and so on. Application sections are available in all special function chapters. In most of the sections the applications are presented in two subsections, Mathematical and Physical, which facilitates the reader in finding applications of interest. Internal links to original papers and monographs, related to the specific applications are available. The inclusion of these application sections is the next important advantage of the NIST handbook, compared to the A&S one.

Section 5 addresses Gamma and related functions, Psi-, Beta, Double Gamma, Polygamma, q-Gamma, and q-Beta functions. The new moments here are q-Gamma, and q-Beta functions, which is a reflection of the intensive development of q-analysis in the last several decades. Among the noted physical applications here are the Rutherford scattering in nonrelativistic quantum mechanics and Veneciano model in particle physics. The Incomplete Gamma and Beta functions are considered in Chapter 8. Error functions and related Dawson's and Fresnel integrals are the subject of Chapter 7, where Cornu's spiral is among the number of noted applications . The Airy functions are subject of a separate chapter, Chapter 9 (unlike the A&S, where they are in a section of a chapter). New special functions included here (compared to A&S) are the Generalized Airy functions and the Incomplete Airy functions. A number of fields and problems in classical and quantum physics are pointed out in the section of Applications, among which there is the problem of quantum particle motion in a homogeneous external field, the exact wave function of which is expressed in terms of Airy function $\mathrm{Ai}(x)$ (example from Landau and Lifshitz textbook on Quantum mechanics). For other examples the reader is recommended to see the book [13].

The next Chapter 10 deals with Bessel Functions. This is the largest chapter in the Handbook. It is stretched over 72 pages, while the average length of a chapter is approximately 22 pages.

Here the properties of Bessel, Hankel, Kelvin, Modified Bessel and Spherical Bessel functions are described in greater detail. A number of integral representations and integrals of these functions is provided. It worths noting at this place that the range $|\mathrm{ph} z| < \pi/4$ of validity of the integral representation 10.32.10 of the modified Bessel function $K_\nu(z)$ can be larger, $|\mathrm{ph} z| < \pi/2$, as noted in [12]. The provided list of applications of Bessel and related functions is also very long: differential equations with poles and turning points, Laplace and Helmholtz equations, small oscillations of a uniform heavy flexible chain, microwave and optical transmission in waveguides etc. We should add that Bessel and related functions appear in a variety of problems in quantum mechanics and quantum optics. For example, the resolution unity measure for the $\mathrm{u}(p,q)$ algebra coherent states (CS) in quadratic boson representation is expressed in terms of $K_\nu$ in a form, which for $q = 1$ is proportional to the product $|z|^{-\nu} K_\nu(2|z|)$ with integer $\nu$ [12]. The Mittag-Leffler function was used in [11] to define new CS of light, called "Mittag-Leffler CS", Quantum oscillator with a singular perturbation $g/x^2$ admits CS with wave function proportional to the product of $\sqrt{x}$, quadratic exponent, and Bessel function $J_{2\kappa-1}(2x)$, $\kappa > 1/2$ [7]. The resolution unity measure density for these CS is equal to $(2/\pi) K_{2\kappa-1}(2|z|) I_{2\kappa-1}(2|z|)$. Many of the other special functions can be found in the expressions of Fock states or CS of quantum systems.

Struve, Parabolic cylinder, and Legendre and related functions are subjects of Chapters 11, 12 and 14. It is noted that, the physical applications of Parabolic cylinder and (associated) Legendre functions arise in connection with solutions to Helmholz and Laplace equations. The associated Legendre functions $P_\nu^\mu(x)$ arise naturally also in solutions to Schrödinger equation. For example, the transition amplitudes between $n$-$m$ oscillator energy levels under parametric excitation are expressed in terms of associated Legendre functions $P^{|n-m|/2}_{(n+m)/2}(\cos\theta/2)$ ([9] and references there in).

The Hypergeometric, Confluent hypergeometric, Generalized hypergeometric, and q-Hypergeometric and related functions are considered in Chapters 15, 13, 16 and 17 respectively. It is noted that these functions play essential role in exactly solvable statistical models, quantum mechanics, deformed symmetries, quantum groups etc. In quantum optics, for example, the $\mathfrak{su}(1,1)$-algebra CS are expressed, in the analytic representations on the complex plane and the unit disc, in terms of Kummer and Gauss hypergeometric functions [6].

The Classical Orthogonal polynomials and several other ones (Krawtchouk, Meixner, Pollaczek, Charlier, q-Jacobi, q-Laguerre, q-Hermite, etc.) are presented in Chapter 18. Covering 50 pages, this is the second largest chapter in the Handbook. A variety of applications are pointed out (with references), among which there are approximation theory, integrable systems, group representations, and Schrödinger equation for a particle in a parabolic potential (the harmonic oscillator), the energy states of which are given in terms of Hermite polynomials. Jacobi polynomials appear, e.g. in the expressions of the transition probabilities between Landau levels [9].

In the recent dacade or so intensive work has been carried out in generalizing the harmonic oscillator, using, instead of Hermite, other types of orthogonal polynomial (see [2–5, 8] and references their in). The classic Glauber CS are given as a series in Hermite polynomial wave functions, which are eigenstates of harmonic oscillator lowering operator. In a similar manner, generalized ladder operator CS associated with other orthogonal polynomials have been recently defined and constructed (for Laguerre [2,8], for Legendre and Chebyshev [2], for Meixner-Polachek [4], for Krawchuk [5]).

Chapters 19-22 deal with Elliptic integrals, Theta and Multidimensional theta functions, and Jacobian elliptic functions correspondingly. Several geometrical applications of elliptic integrals, such as lengths of curves, conformal map onto a rectangle, triaxial ellipsoids, and mutual inductance of coaxial circles are described briefly. Several interesting applications of Jacobian elliptic functions are briefly considered, among which there are nonlinear ODEs and PDEs, lengths and parametrization of plane curves, conformal mapping, classical dynamics of pendulum, quartic oscillator, tops. Recently [10] a new parametrization in terms of elliptic functions for the Mylar balloon was found, which proved efficient in calculations of various geometric quantities of this surface.

The subjects of Chapters 23-25 are Weierstrass elliptic and modular functions, Bernoulli and Euler polynomials, Zeta and related functions. Chapters 28-33 treat

Mathieu, Lamé, Spheroidal wave and Heun functions, Painlevé Transcendents, and Coulomb functions respectively. Functions of Number theory and Functions of matrix argument, which are missing in the A&S, now are included in Chapters 27 and 35. The last Chapter 36 is also new. Its subject are Integrals with coalescing saddles. More specifically, functions covered in this chapter are: cuspoid catastrophes, umbilic catastrophes with codimension three, canonical integrals, diffraction catastrophes. In the Application section the uniform approximation of integrals (in theory of cuspoids) and Kelvin's ship-wave pattern are outlined.

In conclusion, the NIST Handbook provides comprehensive information on hundreds of mathematical functions - from elementary transcendental ones and classical orthogonal polynomials to $q$-deformed special functions, functions of matrix arguments, and catastrophes. Their qualitative features are illustrated by numerous color Figures in two or three dimensions. This is a timely and authoritative modern replacement of the classic A&S, reflecting new results obtained since the year of the first NBS edition and new technological achievements. The associated DLMF may well serve as a model for the effective presentation of highly mathematical reference material on the Web.

The exposition is eminently readable and delightful, and everyone who works with or applies special mathematical functions will profit definitely from this impressive Handbook.

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