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BOOK REVIEW

Lectures on Real Semisimple Lie Algebras and their Representations, by Arkady L. Onischik, European Mathematical Society Publishing House, Zürich 2004, ix + 86 pp., 24.00 €, ISBN 3-03719-002-7

This small volume is an introduction to the subject by one of the great masters. The book contains a detailed study of involutive (anti)automorphisms of semisimple complex Lie algebras and their functorial properties, this is applied to obtain the classification of real forms of complex semisimple Lie algebras and describe the main classes of their representations.

Many topics traditionally belonging to the subject in the title (e.g. parabolic subalgebras, conjugate classes of Cartan subalgebras etc) could not be covered in such limited volume. But the reader who has mastered the material in the book is in a good condition to study and understand the wider treatises and current research, including the books of the author. This is to our knowledge the fifth book by Onishchik on Lie theory.

The theory of real semisimple Lie algebras emerged in the beginning of the twentieth century mainly in the works of Elie Cartan and Herman Weyl. The classification of irreducible symmetric spaces was found out to be equivalent to the classification of real simple Lie algebras, and the nontrivial part of the latter amounts to the classification of real forms of simple complex Lie algebras. The theory originates from the papers of E. Cartan [1], [2]. The "Weyl unitary trick" is the wide practice to reduce questions about the complex semisimple Lie algebra and their real forms to questions about the compact real forms. What makes the theory of the compact real Lie algebras easier to manage, is the fact that their Cartan subalgebras are all conjugate (as in the complex semisimple case) by an inner automorphism of $\mathfrak{g}$, and each element of $\mathfrak{g}$ belongs to some Cartan subalgebra. Neither is true for a noncompact real form.

The lecture notes of Onishchik begin with a short (no proofs) but remarkably complete survey of root systems and the classification of complex simple Lie algebras and their representations based on the combinatorics of roots and weights.

Then real forms of a complex semisimple Lie algebtra $\mathfrak{g}$ are introduced as the fixed subalgebras $\mathfrak{g}_0 = \mathfrak{g}^\sigma$ of an involutive antiautomorphism σ of $\mathfrak{g}$, called the real structure defining $\mathfrak{g}_0$.

Each complex semisimple Lie algebra has exactly one compact real form up to conjugation. This gives a distinguished class of conjugacy class of involutive antiautomorphisms, the compact real structures. The classical theorem of Cartan says thar for each real structure σ there exists a compact one τ, which commutes with σ. Thus we have a Cartan involution $\theta = \sigma \circ \tau \in \mathrm{Aut}(\mathfrak{g})$ which determines the real form $\mathfrak{g}_0$.

The Weyl involution of a complex semisimple Lie algebra $\mathfrak{g}$ is introduced as $\sigma \circ \tau$, where σ is the normal real structure of $\mathfrak{g}$ and τ is a compatible compact real structure. The explicit decomposition of the Weyl involution into a product of an inner automorphism of $\mathfrak{g}$ and a diagram automorphism induced by a symmerty of the Dynkin diagram, is obtained in Section 4.

The Cartan decomposition of a real semisimple Lie algebra $\mathfrak{g}_0$ is discussed in Section 5 on the base of the already obtained information about involutions. The corresponding multiplicative decomposition is studied only for the Lie group $\mathrm{Aut}(\mathfrak{g}_0)$, which is sufficient for the classification of inner and outer involutive automorphisms. A very straightforward proof using the exponent - logarithm bianalytic map between symmetric and positive definite symmetric elements is given.

The closing three sections of the book describe the real representations of real (semi)simple algebras. Section 6 fixes the general functorial properties of involutions under homomorphisms (representations). This gives conditions for inclusion of the image of a real form $\mathfrak{g}_0$ under a representation V in some real form of $\mathfrak{sl}(V)$.

The Cartan index $\epsilon(\rho)$ of a representation $\rho : \mathfrak{g} \leftarrow \mathfrak{gl}(V)$ is $1, -1$, when the representation ρ is real (respectively quaternionic). One of the proclaimed aims of Onishchik's lectures is to present a modernized and generalized version of results obtained by Karpelevich [3] in the early fifties. The Karpelevich index expresses the Cartan index of a representation ρ in terms of the highest weight Λ of ρ.

In the *Appendix* to the book (written by J. Silhan) one can find the theory of restricted roots, and the classification of real simple Lie algebras based on Satake diagrams, including a comprehensive table of Satake diagrams with the value of the Cartan (Karpelevich) index for each dominant weight.

References

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