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ON BOHR-SOMMERFELD-HEISENBERG QUANTIZATION

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Abstract. This paper presents the theory of Bohr-Sommerfeld-Heisenberg quantization of a completely integrable Hamiltonian system in the context of geometric quantization. The theory is illustrated with several examples.

1. Introduction

Most texts on quantum mechanics have a short section on the old quantum theory. They discuss Bohr's quantization of the harmonic oscillator and Sommerfeld's results on the energy spectrum of the hydrogen atom. Usually they mention of Heisenberg's quantum mechanics and give a description of Schrödinger's wave mechanics. Schrödinger's theory is further discussed in the framework of modern quantum mechanics. Heisenberg's theory is relegated to a criptic remark that Dirac proved that the theories of Heisenberg and of Schrödinger are equivalent. In [9], Dirac showed that Heisenberg's matrices can be also obtained in the Schrödinger theory, but he did not state that these theories give the same physical results.

Geometric quantization provides an explanation of Dirac's theory in the framework of modern differential geometry. Within geometric quantization, it is easy to understand Bohr-Sommerfeld quantization rules for completely integrable Hamiltonia systems, see [13]. If a Hamiltonian system with n-degrees of freedom has globally defined action-angle variables (A_i, φ_i), then the Bohr-Sommerfeld conditions define the structure of an n-dimensional lattice on the corresponding basis of the space of quantum states. Moreover, the action functions A_i are quantizable and the quantum operators $\boldsymbol{Q}_{A_k}$, $k = 1, 2, ...n$, corresponding to the action variables $A_1, ...A_n$, are diagonal in this basis.

The lattice structure of the basis defined by the Bohr-Sommerfeld conditions enables us to define n shifting operators that move the basic vectors along the vectors of the lattice. In the case of a regular infinite lattice the shifting operators may be interpreted as the quantization of the functions $\exp(-\mathrm{i}\varphi_1), ..., \exp(-\mathrm{i}\varphi_n)$ and their complex conjugates. In most examples, the lattice is not sufficiently regular

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