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A GENERALIZED CLAIRAUT'S THEOREM IN MINKOWSKI SPACE

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Abstract. In Euclidean space, the geodesics on a surface of revolution can be characterized by means of Clairaut's theorem, which essentially says that the geodesics are curves of fixed angular momentum. A similar result is known for three dimensional Minkowski space for timelike geodesics on surfaces of revolution about the time axis. Here, we extend this result to consider generalizations of surfaces of revolution to those surfaces generated by any one-parameter subgroup of the Lorentz group. We also observe that the geodesic flow in this case is easily seen to be a completely integrable system, and give the explicit formulae for the timelike geodesics.

1. Introduction

The change of signature from Euclidean to Minkowskian geometry results in a fascinating interplay between the two forms of geometry: there exists a formal algebraic similarity in many aspects of the geometry, coupled to important differences between the two, especially in global situations. The lecture notes of López [4], for example, provide a detailed consideration of many of the aspects of three dimensional Minkowski space. The differences arise in various way, and we will here be concerned with some of the consequences of the fact that vectors in Minkowski space can be classified as timelike, null, or spacelike by means of the inner product.

In a previous work [8] we considered surfaces of revolution in the situation with the closest analogy to the Euclidean situation, namely that of the timelike geodesics on surfaces obtained by rotating a timelike curve about the t-axis in Minkowski space. There are, of course, other types of surface of revolution in Minkowski space [3], and in this work we will extend our consideration to these other classes of surface.

We will begin in Section 2 by briefly reviewing Clairaut's theorem in the Euclidean case (for a more detailed exposition, see, for example, Pressley's text [7], and for applications see [1, 5, 6]), and the situation for rotations about the t-axis in Minkowski space. In Section 3 we see how there are three different types of one-parameter subgroup of the three dimensional Lorentz group, giving rise to three different class of surface of revolution. In Section 4 we establish an analogue of

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