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BOOK REVIEW

Lie-Bäcklund-Darboux Transformations by Charles Li and Artyom Yurov, International Press, Somerville 2014, ix+160pp, ISBN: 978-1-57146-288-6.

The Bäcklund-Darboux Transformations (BDT) are named after the pioneering works of Bäcklund and Darboux published at the end of 19-th century considering the sine-Gordon equation which at that time was related only to issues in classical differential geometry of surfaces. They introduced a procedure to find a new solution starting from a given one. The name Bäcklund transformation sometimes is attached to systems of functional equations involving function u and v in such a way that if u satisfies a given fixed partial differential equation the function v satisfies another fixed partial differential equation. The name Darboux transformation is then used in case the two partial differential equations are the same. The functional equation often contains the so-called 'spectral parameter' which importance has been understood much later as relations with some other classical topics in the spectral theory like the Crum-Krein formulas, insertion of additional bound states etc.

Interestingly, the issue, after being a field of active research for some period, gradually has been forgotten and came back again after the discovery in the 70-ties of the past century of the so called completely integrable nonlinear partial differential equations such as now famous Korteweg-de Vries equation and the nonlinear Schrödinger equation. The sine-Gordon equation turned out to be completely integrable also. The new technique applied to solve the completely integrable equations, known as Inverse Scattering Method (ISM) has been intensively developed in the past decades and remains a field of active research until now. The ISM for solving a given Partial differential equation involves a spectral problem $L\psi = 0$ (auxiliary linear problem) such that the differential equation to be solved could be written as compatibility condition $[L, A] = 0$ between the spectral problem $L\psi = 0$ and another spectral problem $A\psi = 0$. The representation of the given partial differential equation as $[L, A] = 0$ is called Lax representation. Thus given the operator L changing the operator A one in principle obtains a hierarchy of integrable equations.

Though the connection between the Inverse Scattering Transform and some integration techniques arising from it, such as the Dressing Method for example, from one side, and BDT from the other side, is quite direct, until now there is no theory that encompasses everything that is known about the BDT and there are many special examples that are very interesting and which are not well understood. To this situation of course contributes also the fact that the mathematical approaches used to study the integrable equations are showing very big variety.

There are several important monograph books treating the subject - for example the classical book of Matveev and Sall' [1], that of Rogers and Schief [2], and a quite recent one published by Gu, Hu and Zhou [3]. In particular, in the book of Matveev and Sall' is described the interpretation of the Darboux transformation as a canonical transformation for Hamiltonian systems.

The book under review treats some of the subjects reflected in the above monographs but it has some specific focuses and contains some new ideas. One of these is to use the Darboux transformations in order to study chaos, in particular in order to obtain the so called homoclinic orbit and Melnikov integral in several specific cases. This idea somewhat characterizes the discussion in Chapters 3-9 so perhaps this is the reason in the Preface the authors speak about first and second part of their book though there is no delimiter in the book where you can say that the first part ends and starts the second. It should be mentioned also that the chapters are indeed short and perhaps they could be called sections instead of chapters.

The book starts with a short introduction, (Chapter 1), then follows a general discussion and examples of Darboux transformations (Chapter 2). Then the study of chaotic behavior obtaining homoclinic orbist and Melnikov integrals is applied for the study of the following equations: Nonlinear Scrödinger Equation (Chapter 3), sine-Gordon Equation (Chapter 4), Heisenberg Ferromagnet Equation (Chapter 5), Vector Nonlinear Scrödinger Equation (Chapter 6), Derivative Nonlinear Scrödinger Equation (Chapter 7), Discrete Nonlinear Scrödinger Equation (Chapter 8), Davey-Stewartson II Equation (Chapter 9). All these equations are considered under periodic boundary conditions.

The second part of the present book gives many interesting higher-dimensional examples and various applications. As it is typical for books on Darboux transformations the number of equations and transformations is quite big but one can nevertheless describe very roughly the contents of the next chapters in the following way. Chapter 10 starts with the so-called Acoustic Spectral Problem, which is linked to the Schrödinger Spectral Problem. This permits to obtain Darboux transformation for the Harry-Dym Equation which is the first equation in the hierarchy of equations solvable using the Acoustic Spectral Problem as L operator. In the same Chapter another technique is applied to the Harry-Dym Equation and the modified Harry-Dym Equation obtaining interesting links to a number of integrable systems. In this chapter is also introduced a special type of Darboux transformation, the so-called Moutard transformation (in this case the initial and the "dressed" eigenfunction correspond to the same spectral value). Chapter 11 discusses the one-dimensional supersymmetric Schrodinger operator and modifications of the classical scheme of level addition. Chapter 12 is about the Darboux transformations for the four-dimensional Dirac equation. Chapter 13 (it is in fact only 4 pages) describes Moutard transformation for two and three dimensional Schrödinger Equations. Chapter 14 is about the Darboux transformations for the Boiti-Leon Pempinelli Equation and transformation permitting to obtain solutions of it starting from solutions of the Burgers Equation. Chapter 15 studies some class of Darboux transformations and another class (called Laplace transformations) of the Goursat Equation and Chapter 16 is about how by applying the so-called Borisov-Zykov's method one can get links between some well known integrable equations.

For a review the book contains not very large bibliography. Explicitly are cited the works of the classics as Darboux, Bäcklund and Lie, some other articles which the authors consider as seminal, as well as several of the authors' articles, related to the specific topics considered in the book and containing original results. It is true however that the most of the well-known monograph books are cited and from them one could trace all the necessary citations. Perhaps this has been done because as has been mentioned the book does not pretend to review the subject in general but rather to pay attention to some particular issues.

The book as a whole will be interesting for the researchers in the field of integrable systems as it will provide them with information about various links between different equations and theories. It could be used also by high graduate level students of Applied Mathematics and Mathematical Physics.

References

[1] Matveev V. and Sall' A., *Darboux Transformations and Solitons*, Springer, Berlin 1991.

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