BOOK REVIEW

A Student's Guide to Lagrangians and Hamiltonians, by Patrick Hamill, Cambridge University Press, Cambridge, 2014, x+173 pp, ISBN: 978-1-107-04288-9.

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The purpose of the book under review is announced in the Introduction: "to give the student of physics a basic overview of Lagrangians and Hamiltonians". The book follows a standard exposition, but nevertheless, it is not a traditional graduate textbook in analytical mechanics. As stated in the title and confirmed on the way of presentation, the book is intended to be a guide to Lagrangian and Hamiltonian techniques. The expected reader is supposed to have been acquainted, at least basically, with the fundamental concepts of classical mechanics, such as Newton's laws, reference frames, work and energy, and as well as of calculus in more than one variables, including scalar and vector operations, coordinate transformations, differentiation and integration of vectors.

The book consists of seven chapters, structured into two relevant parts of equal lengths: "Lagrangian Mechanics" and "Hamiltonian Mechanics". The first two chapters lay the fundamental setup. They comprise all the introductory material that bridges the two approaches to mechanics: the Newtonian vectorial way of exposing the laws of nature and the Lagrangian analytical formulation of mechanics.

Chapter 1, entitled "Fundamental Concepts", begins with a sequence of short sections devoted to generalized coordinates, generalized velocity, constraints, virtual displacements, virtual work, generalized forces, configuration space and phase space. After some historical remarks on the physical and philosophical aspects of the force-versus-energy viewpoint, dating back to Newton and Leibnitz, the author underlines two important properties of the Lagrangian approach in mechanics: "all of the equations in analytical mechanics are scalar quantities" and "the Lagrangian method incorporates constraints into the framework of the analysis". Then the Lagrangian and the Lagrange's equations are introduced without proof – their derivation from the first principles is postponed for the next chapters. Henceforward, till the end of Chapter 1, the Lagrangian formalism is applied to certain

problems in order to demonstrate its practical importance. Four worked examples show how to use the Lagrange's equations as a tool for obtaining the equation of motion.

On further reading of Chapter 1 the student learns about the connection between the symmetry properties inherent to space and time and the invariance of physical quantities. It is shown how the three conservation laws of classical mechanics, namely the laws of conservation of linear momentum, angular momentum and energy, can be derived as a consequence of homogeneity of space, isotropy of space and homogeneity of time, respectively. The notion of cyclic (ignorable) coordinates is also introduced and related to symmetry: "if a coordinate is cyclic, the system is symmetric under changes in that coordinate".

At the end of Chapter 1, the reader is led to the definitions of generalized momentum and Hamiltonian. The new concept of generalized momentum is justified by analogy from the observation that the linear momentum components of a free particle, in rectangular coordinates, are obtained by differentiation of the Lagrangian with respect to the velocity components: $p_x = \partial L/\partial \dot{x}$, $p_y = \partial L/\partial \dot{y}$, $p_z = \partial L/\partial \dot{z}$. The Hamiltonian first appears as a conserved quantity that equals under certain conditions to the total energy of the system. After passing to generalized momentum, the Hamiltonian takes its usual form of a function of the generalized coordinates, generalized momenta and possibly time: $H(q, p, t) = \sum \dot{q}_i p_i - L$. The virial theorem for a bounded conservative system is also considered.

Having passed through Chapter 1, following quite reasonable, intuitive and heuristic way of presentation the student gets a very brief, yet thorough, account of the basic concepts of analytical mechanics.

In the Introduction the author states: "We will focus on what are called variational techniques in mechanics." However, the reader is not expected to be familiar with such techniques. The next Chapter 2, named "The Calculus of Variations", is dedicated to provide with both the technical and the philosophical instrumentarium of the variational method to be used later to formulate the foundation principles of analytical mechanics. In the process of derivation of Euler-Lagrange equation (Section 2.2), the author's first concern is to clarify the notion of a stationary point of integral functional by making an analogy with the stationary point in the ordinary calculus of functions. In order to "make the things more concrete" three examples of determining the path that extremizes the functional are employed, those of the shortest distance (geodesic in a plane and on a sphere) and of the shortest time (the Brachistochrone problem). The Euler-Lagrange equation is derived rigorously, using a simple and clear language. Much care is taken to distinguish the use of the shorthand variational notation δ. Subsection 2.2.1 is designated to disclose the difference between δ, as a symbol of the virtual displacements, and

the differential d, representing the actual changes in the variables. A generalization of the Euler-Lagrange equation to functions with several dependent variables (Section 2.3), as well as, the Lagrange's λ -method for variational problems with constraints – holonomic (Section 2.4.1) and non-holonomic (Section 2.4.2), are considered. Throughout this chapter four solved examples (including the Dido problem), help the student to master the method of Euler-Lagrange equations.

In Chapter 3, named "Lagrangian Mechanics", Lagrange's equations of motion are derived from the fundamental variational principles of classical mechanics. It starts with a derivation based on d'Alembert's principle of virtual work and proceeds with obtaining Lagrange's equations from Hamilton's principle of least action. The virtual work and the Hamilton's principle are observed and clarified in separate sections. Then it is shown how the Lagrange's undetermined multipliers can be used to calculate the forces of constraint. The chapter ends with a discussion of an important and useful property of Lagrange's equations – their invariance (rather co-invariance) under point transformations.

The introduction to Hamiltonian mechanics begins in Chapter 4, "Hamilton's Equations", with the derivation of the Hamilton's "canonical" equations of motion. The transition from Lagrange's to Hamilton's equations is carried out in two different ways: first by using the Legendre transformation and secondly by applying the Hamilton's principle of stationary action. In order to show its usefulness in physics, the Legendre transformation is applied to generating thermodynamic potentials. The notions of phase space, phase curves and general solution of a dynamical problem are developed in analogy to the motion of the molecules in a "phase fluid". The Routhian procedure for elimination of cyclic coordinates is considered and applied to analyzing states of "steady motion". At the end of this chapter the Hamilton's equations of motion are expressed in symplectic matrix notation.

Chapter 5, "Canonical Transformations; Poisson Brackets", encompasses several central concepts of the Hamiltonian mechanics regarding the transformation and the invariance properties of the equations of motion. The canonical transformations are presented by four types of generating functions involving both old and new variables. In the Hamiltonian theory of motion the generalized coordinates q_i and momenta p_i are treated on equal footing, allowing transformations that changes q_i to p_i and vice versa. After giving an example of such a canonical transformation the author explains: "... the Hamiltonian formulation blurs the distinction between momenta and coordinates, and that is why I referred to both of them simply as coordinates." Then, based on an explicit expression of the infinitesimal canonical transformations, it is shown that the whole motion can be interpreted as a continuous transformation problem, in which the Hamiltonian is the generator of the "new" canonical variables at a later time. The canonical transformations can be

characterized by the help of the canonical invariants – quantities that remain unchanged under the action of the canonical transformations. On an advanced level the richness of the transformation properties of the Hamilton's equations is deeply related to in the geometrical structure of the phase space. The canonical invariants considered in this book are the Poisson brackets and the volumes in the phase space for the case of an incompressible phase fluid (Liouville's theorem). The chapter is finished by looking at angular momentum in terms of the Poisson brackets and relating the obtained formulas to the quantum mechanical commutation relations for the angular momentum operator.

The next Chapter 6, named "Hamilton-Jacobi Theory", begins with the derivation of the Hamilton-Jacobi equation. The Hamilton's principle function S is considered as a generating function of a canonical transformation that maps an arbitrary given set of canonical variables (q_i, p_i) to a set of constant phase coordinates (α_i, β_i). Then, the method of separation of variables is demonstrated by integrating the Hamilton-Jacobi equation for the simplest harmonic oscillator. Two short sections are devoted to the relationship between S and the action integral in its indefinite form $S = \int L \mathrm{d}t + \mathrm{const}$ (Section 6.3) and the correspondence between S and the phase of the wave function of material particles leading to the Shrödinger equation (Section 6.4).

The last Chapter 7, "Continuous Systems", gives a short account of the basic steps of a continuous procedure that makes a transition from discrete to continuous systems and to field theory. The Lagrangian and the Hamiltonian formulations of continuous systems are developed from their discrete approximations. The whole scheme of transition from Lagrangian of a particle to Lagrangian per unit length (Lagrangian density) is illustrated on two instructive examples, those of a vibrating string and an elastic rod. Then, the Lagrangian formalism is applied to obtain the Maxwell's equations of the electromagnetic field.

As a recapitulation, this book is a guide, namely it is designed to be a student's companion in his or her first steps in developing of an understanding of the Lagrangian and the Hamiltonian techniques in mechanics. The book is largely based on the classics of Goldstein [1] and Lanczos [4], and the more contemporary textbooks on the subject [2,3,5]. The exposition relies on the student's activity to solve problems, as many of the basic concepts are illustrated throughout the text by carefully selected examples and exercises. There are also problems given at the end of each chapter. Thus, in a logically clear and physically rigorous way the book highlights the landmarks of the analytical mechanics so that the attentive student can be easily prepared for the exam. It is suitable for studying in intermediate and upper-level undergraduate courses of classical mechanics.

References

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