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TWISTOR SPACES AND COMPACT MANIFOLDS ADMITTING BOTH KÄHLER AND NON-KÄHLER STRUCTURES*

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Abstract. In this expository paper we review some twistor techniques and recall the problem of finding compact differentiable manifolds that can carry both Kähler and non-Kähler complex structures. Such examples were constructed independently by Atiyah, Blanchard and Calabi in the 1950's. In the 1980's Tsanov gave an example of a simply connected manifold that admits both Kähler and non-Kähler complex structures - the twistor space of a $K3$ surface. Here we show that the quaternion twistor space of a hyperkähler manifold has the same property.

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1. Introduction

In this paper we discuss a couple of classical approaches to twistor theory. Roughly speaking, the twistor space $Z(M)$ is a family of (almost) complex structures on an orientable Riemannian manifold (M, g) compatible with the given metric g and the orientation. We are going to apply twistor techniques towards the problem of constructing simply connected compact manifolds that carry both Kähler and non-Kähler complex structures.

Atiyah's idea behind his examples in [1] was to consider the set of all complex structures M_n on the real torus $T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ coming from the complex vector space structures on $\mathbb{R}^{2n}$. The space M_n is a complex manifold which is differentially a product of an algebraic variety and the torus T^{2n}, and therefore admits a Kähler structure. On the other hand, there exists a "twisted" complex structure on M_n which is non-Kähler. This rationale works in many other cases, and in particular, one can produce simply connected examples of similar nature. In [22] Tsanov showed that the twistor space of a $K3$ surface is a simply connected 6-dimensional compact manifold which carries both Kähler and non-Kähler complex structures. Here we give examples of twistor spaces of hyperkähler manifolds and show that they also carry both Kähler and non-Kähler complex structures.

*Dedicated to the memory of Professor Vasil V. Tsanov 1948-2017.

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