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LIE ALGEBRA EXTENSIONS AND HIGHER ORDER COCYCLES

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Abstract. In this note we present an abstract approach, based on Lie algebra cohomology, to the Lie algebra extensions associated to symplectic manifolds. We associate to any Lie algebra cocycle of degree at least two an abelian extension by some space $\mathfrak{a}$ and central extensions of subalgebras analogous to the Lie algebras of symplectic, respectively, hamiltonian vector fields. We even obtain a Poisson bracket on $\mathfrak{a}$ compatible with the hamiltonian Lie subalgebra. We then describe how this general approach provides a unified treatment of cocycles defined by closed differential forms on Lie algebras of vector fields on possibly infinite-dimensional manifolds.

Introduction

If (M,ω) is a finite-dimensional symplectic manifold, then we assign to each smooth function $f\colon M \to \mathbb{R}$ its hamiltonian vector field X_f determined uniquely by $i_{X_f}\omega = \mathrm{d}f$, and this leads to a central extension of Lie algebras

$$\mathbf{0} \to H^0_{\mathrm{dR}}(M,\mathbb{R}) \to (C^\infty(M,\mathbb{R}), \{\cdot,\cdot\}) \to \mathfrak{ham}(M,\omega) \to \mathbf{0} \tag{1}$$

where $\mathfrak{ham}(M,\omega)$ denotes the Lie algebra of hamiltonian vector fields on M and the Lie bracket on $C^\infty(M,\mathbb{R})$ is given by the Poisson bracket $\{f,g\} := \omega(X_f, X_g)$. Since a symplectic vector field X on M is hamiltonian if and only if the closed one-form $i_X\omega$ is exact and each closed one-form can be written as $i_X\omega$ for a symplectic vector field X, the exact sequence (1) can be extended to a four term exact sequence

$$\mathbf{0} \to H^0_{\mathrm{dR}}(M,\mathbb{R}) \to C^\infty(M,\mathbb{R}) \to \mathfrak{sp}(M,\omega) \to H^1_{\mathrm{dR}}(M,\mathbb{R}) \to \mathbf{0}. \tag{2}$$

The central extension (1) can also be embedded into an abelian extension

$$\mathbf{0} \to C^\infty(M,\mathbb{R}) \to \widehat{\mathcal{V}}(M) := C^\infty(M,\mathbb{R}) \oplus_\omega \mathcal{V}(M) \to \mathcal{V}(M) \to \mathbf{0} \tag{3}$$