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IVAÏLO M. MLADENOV

Presented by Ivaïlo M. Mladenov

Abstract. Here we derive a bunch of explicit formulas for the circumferences of all types of Cassinian ovals in terms of the complete elliptic integrals of the first kind and their equivalent expressions in terms of the hypergeometric functions.

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1. Introduction

Cassinian ovals were introduced in 1680 by Giovanni Domenico Cassini as an alternative to the Newton's model for the orbit of the Earth around the Sun [8]. Besides being a model for the orbits of planets, Cassinian curves appear in various scientific disciplines such as biology [1,7], nuclear physics [19], demography [20], theory of light [11], matrix theory [4], engineering [10], etc.
Because of their so universal applications it seems reasonable that their geometrical properties should be much better known in general. The aim of the present paper is to present how one should proceed if he/she is interested in the statement of the title. Before that, a little bit history.

More than a century ago Matz [12, p. 222] had derived the formula (the geometrical meaning of these notation will be explained in the next Section)

$$\frac{4c^2}{\sqrt{a^2+c^2}}\int_0^{\pi/2}\Big(1-\frac{4a^2c^2}{(a^2+c^2)^2}\sin^2\phi\Big)^{-1/4}\,\mathrm{d}\phi, \qquad a,c\in\mathbb{R}^+ \tag{1}$$

for the perimeter of the Cassinian oval. Actually, the above formula is nothing more than a definition of the real problem which can be stated as how to evaluate the integral in (1)? From the mathematical point of view the problem behind this question is that the above expression is nothing less then a hyperelliptic integral which is not present in whatever form in the specialized tables of integrals.

That is why Matz proceeds as follows.