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GRASSMANNIAN SIGMA-MODELS

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Abstract. We study solutions of Grassmannian sigma-model both in finite-dimensional and infinite-dimensional settings. Mathematically, such solutions are described by harmonic maps from the Riemann sphere $\mathbb{CP}^1$ or, more generally, compact Riemann surfaces to Grassmannians. We describe first how to construct harmonic maps from compact Riemann surfaces to the Grassmann manifold $\mathrm{G}_r(\mathbb{C}^d)$, using the twistor approach. Then we switch to the infinite-dimensional setting and consider harmonic maps from compact Riemann surfaces to the Hilbert–Schmidt Grassmannian $\mathrm{Gr}_{\mathrm{HS}}(H)$ of a complex Hilbert space H. Solutions of this infinite-dimensional sigma-model are, conjecturally, related to Yang–Mills fields on $\mathbb{R}^4$.

1. Introduction

In this paper we describe classical solutions of Grassmannian sigma-models in finite-dimensional and infinite-dimensional settings. The study of such solutions in the finite-dimensional case was initiated by physicists (cf. e.g., [4,8,13]). Mathematically, sigma-model solutions correspond to harmonic maps from compact Riemann surfaces to Grassmannians $\mathrm{G}_r(\mathbb{C}^d)$.

In the first part of this paper (Sections 2, 3 and 4) we explain how to construct such maps, using the twistor approach. The main idea of this approach, when applied to the construction of harmonic maps from a Riemann surface M to a given Riemannian manifold N, is to construct a certain twistor bundle $\pi : Z \to N$ over N, which has the following property. The twistor space Z is an almost complex manifold such that for any pseudoholomorphic map $\psi : M \to Z$ its projection $\varphi := \pi \circ \psi$ to N is a harmonic map $\varphi : M \to N$. In our case $N = \mathrm{G}_r(\mathbb{C}^d)$ and the role of the twistor bundle over $\mathrm{G}_r(\mathbb{C}^d)$ is played by homogeneous flag bundles $\pi : \mathcal{F}_{\mathbf{r}}(\mathbb{C}^d) \to \mathrm{G}_r(\mathbb{C}^d)$. Using the twistor approach, one can try to reduce the original "real" problem of constructing harmonic maps of compact Riemann surfaces M to $\mathrm{G}_r(\mathbb{C}^d)$ to the "complex" problem of constructing pseudoholomorphic