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# A GRUSS TYPE INEQUALITY FOR SEQUENCES OF VECTORS IN INNER PRODUCT SPACES AND APPLICATIONS

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## Abstract

A Grüss type inequality for sequences of vectors in inner product spaces which complement a recent result from [6] and applications for differentiable convex functions defined on inner product spaces and applications for Fourier and Mellin transforms, are given.

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# Contents

# 1. Introduction

In 1935, G. Grüss proved the following integral inequality (see [11] or [12])

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \le \frac{1}{4}(\Phi-\phi)(\Gamma-\gamma), \tag{1.1}$$

provided that $f$ and $g$ are two integrable functions on $[a,b]$ and satisfy the condition

$$\phi \le f(x) \le \Phi \text{ and } \gamma \le g(x) \le \Gamma \text{ for all } x\in[a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is the *best possible* and is achieved for

$$f(x)=g(x)=sgn\left(x-\frac{a+b}{2}\right).$$

The discrete version of (1.1) states that:

If $a\le a_i\le A$, $b\le b_i\le B$ $(i=1,\dots,n)$ where $a,\ A,\ a_i,\ b,\ B,\ b_i$ are real numbers, then

$$\left|\frac{1}{n}\sum_{i=1}^n a_ib_i-\frac{1}{n}\sum_{i=1}^n a_i\,\frac{1}{n}\sum_{i=1}^n b_i\right|\le\frac{1}{4}(A-a)(B-b) \tag{1.3}$$

and the constant $\frac{1}{4}$ is the best possible.

In the recent paper [2], the author proved the following generalisation in inner product spaces.

**Theorem 1.1.** *Let* $(X;\langle\cdot,\cdot\rangle)$ *be an inner product space over* $\mathbb{K}, \mathbb{K}=\mathbb{C},\mathbb{R}$, *and* $e\in X$, $\|e\|=1$. *If* $\phi,\Phi,\gamma,\Gamma\in\mathbb{K}$ *and* $x,y\in X$ *such that*

$$\operatorname{Re}\langle\Phi e-x,x-\phi e\rangle\geq 0 \text{ and } \operatorname{Re}\langle\Gamma e-y,y-\gamma e\rangle\geq 0 \tag{1.4}$$

*holds, then we have the inequality*

$$|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|\leq\frac{1}{4}|\Phi-\phi||\Gamma-\gamma|. \tag{1.5}$$

*The constant* $\frac{1}{4}$ *is the best possible.*

It has been shown in [1] that the above theorem, for the real case, contains the usual integral and discrete Grüss inequality and also some Grüss type inequalities for mappings defined on infinite intervals.

Namely, if $\rho:(-\infty,+\infty)\to(-\infty,+\infty)$ is a probability density function, i.e., $\int_{-\infty}^{\infty}\rho(t)\,dt=1$, then $\rho^{\frac{1}{2}}\in L^2(-\infty,\infty)$ and obviously $\|\rho^{\frac{1}{2}}\|_2=1$. Consequently, if we assume that $f,g\in L^2(-\infty,\infty)$ and

$$\alpha\rho^{\frac{1}{2}}\leq f\leq\psi\rho^{\frac{1}{2}},\ \beta\rho^{\frac{1}{2}}\leq g\leq\theta\rho^{\frac{1}{2}} \text{ a.e. on } (0,\infty), \tag{1.6}$$

then we have the inequality

$$\left|\int_{-\infty}^{\infty}f(t)\,g(t)\,dt-\int_{-\infty}^{\infty}f(t)\,\rho^{\frac{1}{2}}(t)\,dt\int_{-\infty}^{\infty}g(t)\,\rho^{\frac{1}{2}}(t)\,dt\right| \leq\frac{1}{4}(\psi-\alpha)(\theta-\beta). \tag{1.7}$$

In a similar way, if $e = (e_i)_{i\in\mathbb{N}} \in l^2(\mathbb{R})$ with $\sum_{i\in\mathbb{N}} |e_i|^2 = 1$ and $x = (x_i)_{i\in\mathbb{N}}$, $y = (y_i)_{i\in\mathbb{N}} \in l^2(\mathbb{R})$ are such that

$$\alpha e_i \le x_i \le \psi e_i, \quad \beta e_i \le y_i \le \theta e_i \tag{1.8}$$

for all $i \in \mathbb{N}$, then we have

$$\left| \sum_{i\in\mathbb{N}} x_i y_i - \sum_{i\in\mathbb{N}} x_i e_i \sum_{i\in\mathbb{N}} y_i e_i \right| \le \frac{1}{4} (\psi - \alpha)(\theta - \beta). \tag{1.9}$$

In the recent paper [6], the author also proved the following discrete inequality in inner product spaces:

$$\left\| \sum_{i=1}^{n} p_i a_i x_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i x_i \right\| \le \frac{1}{4} |A - a| \, \|X - x\| \tag{1.10}$$

provided $x_i \in H$, $a_i \in \mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ and $a, A \in \mathbb{K}$, $x, X \in H$ are such that

$$\operatorname{Re}\left[(A - a_i)(\bar{a}_i - \bar{a})\right] \ge 0 \text{ and } \quad \operatorname{Re}\langle X - x_i, x_i - x\rangle \ge 0 \text{ for all } i \in \{1, \dots, n\}. \tag{1.11}$$

The constant $\frac{1}{4}$ is sharp.

For other recent developments of the Grüss inequality, see the papers [1]-[6], [10] and the website `http://rgmia.vu.edu.au/Gruss.html`

In this paper we point out some other Grüss type inequalities in inner product spaces which will complement the above result (1.10).

## 2. Preliminary Results

The following lemma is of interest in itself (see also [6]).

**Lemma 2.1.** *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $x_i \in H$ and $p_i \geq 0$ $(i=1,\dots,n)$ such that $\sum_{i=1}^{n} p_i = 1$ $(n \geq 2)$. If $x, X \in H$ are such that*

$$\operatorname{Re}\langle X - x_i, x_i - x\rangle \geq 0 \text{ for all } i \in \{1,\dots,n\}, \tag{2.1}$$

*then we have the inequality*

$$0 \leq \sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2 \leq \frac{1}{4}\|X - x\|^2. \tag{2.2}$$

*The constant $\frac{1}{4}$ is sharp.*

*Proof.* Define

$$I_1 := \left\langle X - \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i x_i - x \right\rangle$$

and

$$I_2 := \sum_{i=1}^{n} p_i \langle X - x_i, x_i - x\rangle.$$

Then

$$I_1 = \sum_{i=1}^{n} p_i \langle X, x_i\rangle - \langle X, x\rangle - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2 + \sum_{i=1}^{n} p_i \langle x_i, x\rangle$$

and

$$I_2 = \sum_{i=1}^{n} p_i \langle X, x_i \rangle - \langle X, x \rangle - \sum_{i=1}^{n} p_i \|x_i\|^2 + \sum_{i=1}^{n} p_i \langle x_i, x \rangle .$$

Consequently

$$I_1 - I_2 = \sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2 . \tag{2.3}$$

Taking the real value in (2.3) we can state

$$\sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2 = \operatorname{Re} \left\langle X - \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i x_i - x \right\rangle - \sum_{i=1}^{n} p_i \operatorname{Re} \langle X - x_i, x_i - x \rangle , \tag{2.4}$$

which is an identity of interest in itself.

Using the assumption (2.1), we can conclude, by (2.4), that

$$\sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2 \le \operatorname{Re} \left\langle X - \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i x_i - x \right\rangle . \tag{2.5}$$

It is known that if $y, z \in H$, then

$$4 \operatorname{Re} \langle z, y \rangle \le \|z + y\|^2 , \tag{2.6}$$

with equality iff $z = y$.

Now, by (2.6), we can state that

$$\operatorname{Re}\left\langle X - \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i x_i - x \right\rangle \leq \frac{1}{4} \left\| X - \sum_{i=1}^{n} p_i x_i + \sum_{i=1}^{n} p_i x_i - x \right\|^2 = \frac{1}{4} \|X - x\|^2 .$$

Using (2.5), we can easily deduce (2.2).

To prove the sharpness of the constant $\frac{1}{4}$, let us assume that the inequality (2.2) holds with a constant $c > 0$, i.e.,

$$0 \leq \sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2 \leq c \|X - x\|^2 \tag{2.7}$$

for all $p_i, x_i$ and $x, X$ as in the hypothesis of Lemma 2.1.

Assume that $n = 2$, $p_1 = p_2 = \frac{1}{2}$, $x_1 = x$ and $x_2 = X$ with $x, X \in H$ and $x \neq X$. Then, obviously,

$$\langle X - x_1, x_1 - x \rangle = \langle X - x_2, x_2 - x \rangle = 0,$$

which shows that the condition (2.1) holds.

If we replace $n$, $p_1$, $p_2$, $x_1$, $x_2$ in (2.7), we obtain

$$\begin{aligned} \sum_{i=1}^{2} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{2} p_i x_i \right\|^2 &= \frac{1}{2} \left( \|x\|^2 + \|X\|^2 - \left\| \frac{x + X}{2} \right\|^2 \right) \\ &= \frac{\|X - x\|^2}{4} \leq c \|X - x\|^2 , \end{aligned}$$

from where we deduce $c \geq \frac{1}{4}$, which proves the sharpness of the constant factor $\frac{1}{4}$. □

**Remark 2.1.** *The assumption (2.1) can be replaced by the more general condition*

$$\sum_{i=1}^{n} p_i \operatorname{Re} \langle X - x_i, x_i - x \rangle \geq 0 \tag{2.8}$$

*and the conclusion (2.2) will still remain valid.*

The following corollary is natural.

**Corollary 2.2.** *Let* $a_i \in \mathbb{K}$, $p_i \geq 0$ $(i = 1, \ldots, n)$ $(n \geq 2)$ *with* $\sum_{i=1}^{n} p_i = 1$. *If* $a, A \in \mathbb{K}$ *are such that*

$$\operatorname{Re}\left[(A - a_i)(\bar{a}_i - \bar{a})\right] \geq 0 \text{ for all } i \in \{1, \ldots, n\}, \tag{2.9}$$

*then we have the inequality*

$$0 \leq \sum_{i=1}^{n} p_i |a_i|^2 - \left| \sum_{i=1}^{n} p_i a_i \right|^2 \leq \frac{1}{4} |A - a|^2. \tag{2.10}$$

*The constant $\frac{1}{4}$ is sharp.*

The proof follows by the above Lemma 2.1 by choosing $H = \mathbb{K}$, $\langle x, y \rangle := x\bar{y}$, $x_i = a_i$, $x = a$, $X = A$. We omit the details.

**Remark 2.2.** *The condition (2.9) can be replaced by the more general assumption*

$$\sum_{i=1}^{n} p_i \operatorname{Re}\left[(A - a_i)\left(\bar{a}_i - \bar{a}\right)\right] \geq 0. \tag{2.11}$$

**Remark 2.3.** *If we assume that* $\mathbb{K} = \mathbb{R}$*, then (2.8) is equivalent with*

$$a \leq a_i \leq A \ \text{ for all } i \in \{1, ..., n\} \tag{2.12}$$

*and then, with the assumption (2.12), we get the discrete Grüss type inequality*

$$0 \leq \sum_{i=1}^{n} p_i a_i^2 - \left(\sum_{i=1}^{n} p_i a_i\right)^2 \leq \frac{1}{4}(A - a)^2 \tag{2.13}$$

*and the constant* $\frac{1}{4}$ *is sharp.*

## 3. A Discrete Inequality of Grüss Type

The following Grüss type inequality holds.

**Theorem 3.1.** *Let* $(H;\langle\cdot,\cdot\rangle)$ *be an inner product space over* $\mathbb{K}$; $\mathbb{K}=\mathbb{C},\mathbb{R}$, $x_i,y_i\in H$, $p_i\geq 0$ $(i=0,\dots,n)$ $(n\geq 2)$ *with* $\sum_{i=1}^{n}p_i=1$. *If* $x,X,y,Y\in H$ *are such that*

$$\operatorname{Re}\langle X-x_i,x_i-x\rangle\geq 0 \text{ and } \operatorname{Re}\langle Y-y_i,y_i-y\rangle\geq 0 \text{ for all } i\in\{1,\dots,n\}, \tag{3.1}$$

*then we have the inequality*

$$\left|\sum_{i=1}^{n}p_i\langle x_i,y_i\rangle-\left\langle\sum_{i=1}^{n}p_ix_i,\sum_{i=1}^{n}p_iy_i\right\rangle\right|\leq\frac{1}{4}\|X-x\|\,\|Y-y\|. \tag{3.2}$$

*The constant* $\frac{1}{4}$ *is sharp.*

*Proof.* A simple calculation shows that

$$\sum_{i=1}^{n}p_i\langle x_i,y_i\rangle-\left\langle\sum_{i=1}^{n}p_ix_i,\sum_{i=1}^{n}p_iy_i\right\rangle=\frac{1}{2}\sum_{i,j=1}^{n}p_ip_j\langle x_i-x_j,y_i-y_j\rangle. \tag{3.3}$$

Taking the modulus in both parts of (3.3), and using the generalized triangle inequality, we obtain

$$\left|\sum_{i=1}^{n}p_i\langle x_i,y_i\rangle-\left\langle\sum_{i=1}^{n}p_ix_i,\sum_{i=1}^{n}p_iy_i\right\rangle\right|\leq\frac{1}{2}\sum_{i,j=1}^{n}p_ip_j\left|\langle x_i-x_j,y_i-y_j\rangle\right|. \tag{3.4}$$

By Schwartz's inequality in inner product spaces we have

$$|\langle x_i - x_j, y_i - y_j \rangle| \leq \|x_i - x_j\| \, \|y_i - y_j\| \tag{3.5}$$

for all $i, j \in \{1, ..., n\}$, and therefore

$$\left| \sum_{i=1}^{n} p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i \right\rangle \right| \leq \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\| \, \|y_i - y_j\| . \tag{3.6}$$

Using the Cauchy-Buniakowsky-Schwartz inequality for double sums, we can state that

$$\begin{aligned} \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\| \, \|y_i - y_j\| & \\ \leq \left( \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\|^2 \right)^{\frac{1}{2}} & \times \left( \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \|y_i - y_j\|^2 \right)^{\frac{1}{2}} \end{aligned} \tag{3.7}$$

and, a simple calculation shows that,

$$\frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\|^2 = \sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2$$

and

$$\frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \|y_i - y_j\|^2 = \sum_{i=1}^{n} p_i \|y_i\|^2 - \left\| \sum_{i=1}^{n} p_i y_i \right\|^2 .$$

We obtain

$$\left|\sum_{i=1}^{n} p_i \langle x_i, y_i\rangle - \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i \right\rangle\right| \leq \left(\sum_{i=1}^{n} p_i \|x_i\|^2 - \left\|\sum_{i=1}^{n} p_i x_i\right\|^2\right)^{\frac{1}{2}} \times \left(\sum_{i=1}^{n} p_i \|y_i\|^2 - \left\|\sum_{i=1}^{n} p_i y_i\right\|^2\right)^{\frac{1}{2}}. \tag{3.8}$$

Using Lemma 2.1, we know that

$$\left(\sum_{i=1}^{n} p_i \|x_i\|^2 - \left\|\sum_{i=1}^{n} p_i x_i\right\|^2\right)^{\frac{1}{2}} \leq \frac{1}{2}\|X - x\|$$

and

$$\left(\sum_{i=1}^{n} p_i \|y_i\|^2 - \left\|\sum_{i=1}^{n} p_i y_i\right\|^2\right)^{\frac{1}{2}} \leq \frac{1}{2}\|Y - y\|.$$

Therefore, by (3.8) we may deduce the desired inequality (3.3).

To prove the sharpness of the constant $\frac{1}{4}$, let us assume that (3.2) holds with a constant $c > 0$, i.e.,

$$\left|\sum_{i=1}^{n} p_i \langle x_i, y_i\rangle - \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i \right\rangle\right| \leq c\|X - x\|\|Y - y\| \tag{3.9}$$

under the above assumptions for $p_i,\ x_i,\ y_i,\ x,\ X,\ y,\ Y$ and $n \geq 2$.

If we choose $n=2$, $x_1=x,\ x_2=X$, $y_1=y,\ y_2=Y\ \ (x\neq X,\ y\neq Y)$ and $p_1=p_2=\frac{1}{2}$, then

$$\begin{aligned}\sum_{i=1}^{2}p_i\left\langle x_i,y_i\right\rangle-\left\langle\sum_{i=1}^{2}p_ix_i,\sum_{i=1}^{2}p_iy_i\right\rangle &= \frac{1}{2}\sum_{i,j=1}^{2}p_ip_j\left\langle x_i-x_j,y_i-y_j\right\rangle\\ &= \sum_{1\leq i<j\leq 2}p_ip_j\left\langle x_i-x_j,y_i-y_j\right\rangle\\ &= \frac{1}{4}\left\langle x-X,y-Y\right\rangle\end{aligned}$$

and then

$$\left|\sum_{i=1}^{2}p_i\left\langle x_i,y_i\right\rangle-\left\langle\sum_{i=1}^{2}p_ix_i,\sum_{i=1}^{2}p_iy_i\right\rangle\right|=\frac{1}{4}\left|\left\langle x-X,y-Y\right\rangle\right|.$$

Choose $X-x=z$, $Y-y=z$, $z\neq 0$. Then using (3.9), we derive

$$\frac{1}{4}\left\|z\right\|^2\leq c\left\|z\right\|^2,\quad z\neq 0$$

which implies that $c\geq\frac{1}{4}$, and the theorem is proved. □

**Remark 3.1.** *The condition (3.1) can be replaced by the more general assumption*

$$\sum_{i=1}^{n}p_i\operatorname{Re}\left\langle X-x_i,x_i-x\right\rangle\geq 0,\quad \sum_{i=1}^{n}p_i\operatorname{Re}\left\langle Y-y_i,y_i-y\right\rangle\geq 0 \tag{3.10}$$

*and the conclusion (3.2) still remains valid.*

The following corollary for real or complex numbers holds.

**Corollary 3.2.** *Let* $a_i, b_i \in \mathbb{K}\ (\mathbb{K} = \mathbb{C}, \mathbb{R})$, $p_i \geq 0\ (i = 1, \dots, n)$ *with* $\sum_{i=1}^{n} p_i = 1$. *If* $a, A, b, B \in \mathbb{K}$ *are such that*

$$\operatorname{Re}\left[(A - a_i)\left(\bar{a}_i - \bar{a}\right)\right] \geq 0 \ \text{ and } \ \operatorname{Re}\left[(B - b_i)\left(\bar{b}_i - \bar{b}\right)\right] \geq 0, \tag{3.11}$$

*then we have the inequality*

$$\left| \sum_{i=1}^{n} p_i a_i \bar{b}_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i \bar{b}_i \right| \leq \frac{1}{4} |A - a| \, |B - b| \tag{3.12}$$

*and the constant* $\frac{1}{4}$ *is sharp.*

The proof is obvious by Theorem 3.1 applied for the inner product space $(\mathbb{C}, \langle \cdot, \cdot \rangle)$ where $\langle x, y \rangle = x \cdot \bar{y}$. We omit the details.

**Remark 3.2.** *The condition (3.11) can be replaced by the more general condition*

$$\sum_{i=1}^{n} p_i \operatorname{Re}\left[(A - a_i)\left(\bar{a}_i - \bar{a}\right)\right] \geq 0, \quad \sum_{i=1}^{n} p_i \operatorname{Re}\left[(B - b_i)\left(\bar{b}_i - \bar{b}\right)\right] \geq 0 \tag{3.13}$$

*and the conclusion of the above corollary will still remain valid.*

**Remark 3.3.** *If we assume that* $a_i$, $b_i$, $a$, $b$, $A$, $B$ *are real numbers, then (3.11) is equivalent to*

$$a \leq a_i \leq A, \quad b \leq b_i \leq B \ \text{ for all } i \in \{1, \dots, n\} \tag{3.14}$$

*and (3.12) becomes*

$$0 \leq \left| \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i \right| \leq \frac{1}{4} (A - a)(B - b), \tag{3.15}$$

*which is the classical Grüss inequality for sequences of real numbers.*

## 4. Applications for Convex Functions

Let $(H;\langle\cdot,\cdot\rangle)$ be a real inner product space and $F:H\to\mathbb{R}$ a Fréchet differentiable convex mapping on $H$. Then we have the "gradient inequality"

$$F(x)-F(y)\geq\langle\nabla F(y),x-y\rangle \tag{4.1}$$

for all $x,y\in H$, where $\nabla F:H\to H$ is the gradient operator associated to the differentiable convex function $F$.

The following theorem holds.

**Theorem 4.1.** *Let $F:H\to\mathbb{R}$ be as above and $x_i\in H$ $(i=1,\dots,n)$. Suppose that there exists the vectors $\gamma,\phi\in H$ such that $\langle x_i-\gamma,\phi-x_i\rangle\geq 0$ for all $i\in\{1,\dots,m\}$ and $m,M\in H$ such that $\langle\nabla F(x_i)-m,\ M-\nabla F(x_i)\rangle\geq 0$ for all $i\in\{1,\dots,m\}$. Then for all $p_i\geq 0$ $(i=1,\dots,m)$ with $P_m:=\sum_{i=1}^{m}p_i>0$, we have the inequality*

$$0\leq\frac{1}{P_m}\sum_{i=1}^{m}p_iF(x_i)-F\left(\frac{1}{P_m}\sum_{i=1}^{m}p_ix_i\right)\leq\frac{1}{4}\|\phi-\gamma\|\,\|M-m\|. \tag{4.2}$$

*Proof.* Choose in (4.1), $x=\frac{1}{P_M}\sum_{i=1}^{m}p_ix_i$ and $y=x_j$ to obtain

$$F\left(\frac{1}{P_m}\sum_{i=1}^{m}p_ix_i\right)-F(x_j)\geq\left\langle\nabla F(x_j),\frac{1}{P_m}\sum_{i=1}^{m}p_ix_i-x_j\right\rangle \tag{4.3}$$

for all $j\in\{1,\dots,n\}$.

If we multiply (4.3) by $p_j \geq 0$ and sum over $j$ from $1$ to $m$, we have

$$P_m F\left(\frac{1}{P_m}\sum_{i=1}^{m} p_i x_i\right) - \sum_{j=1}^{m} p_j F(x_j) \geq \frac{1}{P_m}\left\langle \sum_{j=1}^{m} p_j \nabla F(x_j), \sum_{i=1}^{m} p_i x_i \right\rangle - \sum_{i=1}^{m} \langle \nabla F(x_i), x_i\rangle .$$

Dividing by $P_m > 0$, we obtain the inequality

$$\begin{aligned} (4.4) \qquad 0 &\leq \frac{1}{P_m}\sum_{i=1}^{m} p_i F(x_i) - F\left(\frac{1}{P_m}\sum_{i=1}^{m} p_i x_i\right) \\ &\leq \frac{1}{P_m}\sum_{i=1}^{m} p_i \langle \nabla F(x_i), x_i\rangle \\ &\quad - \left\langle \frac{1}{P_m}\sum_{i=1}^{m} p_i \nabla F(x_i), \frac{1}{P_m}\sum_{i=1}^{m} p_i x_i \right\rangle \end{aligned}$$

which is a generalisation for the case of inner product spaces of the result by Dragomir and Goh established in 1996 for the case of differentiable mappings defined on $\mathbb{R}^n$ [9].

Applying Theorem 3.1 for real inner product spaces, $X = \phi$, $x = \gamma$, $y_i =$

$\nabla F(x_i)$, $y = m$, $Y = M$ and $n = m$, we easily deduce

$$\frac{1}{P_m}\sum_{i=1}^{m} p_i \langle x_i, \nabla F(x_i)\rangle - \left\langle \frac{1}{P_m}\sum_{i=1}^{m} p_i x_i, \frac{1}{P_m}\sum_{i=1}^{m} p_i \nabla F(x_i)\right\rangle \leq \frac{1}{4}\|\Phi - \phi\| \|M - m\| \tag{4.5}$$

and then, by (4.4) and (4.5) we can conclude that the desired inequality (4.2) holds. □

**Remark 4.1.** *The conditions*

$$\langle x_i - \gamma, \phi - x_i\rangle \geq 0, \quad \langle \nabla F(x_i) - m, M - \nabla F(x_i)\rangle \geq 0, \tag{4.6}$$

*for all* $i \in \{1, ..., m\}$ *can be replaced by the more general conditions*

$$\sum_{i=1}^{m} p_i \langle x_i - \gamma, \phi - x_i\rangle \geq 0 \text{ and } \sum_{i=1}^{m} p_i \langle \nabla F(x_i) - m, M - \nabla F(x_i)\rangle \geq 0 \tag{4.7}$$

*and the conclusion (4.2) will still be valid.*

**Remark 4.2.** *Even if the inequality (4.2) is not as sharp as (4.4), it can be more useful in practice when only some bounds of the gradient operator* $\nabla F$ *and of the vectors* $x_i$ $(i = 1, ..., n)$ *are known. In other words, it provides the opportunity to estimate the difference*

$$\Delta(F, x, p) := \frac{1}{P_m}\sum_{i=1}^{m} p_i F(x_i) - F\left(\frac{1}{P_m}\sum_{i=1}^{m} p_i x_i\right),$$

*where the differences* $\|\phi - \gamma\|$ *and* $\|M - m\|$ *are known.*

**Remark 4.3.** *For example, if we know that* $\langle \nabla F(x_i) - m,\ M - \nabla F(x_i) \rangle \geq 0$ *for all* $i \in \{1, ..., m\}$ *and the vectors* $x_i$ $(i = 1, ..., n)$ *are not too far from each other in the sense that* $\langle x_i - \gamma, \phi - x_i \rangle \geq 0$ *for all* $i \in \{1, ..., m\}$ *and* $\|\phi - \gamma\| \leq \frac{4\varepsilon}{\|M-m\|}$ $(\varepsilon > 0)$, *then by (4.2), we can conclude that*

$$0 \leq \Delta(F, x, p) \leq \varepsilon.$$

# 5. Applications for Some Discrete Transforms

Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{K}$, $\mathbb{K} = \mathbb{C}, \mathbb{R}$ and $\bar{x} = (x_1, \dots, x_n)$ be a sequence of vectors in $H$.

For a given $m \in \mathbb{K}$, define the *discrete Fourier transform*

$$\mathcal{F}_w(\bar{x})(m) = \sum_{k=1}^{n} \exp(2wimk) \times x_k, \quad m = 1, \dots, n. \tag{5.1}$$

The complex number $\sum_{k=1}^{n} \exp(2wimk) \langle x_k, y_k \rangle$ is actually the usual Fourier transform of the vector $(\langle x_1, y_1 \rangle, \dots, \langle x_n, y_n \rangle) \in \mathbb{K}^n$ and will be denoted by

$$\mathcal{F}_w(\bar{x} \cdot \bar{y})(m) = \sum_{k=1}^{n} \exp(2wimk) \langle x_k, y_k \rangle, \quad m = 1, \dots, n. \tag{5.2}$$

The following result holds.

**Theorem 5.1.** *Let $\bar{x}, \bar{y} \in H^n$ be sequences of vectors such that there exists the vectors $c, C, y, Y \in H$ with the properties*

$$\operatorname{Re} \langle C - \exp(2wimk) x_k, \exp(2wimk) x_k - c \rangle \geq 0, \quad k, m = 1, \dots, n \tag{5.3}$$

*and*

$$\operatorname{Re} \langle Y - y_k, y_k - y \rangle \geq 0, \quad k = 1, \dots, n. \tag{5.4}$$

*Then we have the inequality*

$$\left|\mathcal{F}_w(\bar{x}\cdot\bar{y})(m)-\left\langle \mathcal{F}_w(\bar{x})(m),\frac{1}{n}\sum_{k=1}^{n}y_k\right\rangle\right|\le\frac{n}{4}\|C-c\|\,\|Y-y\|, \tag{5.5}$$

*for all* $m\in\{1,...,n\}$.

The proof follows by Theorem 3.1 applied for $p_k=\frac{1}{n}$ and for the sequences $x_k\to c_k=\exp(2wimk)\,x_k$ and $y_k\ \ (k=1,...,n)$. We omit the details.

We can also consider the *Mellin transform*

$$\mathcal{M}(\bar{x})(m):=\sum_{k=1}^{n}k^{m-1}x_k,\quad m=1,...,n, \tag{5.6}$$

of the sequence $\bar{x}=(x_1,...,x_n)\in H^n$.

We remark that the complex number $\sum_{k=1}^{n}k^{m-1}\langle x_k,y_k\rangle$ is actually the Mellin transform of the vector $(\langle x_1,y_1\rangle,...,\langle x_n,y_n\rangle)\in\mathbb{K}^n$ and will be denoted by

$$\mathcal{M}(\bar{x}\cdot\bar{y})(m):=\sum_{k=1}^{n}k^{m-1}\langle x_k,y_k\rangle. \tag{5.7}$$

The following theorem holds.

**Theorem 5.2.** *Let* $\bar{x},\bar{y}\in H^n$ *be sequences of vectors such that there exist the vectors* $d,D,y,Y\in H$ *with the properties*

$$\operatorname{Re}\left\langle D-k^{m-1}x_k,k^{m-1}x_k-d\right\rangle\ge 0 \tag{5.8}$$

*for all* $k, m \in \{1, ..., n\}$ *, and (5.4) is fulfilled.*
*Then we have the inequality*

$$\left| \mathcal{M}(\bar{x} \cdot \bar{y})(m) - \left\langle \mathcal{M}(\bar{x})(m), \frac{1}{n} \sum_{k=1}^{n} y_k \right\rangle \right| \leq \frac{n}{4} \|D - d\| \|Y - y\| \tag{5.9}$$

*for all* $m \in \{1, ..., n\}$*.*

The proof follows by Theorem 3.1 applied for $p_k = \frac{1}{n}$ and for the sequences $x_k \to d_k = k x_k$ and $y_k$ $(k = 1, ..., n)$. We omit the details.

Another result which connects the Fourier transforms for different parameters $w$ also holds.

**Theorem 5.3.** *Let* $\bar{x}, \bar{y} \in H^n$ *and* $w, z \in \mathbb{K}$*. If there exists the vectors* $e, E, f, F \in H$ *such that*

$$\operatorname{Re} \langle E - \exp(2wimk) x_k, \exp(2wimk) x_k - e \rangle \geq 0, \quad k, m = 1, ..., n$$

*and*

$$\operatorname{Re} \langle F - \exp(2zimk) y_k, \exp(2zimk) y_k - f \rangle \geq 0, \quad k, m = 1, ..., n$$

*then we have the inequality:*

$$\left| \frac{1}{n} \mathcal{F}_{w+z}(\bar{x} \cdot \bar{y})(m) - \left\langle \frac{1}{n} \mathcal{F}_w(\bar{x})(m), \frac{1}{n} \mathcal{F}_z(\bar{y})(m) \right\rangle \right| \leq \frac{1}{4} \|E - e\| \|F - f\|,$$

*for all* $m \in \{1, ..., n\}$*.*

The proof follows by Theorem 3.1 for the sequences $\exp(2wimk) x_k$, $\exp(2zimk) y_k$ $(k = 1, ..., n)$. We omit the details.

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