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STRONGLY ELLIPTIC OPERATORS FOR A PLANE WAVE DIFFRACTION PROBLEM IN BESSEL POTENTIAL SPACES

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ABSTRACT. We consider a plane wave diffraction problem by a union of several infinite strips. The problem is formulated as a boundary-transmission one for the Helmholtz equation in a Bessel potential space setting and where Neumann conditions are assumed on the strips. Using arguments of strong ellipticity and different kinds of operator relations between convolution type operators, it is shown the well-posedness of the problem in a smoothness neighborhood of the Bessel potential space with finite energy norm.

Key words and phrases: Diffraction problem, Strong ellipticity, Convolution type operator, Wiener-Hopf operator, Equivalence after extension.

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1. FORMULATION OF THE PROBLEM

This paper deals with the problem of diffraction of an electromagnetic wave by a union of n infinite magnetic strips from an operator point of view.

We will present a formulation that results from the investigation of plane waves which propagate in a direction orthogonal to the edge $x = y = 0$, $z \in \mathbb{R}$. Thus, the problem will be posed as a boundary-transmission one for the two-dimensional Helmholtz equation where the dependence on one dimension is dropped already. Moreover, also due to perpendicular wave incidence, the union of n infinite strips will be represented by

$$\Omega =]\gamma_1, \gamma_2[\cup \cdots \cup]\gamma_{2n-1}, \gamma_{2n}[,$$

with $0 = \gamma_1 < \cdots < \gamma_{2n}$ and $n \in \mathbb{N}$.

We will use the Bessel potential spaces $H^\sigma(\mathbb{R})$, with $\sigma \in \mathbb{R}$, formed by the tempered distributions φ such that $\|\varphi\|_{H^\sigma(\mathbb{R})} = \|\mathcal{F}^{-1}(1+\xi^2)^{\sigma/2} \cdot \mathcal{F}\varphi\|_{L^2(\mathbb{R})}$ is finite (here $\mathcal{F}$ denotes the Fourier transformation). In addition, we denote by $\widetilde{H}^\sigma(\Omega)$ [17, §2.10.3] the closed subspace of $H^\sigma(\mathbb{R})$

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defined by the distributions with support contained in $\overline{\Omega}$ and $H^\sigma(\Omega)$ will denote the space of generalized functions on Ω which have extensions into $\mathbb{R}$ that belong to $H^\sigma(\mathbb{R})$. The space $\widetilde{H}^\sigma(\Omega)$ is endowed with the subspace topology, and on $H^\sigma(\Omega)$ we put the norm of the quotient space $H^\sigma(\mathbb{R})/\widetilde{H}^\sigma\left(\mathbb{R}\backslash\overline{\Omega}\right)$. In particular, we shall denote by $L^2(\mathbb{R}_+)$ and $L^2_+(\mathbb{R})$, the spaces $H^0(\mathbb{R}_+)$ and $\widetilde{H}^0(\mathbb{R}_+)$, respectively. All those definitions can be extended to the multi-index case $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m) \in \mathbb{R}^m$ by taking the product topology.

The problem is inspired by the classical Sommerfeld type problems considered, for instance in [7, 9, 10, 12, 13, 15, 16], for the half-line case instead of the present Ω. In fact, we present here a generalization of the problem treated in [4] where a corresponding problem was taken into consideration for only one strip. Several changes take place here, in particular, we notice the necessity of different constructions of operator relations that can be found in the next sections.

More concretely, we are interested in studying well-posedness of the problem to find $u \in L^2(\mathbb{R}^2)$, with $u_{|\mathbb{R}^2_\pm} \in H^\epsilon(\mathbb{R}^2_\pm)$, $\epsilon \in]1/2, 3/2[$, so that

$$\left(\Delta + k^2\right) u = 0 \quad \text{in} \quad \mathbb{R}^2_\pm, \tag{1.1}$$

$$u_1^\pm = h \quad \text{on} \quad \Omega, \tag{1.2}$$

$$\begin{cases} u_0^+ - u_0^- = 0 \\ u_1^+ - u_1^- = 0 \end{cases} \quad \text{on} \quad \mathbb{R} \setminus \overline{\Omega}, \tag{1.3}$$

where $\mathbb{R}^2_\pm$ represents the upper/lower half-plane, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ stands for the Laplace operator, $k \in \mathbb{C}$ is the wave number, which, due to the assumption of a lossy medium, is assumed to fulfill

$$\Im m k > 0,$$

$u_0^\pm = u_{|y=\pm 0}$, $u_1^\pm = (\partial u/\partial y)_{|y=\pm 0}$ and the element $h \in H^{\epsilon-3/2}(\Omega)$ is arbitrarily given.

2. The Problem from an Operator Point of View

In order to study the existence and uniqueness of the solution of the problem, as well as continuous dependence on the data, we will construct several operators that are shown to be, in a sense, connected with the problem.

In the first stage, the problem can be described by the use of a linear operator

$$\mathcal{P} \;:\; D(\mathcal{P}) \to H^{\epsilon-3/2}(\Omega),$$

if we define $D(\mathcal{P})$ as the subspace of $H^\epsilon(\mathbb{R}^2_+) \times H^\epsilon(\mathbb{R}^2_-)$ whose functions fulfill the Helmholtz equation (1.1) and all the remaining homogeneous transmission conditions that appear from (1.2) – (1.3) whereas the action $\mathcal{P}u = h$ results from the non-homogeneous conditions (1.2).

In this sense, we will say that the operator $\mathcal{P}$ is *associated* to the problem and our aim is to prove that $\mathcal{P}$ is bounded and invertible for suitable orders of smoothness ϵ. This goal will be achieved by the construction of several operator relations that will allow us to understand better the structure of $\mathcal{P}$.

To this end, we begin by introducing some notation. Let $r_{\mathbb{R}\to\Omega} : H^\sigma(\mathbb{R}) \to H^\sigma(\Omega)$ be the restriction operator and let

$$t(\xi) = \left(\xi^2 - k^2\right)^{1/2}, \qquad \xi \in \mathbb{R},$$

denote the branch of the square root that tends to $+\infty$ as $\xi \to +\infty$ with branch cuts along $\pm k \pm i\eta$, $\eta \geq 0$.

Theorem 2.1. *The operator $\mathcal{P}$ is equivalent to the convolution type operator*

$$W_{t,\Omega} = r_{\mathbb{R}\to\Omega}\mathcal{F}^{-1} t \cdot \mathcal{F} \;:\; \widetilde{H}^{\epsilon-1/2}(\Omega) \to H^{\epsilon-3/2}(\Omega), \tag{2.1}$$

i.e. there are bounded invertible linear operators E and F so that $\mathcal{P} = EW_{t,\Omega}F$.

Proof. Analogously to what happens in the half-line case [10], a function $u \in L^2(\mathbb{R}^2)$, with $u_{|\mathbb{R}^2_\pm} \in H^\epsilon(\mathbb{R}^2_\pm)$, satisfies the Helmholtz equation (1.1) if and only if it is representable by

$$u(x,y) = \mathcal{F}^{-1}_{\xi\mapsto x} e^{-t(\xi)y} \mathcal{F}_{x\mapsto\xi} u_0^+(x)\chi_+(y) + \mathcal{F}^{-1}_{\xi\mapsto x} e^{t(\xi)y} \mathcal{F}_{x\mapsto\xi} u_0^-(x)\chi_-(y) \tag{2.2}$$

for $(x,y) \in \mathbb{R}^2$, where $\mathcal{F}_{x\mapsto\xi} u(x,y) = \int_{\mathbb{R}} u(x,y)e^{i\xi x}dx$ and χ_+, χ_- denote the characteristic functions of the positive and negative half-line, respectively.

Let

$$Z = \left\{ (\phi,\psi) \in \left[H^{\epsilon-1/2}(\mathbb{R})\right]^2 \,:\, \phi - \psi \in \widetilde{H}^{\epsilon-1/2}(\Omega),\ \mathcal{F}^{-1} t \cdot \mathcal{F}(\phi+\psi) = 0 \right\}.$$

Taking into account the representation formula (2.2), we have that the trace operator,

$$\begin{array}{rcccl} T_0 & : & D(\mathcal{P}) & \to & Z \\ & & u & \mapsto & u_0 = \begin{bmatrix} u_0^+ \\ u_0^- \end{bmatrix}, \end{array}$$

is continuously invertible by the Poisson operator $K : u_0 \mapsto u$ defined by (2.2).

On the other hand, a direct computation leads us to

$$\mathcal{P} \;=\; -\frac{1}{2} W_{t,\Omega}\, R_1\, \mathcal{C}\, T_0, \tag{2.3}$$

where R_1 is the restriction operator to the first component and $\mathcal{C}$ is the convolution operator (on the full line)

$$\mathcal{C} = \mathcal{F}^{-1} \begin{bmatrix} 1 & -1 \\ -t & -t \end{bmatrix} \cdot \mathcal{F} \quad : \quad Z \to \widetilde{H}^{\epsilon-1/2}(\Omega) \times \{0\}.$$

Thus, (2.3) exhibits an operator equivalence between $\mathcal{P}$ and the convolution type operator $W_{t,\Omega}$, defined in (2.1), because $R_1\mathcal{C}T_0$ is continuously invertible by $K\mathcal{C}^{-1}[I \;\; 0]^T$ (please note that, for $u \in D(\mathcal{P})$, we have $u_1^+ = u_1^-$ due to (1.2) and (1.3)). □

3. THE EXTENSION TO A HALF-LINE SETTING

In this section we will perform some operator extension methods [2, 3, 5, 6, 8], translated by several operator matrix identities.

As a consequence, we will obtain certain operator relations that will help us to arrive at the desired invertibility conditions.

Theorem 3.1. *Let us consider the Wiener-Hopf operator*

$$W_{\Psi,\mathbb{R}_+} = r_{\mathbb{R}\to\mathbb{R}_+} \mathcal{F}^{-1} \Psi \cdot \mathcal{F} \quad : \quad \widetilde{H}^\sigma(\mathbb{R}_+) \to H^\sigma(\mathbb{R}_+), \tag{3.1}$$

$$\Psi = \begin{bmatrix} \tau_{-\gamma_2} & & & & & & & \\ & \tau_{-(\gamma_4-\gamma_3)} & & & & & & \\ & & \ddots & & & & & \\ & & & \tau_{-(\gamma_{2n}-\gamma_{2n-1})} & & & & \\ & & & & \tau_{-(\gamma_3-\gamma_2)} & & & \\ & & & & & \ddots & & \\ & & & & & & \tau_{-(\gamma_{2n-1}-\gamma_{2n-2})} & \\ t & t\,\tau_{\gamma_3} & \cdots & t\,\tau_{\gamma_{2n-1}} & \tau_{\gamma_2} & \cdots & \tau_{\gamma_{2n-2}} & \tau_{\gamma_{2n}} \end{bmatrix},$$

where the empty entries (i.e., outside of the main diagonal and the last row) are zero, $\tau_a(\xi) = e^{i\xi a}$, $\xi \in \mathbb{R}$ and $\sigma = (\epsilon - 1/2, \ldots, \epsilon - 1/2, \epsilon - 3/2, \ldots, \epsilon - 3/2)$.

There are Banach spaces X_1, Y_1 and linear homeomorphisms E_1 and F_1 so that

$$\begin{bmatrix} W_{t,\Omega} & 0 \\ 0 & I_{X_1} \end{bmatrix} = E_1 \begin{bmatrix} W_{\Psi,\mathbb{R}_+} & 0 \\ 0 & I_{Y_1} \end{bmatrix} F_1. \tag{3.2}$$

I.e., $W_{t,\Omega}$ and $W_{\Psi,\mathbb{R}_+}$ are operators which are equivalent after extension.

Proof. An algebraic equivalence after extension between $W_{t,\Omega}$ and $W_{\Psi,\mathbb{R}_+}$ is known from [2]. This means that we already have an identity like (3.2) but without the guarantee that the invertible linear operators E_1 and F_1 are bounded.

Now, taking into account that bounded linear operators with closed ranges and acting between Hilbert spaces are generalized invertible, the result is derived from [1, Theorem 2] that claims that generalized invertible operators in Banach spaces are equivalent after extension if and only if their defect spaces are homeomorphic. □

We shall use the functions

$$\lambda^\nu_\pm(\xi) = \operatorname{diag}\left[(\xi \pm k)^{\nu_1}, \ldots, (\xi \pm k)^{\nu_m}\right], \quad \xi \in \mathbb{R},$$

as well as $\zeta^\nu = \lambda^\nu_- \, \lambda^{-\nu}_+$.

Theorem 3.2. *The Wiener-Hopf operator $W_{\Psi,\mathbb{R}_+}$ is equivalent to*

$$W_{\Psi_0,\mathbb{R}_+} = r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\Psi_0 \cdot \mathcal{F} \quad : \quad \left[L^2_+(\mathbb{R})\right]^{2n} \to \left[L^2(\mathbb{R}_+)\right]^{2n}, \tag{3.3}$$

where

$$\Psi_0 = \zeta^\alpha \begin{bmatrix} \zeta^s\tau_{-\gamma_2} & & & & & & & \\ & \zeta^s\tau_{-(\gamma_4-\gamma_3)} & & & & & & \\ & & \ddots & & & & & \\ & & & \zeta^s\tau_{-(\gamma_{2n}-\gamma_{2n-1})} & & & & \\ & & & & \zeta^{-s}\tau_{-(\gamma_3-\gamma_2)} & & & \\ & & & & & \ddots & & \\ & & & & & & \zeta^{-s}\tau_{-(\gamma_{2n-1}-\gamma_{2n-2})} & \\ 1 & \tau_{\gamma_3} & \cdots & \tau_{\gamma_{2n-1}} & \zeta^{-s}\tau_{\gamma_2} & \cdots & \zeta^{-s}\tau_{\gamma_{2n-2}} & \zeta^{-s}\tau_{\gamma_{2n}} \end{bmatrix},$$

with $\alpha = (\epsilon - 1, \ldots, \epsilon - 1)$ and $s = 1/2$.

Proof. Let us consider the operators

$$\begin{aligned} E_2 &= r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\lambda^{-\sigma}_- \cdot \mathcal{F} l_0 \quad : \quad \left[L^2(\mathbb{R}_+)\right]^{2n} \to H^\sigma(\mathbb{R}_+) \\ F_2 &= l_0 r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\lambda^{\sigma}_+ \cdot \mathcal{F} \quad : \quad \widetilde{H}^\sigma(\mathbb{R}_+) \to \left[L^2_+(\mathbb{R})\right]^{2n}, \end{aligned}$$

where $l_0 : \left[L^2(\mathbb{R}_+)\right]^{2n} \to \left[L^2_+(\mathbb{R})\right]^{2n}$ is the zero extension operator.

These operators are bounded invertible (see [17, §2.10.3]). Moreover, attending to the structure of E_2 and F_2 [17], we have

$$W_{\Psi,\mathbb{R}_+} = E_2 \, W_{\Psi_0,\mathbb{R}_+} \, F_2,$$

which demonstrates operator equivalence between $W_{\Psi,\mathbb{R}_+}$ and $W_{\Psi_0,\mathbb{R}_+}$. □

Corollary 3.3. *The convolution type operator $W_{t,\Omega}$ and any of the Wiener-Hopf operators $W_{\Psi,\mathbb{R}_+}$, $W_{\Psi_0,\mathbb{R}_+}$ belong to the same regularity class* [5]*. More precisely, any of these three operators is invertible, one-sided invertible, Fredholm, semi-Fredholm, one-sided regularizable, generalized invertible or normally solvable, if and only if one of the others enjoys that property.*

Proof. From the above relations we derive that $\ker W_{t,\Omega}$, $\ker W_{\Psi,\mathbb{R}_+}$ and $\ker W_{\Psi_0,\mathbb{R}_+}$ are isomorphic and that the ranges of these operators are closed only at the same time. In addition, the presented relations allow us to conclude that

$$H^{\epsilon-3/2}(\Omega)/\operatorname{im} W_{t,\Omega}, \quad H^{\sigma}(\mathbb{R}_+)/\operatorname{im} W_{\Psi,\mathbb{R}_+}, \quad \left[L^2(\mathbb{R}_+)\right]^{2n}/\operatorname{im} W_{\Psi_0,\mathbb{R}_+}$$

are also isomorphic. Thus, the statement follows. □

4. STRONG ELLIPTICITY AND CORRESPONDING WELL-POSEDNESS OF THE PROBLEM

Let us consider the $2n \times 2n$ matrix function

$$\zeta^{\beta} = \operatorname{diag}\left[\zeta^{-\epsilon+1/2}, \ldots, \zeta^{-\epsilon+1/2}, \zeta^{-\epsilon+3/2}, \ldots, \zeta^{-\epsilon+3/2}\right]$$

(where the first n elements in the main diagonal are equal and the last n elements are also all equal). We introduce the auxiliary operator

$$W_{\zeta^{\beta}\Psi_0,\mathbb{R}_+} = r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\zeta^{\beta}\Psi_0 \cdot \mathcal{F} \quad : \quad \left[L^2_+(\mathbb{R})\right]^{2n} \to \left[L^2(\mathbb{R}_+)\right]^{2n}. \tag{4.1}$$

This new operator will help us to arrive at the desired invertibility conditions. For this purpose, first, let us present some symmetries in the structure of $W_{\zeta^{\beta}\Psi_0,\mathbb{R}_+}$.

Theorem 4.1. *The Wiener-Hopf operator $W_{\zeta^{\beta}\Psi_0,\mathbb{R}_+}$ is equivalent to*

$$W_{\Psi_1,\mathbb{R}_+}\, l_0\, W_{\Psi_2,\mathbb{R}_+}\, l_0\, W_{\overline{\Psi_1},\mathbb{R}_+} \quad : \quad \left[L^2_+(\mathbb{R})\right]^{2n} \to \left[L^2(\mathbb{R}_+)\right]^{2n}$$

where, for $j = 1, 2$,

$$W_{\Psi_j,\mathbb{R}_+} = r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\Psi_j \cdot \mathcal{F} \quad : \quad \left[L^2_+(\mathbb{R})\right]^{2n} \to \left[L^2(\mathbb{R}_+)\right]^{2n}$$

and

$$\Psi_1 = \left[\begin{array}{c|c} & \begin{array}{cccc} \tau_{-\gamma_{2n}} & & & \\ & \tau_{-\gamma_{2n-2}} & & \\ & & \ddots & \\ & & & \tau_{-\gamma_2} \end{array} \\ \hline \begin{array}{c|cccc} & \tau_{-\gamma_{2n-1}} & & & \\ & & \tau_{-\gamma_{2n-3}} & & \\ & & & \ddots & \\ & & & & \tau_{-\gamma_3} \\ \hline n & 1 & 1 & \cdots & 1 \end{array} & \end{array}\right],$$

$$\Psi_2 = \left[\begin{array}{ccc|ccc} 1 & & & \zeta^{1/2}/n & \cdots & \zeta^{1/2}/n \\ & \ddots & & & & \\ & & 1 & & & \\ \hline & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{array}\right].$$

Proof. From the identity

$$
\zeta^{\beta}\Psi_0 = \left[\begin{array}{c|c}
\begin{matrix} & & 1 \\ & \iddots & \\ 1 & & \end{matrix} & \\ \hline
 & \begin{array}{c|c} \begin{matrix} & & 1 \\ & \iddots & \\ 1 & & \end{matrix} & \\ \hline & 1 \end{array}
\end{array}\right] \Psi_1 \; \Psi_2 \; \overline{\Psi_1} \times
$$

$$
\left[\begin{array}{c|c}
\begin{array}{c|ccc} 1/n & -1/n & \cdots & -1/n \\ \hline & & & 1 \\ & & \iddots & \\ & 1 & & \end{array} & \\ \hline
 & \begin{matrix} & & & 1/n \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{matrix}
\end{array}\right]
$$

it follows that $W_{\zeta^{\beta}\Psi_0,\mathbb{R}_+}$ and

$$
W_{\Psi_1\Psi_2\overline{\Psi_1},\mathbb{R}_+} = r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\Psi_1\Psi_2\overline{\Psi_1}\cdot\mathcal{F} \quad : \quad \left[L^2_+(\mathbb{R})\right]^{2n} \to \left[L^2(\mathbb{R}_+)\right]^{2n}
$$

are equivalent operators.

Therefore, in order to obtain the present result, we only have to observe that the “minus" and “plus" semi-almost periodic entries of Ψ_1 and $\overline{\Psi_1}$, respectively, lead us to

$$
\begin{aligned}
W_{\Psi_1\Psi_2\overline{\Psi_1},\mathbb{R}_+} &= \left(r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\Psi_1\cdot\mathcal{F}\right) l_0 \left(r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\Psi_2\cdot\mathcal{F}\right) l_0 \left(r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\overline{\Psi_1}\cdot\mathcal{F}\right) \\
&= W_{\Psi_1,\mathbb{R}_+} \; l_0 \; W_{\Psi_2,\mathbb{R}_+} \; l_0 \; W_{\overline{\Psi_1},\mathbb{R}_+}.
\end{aligned}
$$

□

We recall that an essentially bounded $m\times m$ matrix-valued function $\psi(\xi) = [\psi_{ij}(\xi)]$ is said to be *strongly elliptic* if there exist constants $\eta \in \mathbb{C}$ and $C > 0$ so that

$$
\Re e \;\; \eta \sum_{i,j=1}^{m} \psi_{ij}(\xi)\mu_j\overline{\mu_i} \geq C \sum_{i=1}^{m} |\mu_i|^2, \qquad \forall \mu = (\mu_1,\ldots,\mu_m) \in \mathbb{C}^m.
$$

Theorem 4.2. *The Wiener-Hopf operator introduced in (4.1),*

$$
W_{\zeta^{\beta}\Psi_0,\mathbb{R}_+} \quad : \quad \left[L^2_+(\mathbb{R})\right]^{2n} \to \left[L^2(\mathbb{R}_+)\right]^{2n},
$$

is an invertible operator.

Proof. The nonnegative mean motions [14] of the semi-almost periodic entries of $\overline{\Psi_1}$ ensure us (see [14, Theorem 1]) the left invertibility of $W_{\overline{\Psi_1},\mathbb{R}_+}$.

Moreover, we observe that Ψ_2 is strongly elliptic. In fact, from

$$
\left|\frac{\zeta^{1/2}}{n}\,\mu_j\overline{\mu_1}\right| \leq \frac{1}{2n}\left(|\mu_j|^2 + |\mu_1|^2\right), \qquad j = n+1,\ldots,2n,
$$

it follows that, for $\Psi_2(\xi) = [\Theta_{ij}(\xi)]$,

$$\begin{aligned}
\Re e \sum_{i,j=1}^{2n} \Theta_{ij}(\xi)\mu_j\overline{\mu_i} &\geq \sum_{j=1}^{2n} |\mu_j|^2 - \frac{1}{2n}\sum_{j=n+1}^{2n}\left(|\mu_j|^2 + |\mu_1|^2\right) \\
&= \frac{1}{2}|\mu_1|^2 + \sum_{j=2}^{n}|\mu_j|^2 + \left(1 - \frac{1}{2n}\right)\sum_{j=n+1}^{2n}|\mu_j|^2 \\
&\geq \frac{1}{2}\sum_{j=1}^{2n}|\mu_j|^2,
\end{aligned}$$

for all $(\mu_1, \ldots, \mu_{2n}) \in \mathbb{C}^{2n}$.

Consequently, for $\psi \in [L^2(\mathbb{R}_+)]^{2n}$, we have

$$\begin{aligned}
&\Re e\langle (W_{\Psi_1,\mathbb{R}_+} l_0)(W_{\Psi_2,\mathbb{R}_+} l_0)(W_{\overline{\Psi_1},\mathbb{R}_+} l_0)\psi, \psi\rangle \\
&\quad= \Re e\langle (W_{\Psi_2,\mathbb{R}_+} l_0)(W_{\overline{\Psi_1},\mathbb{R}_+} l_0)\psi, (W_{\overline{\Psi_1},\mathbb{R}_+} l_0)\psi\rangle \\
&\quad= \Re e\langle \left(r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\Psi_2 \cdot \mathcal{F} l_0\right)(W_{\overline{\Psi_1},\mathbb{R}_+} l_0)\psi, (W_{\overline{\Psi_1},\mathbb{R}_+} l_0)\psi\rangle \\
&\quad= \Re e\langle \Psi_2 \cdot \mathcal{F} l_0 (W_{\overline{\Psi_1},\mathbb{R}_+} l_0)\psi, \mathcal{F} l_0 (W_{\overline{\Psi_1},\mathbb{R}_+} l_0)\psi\rangle \\
&\quad\geq \frac{1}{2}\,\|\mathcal{F} l_0 (W_{\overline{\Psi_1},\mathbb{R}_+} l_0)\psi\|^2 \\
&\quad= \frac{1}{2}\,\|W_{\overline{\Psi_1},\mathbb{R}_+} l_0 \psi\|^2 \\
&\quad\geq \frac{1}{2}\,C_1\|\psi\|^2,
\end{aligned}$$

where $C_1 > 0$ is provided by the left invertibility of $W_{\overline{\Psi_1},\mathbb{R}_+}$. This inequality allows us to conclude that $W_{\Psi_1,\mathbb{R}_+} l_0 W_{\Psi_2,\mathbb{R}_+} l_0 W_{\overline{\Psi_1},\mathbb{R}_+} : [L^2_+(\mathbb{R})]^{2n} \to [L^2(\mathbb{R}_+)]^{2n}$ is a left invertible operator.

Applying the same reasoning to the conjugate operator

$$\begin{aligned}
\left(l_0 W_{\Psi_1,\mathbb{R}_+} l_0 W_{\Psi_2,\mathbb{R}_+} l_0 W_{\overline{\Psi_1},\mathbb{R}_+}\right)^* &= \left(l_0 W_{\overline{\Psi_1},\mathbb{R}_+}\right)^*\left(l_0 W_{\Psi_2,\mathbb{R}_+}\right)^*\left(l_0 W_{\Psi_1,\mathbb{R}_+}\right)^* \\
&= l_0 W_{\Psi_1,\mathbb{R}_+} l_0 W_{\overline{\Psi_2},\mathbb{R}_+} l_0 W_{\overline{\Psi_1},\mathbb{R}_+}
\end{aligned}$$

we obtain that this is also a left invertible operator.

Thus $W_{\Psi_1,\mathbb{R}_+} l_0 W_{\Psi_2,\mathbb{R}_+} l_0 W_{\overline{\Psi_1},\mathbb{R}_+}$ is an invertible operator and from Theorem 4.1 our goal is achieved. □

Corollary 4.3. *The Wiener-Hopf operator $W_{\Psi_0,\mathbb{R}_+}$, defined in (3.3), is a Fredholm operator with zero Fredholm index.*

Proof. From Theorem 4.2 we know that $W_{\zeta^\beta\Psi_0,\mathbb{R}_+}$ is an invertible operator. Thus (see e.g. [11, Chapter 1, Theorem 3.11]), the result is a consequence of $W_{\Psi_0,\mathbb{R}_+}$ and $W_{\zeta^\beta\Psi_0,\mathbb{R}_+}$ being homotopic operators in the class of Fredholm operators acting from $[L^2_+(\mathbb{R})]^{2n}$ to $[L^2(\mathbb{R}_+)]^{2n}$. □

Theorem 4.4. *The operator $\mathcal{P}$ (associated to the problem) is bounded invertible and, therefore, our problem is well-posed for all orders of smoothness $\epsilon \in]1/2, 3/2[$.*

Proof. Due to the fact that, for negative parameters $-\gamma$ and any $s \in \mathbb{R}$,

$$r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\tau_{-\gamma}\cdot\mathcal{F} \quad : \quad \widetilde{H}^s(\mathbb{R}_+) \to H^s(\mathbb{R}_+)$$

are nothing more than left shift operators composed with the restriction operator $r_{\mathbb{R}\to\mathbb{R}_+}$, we have that these operators are surjective. Therefore, taking into account the structure of Ψ (see (3.1)), we obtain that $W_{\Psi,\mathbb{R}_+}$ is a surjective operator whenever

$$W_{t,\mathbb{R}_+} = r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}t\cdot\mathcal{F} \quad : \quad \widetilde{H}^{\epsilon-1/2}(\mathbb{R}_+) \to H^{\epsilon-3/2}(\mathbb{R}_+)$$

is a surjective operator, which is true for all ϵ. As a consequence, the codimension of the image of $W_{\Psi,\mathbb{R}_+}$ is zero.

Thus, from Corollary 3.3 and Corollary 4.3, we have that $W_{t,\Omega}$ is invertible. Therefore, the result is obtained if we take into consideration Theorem 2.1. □

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