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A MONOTONICITY PROPERTY OF THE Γ FUNCTION

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The starting point of this note was an inequality,

$$1 \le \frac{\Gamma\left(\frac{n}{2}+1\right)^{\frac{n-d}{n}}}{\Gamma\left(\frac{n-d}{2}+1\right)} \le e^{\frac{d}{2}}, \tag{1}$$

for all pairs of integers $0 \le d \le n$, in [5, Lemma 2.1]. Note that the left hand side of this inequality is an immediate consequence of the logarithmic convexity of the Γ-function; see [5]. Looking for a stream-lined proof of inequality (1), we first found a proof of the more general inequality

$$\frac{\Gamma(p+1)^{\frac{1}{p}}}{\Gamma(q+1)^{\frac{1}{q}}} \le e^{\frac{p}{q}-1}, \tag{2}$$

valid for all $0 < q \le p$, and finally showed

$$\frac{\Gamma(p+1)^{\frac{1}{p}}}{\Gamma(q+1)^{\frac{1}{q}}} \le \frac{p+1}{q+1}, \tag{3}$$

for all $-1 < q \le p$. These inequalities will be immediate consequences of the following result.

Theorem 1. *The function* $f(x) := 1 + \frac{1}{x}\ln\Gamma(x+1) - \ln(x+1)$ *is strictly completely monotone on* $(-1,\infty)$,

$$\lim_{x\to-1} f(x) = 1, \quad \lim_{x\to\infty} f(x) = 0,$$
$$f(0) = \lim_{x\to 0} f(x) = 1-\gamma.$$

(Here, γ *is the Euler-Mascheroni constant, and* strictly completely monotone *means* $(-1)^n f^{(n)}(x) > 0$ *for all* $x \in (-1,\infty)$, $n \in \mathbb{N}_0$*).*

Proof. The main ingredient of the proof is the integral representation

$$\ln\Gamma(x+1) = x\ln(x+1) - x + \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t-1}\right) e^{-t}\frac{1}{t}(1-e^{-xt})\,dt,$$

which is an immediate consequence of [6, formula 1.9 (2) (p. 21)] and [6, formula 1.7.2 (18) (p. 17)]. We obtain

$$f(x) = \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t-1}\right) e^{-t}\frac{1}{xt}(1-e^{-xt})\,dt.$$

The function

$$g(y) := \frac{1}{y}(1-e^{-y}) = \int_0^1 e^{-sy}\,ds$$

is strictly completely monotone on $\mathbb{R}$. Since $\frac{1}{t} - \frac{1}{e^t-1} > 0$ for all $t > 0$, we conclude that f is strictly completely monotone. As $y \to \infty$, $g(y)$ tends to zero, and hence $\lim_{x\to\infty} f(x) = 0$. The definition of f shows $\lim_{x\to 0} f(x) = 1 + \psi(1) = 1 - \gamma$; cf. [6, formula 1.7 (4) (p. 15)]. Finally,

$$\lim_{x\to -1} f(x) = 1 + \lim_{x\to -1}\left(\frac{1}{x}\big(\ln\Gamma(x+2) - \ln(x+1)\big) - \ln(x+1)\right) = 1.$$

$\square$

Corollary 2. *Inequalities* (3), (2) *and* (1) *are valid for the indicated ranges.*

Proof. Inequality (3) is just a reformulation of the monotonicity of the function f from Theorem 1. Continuing (3) to the right,

$$\frac{p+1}{q+1} \le \frac{p}{q} \le e^{\frac{p}{q}-1} \quad (0 < q \le p),$$

we obtain (2). Setting $q = \frac{n-d}{2}$, $p = \frac{n}{2}$ we get (1). $\square$

Remark 1.

(a) *In [4] it was shown that the function* $\xi \mapsto \xi\left(\Gamma\left(1+\frac{1}{\xi}\right)\right)^{\xi}$ *is increasing on* $(0,\infty)$. *This fact follows immediately from our Theorem 1, because of*

$$\ln\left(\frac{1}{x}\Gamma(x+1)^{\frac{1}{x}}\right)+1 = -\ln x+\frac{1}{x}\Gamma(x+1)+1 = \ln(x+1)-\ln x+f(x).$$

(In fact, the latter function even is strictly completely monotone as well.)

(b) *For other recent results on (complete) monotonicity properties of the* Γ-*function we refer to [1, 2, 3].*

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