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A MONOTONICITY PROPERTY OF THE Γ FUNCTION

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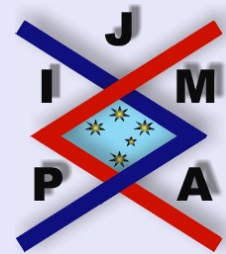
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The starting point of this note was an inequality,

$$(1) \quad 1 \leq \frac{\Gamma\left(\frac{n}{2} + 1\right)^{\frac{n-d}{n}}}{\Gamma\left(\frac{n-d}{2} + 1\right)} \leq e^{\frac{d}{2}},$$

for all pairs of integers $0 \leq d \leq n$, in [5, Lemma 2.1]. Note that the left hand side of this inequality is an immediate consequence of the logarithmic convexity of the Γ -function; see [5]. Looking for a stream-lined proof of inequality (1), we first found a proof of the more general inequality

$$(2) \quad \frac{\Gamma(p+1)^{\frac{1}{p}}}{\Gamma(q+1)^{\frac{1}{q}}} \leq e^{\frac{p}{q}-1},$$

valid for all $0 < q \leq p$, and finally showed

$$(3) \quad \frac{\Gamma(p+1)^{\frac{1}{p}}}{\Gamma(q+1)^{\frac{1}{q}}} \leq \frac{p+1}{q+1},$$

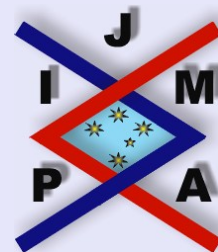
for all $-1 < q \leq p$. These inequalities will be immediate consequences of the following result.

Theorem 1. *The function $f(x) := 1 + \frac{1}{x} \ln \Gamma(x+1) - \ln(x+1)$ is strictly completely monotone on $(-1, \infty)$,*

$$\lim_{x \rightarrow -1} f(x) = 1, \quad \lim_{x \rightarrow \infty} f(x) = 0,$$

$$f(0) = \lim_{x \rightarrow 0} f(x) = 1 - \gamma.$$

(Here, γ is the Euler-Mascheroni constant, and strictly completely monotone means $(-1)^n f^{(n)}(x) > 0$ for all $x \in (-1, \infty)$, $n \in \mathbb{N}_0$).



Integral Means Inequalities for Fractional Derivatives of Some General Subclasses of Analytic Functions

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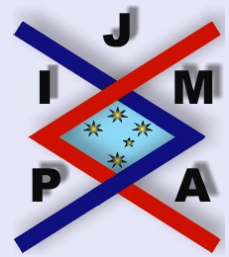


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Proof. The main ingredient of the proof is the integral representation

$$\ln \Gamma(x+1) = x \ln(x+1) - x + \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1} \right) e^{-t} \frac{1}{t} (1 - e^{-xt}) dt,$$

which is an immediate consequence of [6, formula 1.9 (2) (p. 21)] and [6, formula 1.7.2 (18) (p. 17)]. We obtain

$$f(x) = \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1} \right) e^{-t} \frac{1}{xt} (1 - e^{-xt}) dt.$$

The function

$$g(y) := \frac{1}{y} (1 - e^{-y}) = \int_0^1 e^{-sy} ds$$

is strictly completely monotone on \mathbb{R} . Since $\frac{1}{t} - \frac{1}{e^t - 1} > 0$ for all $t > 0$, we conclude that f is strictly completely monotone. As $y \rightarrow \infty$, $g(y)$ tends to zero, and hence $\lim_{x \rightarrow \infty} f(x) = 0$. The definition of f shows $\lim_{x \rightarrow 0} f(x) = 1 + \psi(1) = 1 - \gamma$; cf. [6, formula 1.7 (4) (p. 15)]. Finally,

$$\lim_{x \rightarrow -1} f(x) = 1 + \lim_{x \rightarrow -1} \left(\frac{1}{x} (\ln \Gamma(x+2) - \ln(x+1)) - \ln(x+1) \right) = 1.$$

□

Corollary 2. *Inequalities (3), (2) and (1) are valid for the indicated ranges.*

Proof. Inequality (3) is just a reformulation of the monotonicity of the function f from Theorem 1. Continuing (3) to the right,

$$\frac{p+1}{q+1} \leq \frac{p}{q} \leq e^{\frac{p}{q}-1} \quad (0 < q \leq p),$$

we obtain (2). Setting $q = \frac{n-d}{2}$, $p = \frac{n}{2}$ we get (1). □

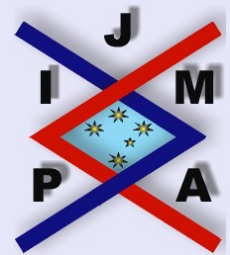
Remark 1.

(a) In [4] it was shown that the function $\xi \mapsto \xi \left(\Gamma \left(1 + \frac{1}{\xi} \right) \right)^\xi$ is increasing on $(0, \infty)$. This fact follows immediately from our Theorem 1, because of

$$\ln \left(\frac{1}{x} \Gamma(x+1)^{\frac{1}{x}} \right) + 1 = -\ln x + \frac{1}{x} \Gamma(x+1) + 1 = \ln(x+1) - \ln x + f(x).$$

(In fact, the latter function even is strictly completely monotone as well.)

(b) For other recent results on (complete) monotonicity properties of the Γ -function we refer to [1, 2, 3].



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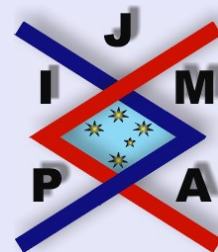
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