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APPROXIMATION OF ENTIRE FUNCTIONS OF TWO COMPLEX VARIABLES IN BANACH SPACES

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ABSTRACT. In the present paper, we study the polynomial approximation of entire functions of two complex variables in Banach spaces. The characterizations of order and type of entire functions of two complex variables have been obtained in terms of the approximation errors.

Key words and phrases: Entire function, Order, type, Approximation, Error.

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1. INTRODUCTION

Let $f(z_1, z_2) = \sum a_{m_1 m_2} z_1^{m_1} z_2^{m_2}$ be a function of the complex variables z_1 and z_2, regular for $\mid z_t \mid \leq r_t$, $t = 1, 2$. If r_1 and r_2 can be taken arbitrarily large, then $f(z_1, z_2)$ represents an entire function of the complex variables z_1 and z_2. Following Bose and Sharma [1], we define the maximum modulus of $f(z_1, z_2)$ as

$$M(r_1, r_2) = \max_{|z_t| \leq r_t} |f(z_1, z_2)|, \quad t = 1, 2.$$

The order ρ of the entire function $f(z_1, z_2)$ is defined as [1, p. 219]:

$$\limsup_{r_1, r_2 \to \infty} \frac{\log \log M(r_1, r_2)}{\log(r_1 r_2)} = \rho.$$

For $0 < \rho < \infty$, the type τ of an entire function $f(z_1, z_2)$ is defined as [1, p. 223]:

$$\limsup_{r_1, r_2 \to \infty} \frac{\log M(r_1, r_2)}{r_1^{\rho} + r_2^{\rho}} = \tau.$$

Bose and Sharma [1], obtained the following characterizations for order and type of entire functions of two complex variables.

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Theorem 1.1. *The entire function* $f(z_1, z_2) = \sum_{m_1,m_2=0}^{\infty} a_{m_1m_2} z_1^{m_1} z_2^{m_2}$ *is of finite order if and only if*

$$\mu = \limsup_{m_1,m_2\to\infty} \frac{\log(m_1^{m_1} m_2^{m_2})}{\log\left(|a_{m_1m_2}|^{-1}\right)}$$

is finite and then the order ρ *of* $f(z_1, z_2)$ *is equal to* μ*.*

Define

$$\alpha = \limsup_{m_1,m_2\to\infty} \sqrt[(m_1+m_2)]{m_1^{m_1} m_2^{m_2} |a_{m_1m_2}|^{\rho}}.$$

We have

Theorem 1.2. *If* $0 < \alpha < \infty$*, the function* $f(z_1, z_2) = \sum_{m_1,m_2=0}^{\infty} a_{m_1m_2} z_1^{m_1} z_2^{m_2}$ *is an entire function of order* ρ *and type* τ *if and only if* $\alpha = e\rho\tau$*.*

Let $H_q, q > 0$ denote the space of functions $f(z_1, z_2)$ analytic in the unit bi-disc $U = \{z_1, z_2 \in C : |z_1| < 1, |z_2| < 1\}$ such that

$$\|f\|_{H_q} = \lim_{r_1,r_2\to 1-0} M_q(f; r_1, r_2) < \infty,$$

where

$$M_q(f; r_1, r_2) = \left\{ \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left| f(r_1 e^{it_1}, r_2 e^{it_2}) \right|^q dt_1 dt_2 \right\}^{\frac{1}{q}},$$

and let $H_q^{'}$, $q > 0$ denote the space of functions $f(z_1, z_2)$ analytic in U and satisfying the condition

$$\|f\|_{H_q^{'}} = \left\{ \frac{1}{\pi^2} \int_{|z_1|<1} \int_{|z_2|<1} |f(z_1, z_2)|^q dx_1 dy_1 dx_2 dy_2 \right\}^{\frac{1}{q}} < \infty.$$

Set

$$\|f\|_{H_\infty^{'}} = \|f\|_{H_\infty} = \sup\{|f(z_1, z_2)| : z_1, z_2 \in U\}.$$

H_q and $H_q^{'}$ are Banach spaces for $q \geq 1$. In analogy with spaces of functions of one variable, we call H_q and $H_q^{'}$ the Hardy and Bergman spaces respectively.

The function $f(z_1, z_2)$ analytic in U belongs to the space $\mathbf{B}(p, q, \kappa)$, where $0 < p < q \leq \infty$, and $0 < \kappa \leq \infty$, if

$$\|f\|_{p,q,\kappa} = \left\{ \int_0^1 \int_0^1 \{(1 - r_1)(1 - r_2)\}^{\kappa(1/p-1/q)-1} M_q^{\kappa}(f, r_1, r_2) dr_1 dr_2 \right\}^{\frac{1}{\kappa}} < \infty,$$
$$0 < \kappa < \infty,$$

$$\|f\|_{p,q,\infty} = \sup\left\{[(1 - r_1)(1 - r_2)\}^{(1/p-1/q)-1} M_q(f, r_1, r_2) : 0 < r_1, r_2 < 1\right\} < \infty.$$

The space $\mathbf{B}(p, q, \kappa)$ is a Banach space for $p > 0$ and $q, \kappa \geq 1$, otherwise it is a Fréchet space. Further, we have

$$H_q \subset H_q^{'} = \mathbf{B}\left(\frac{q}{2}, q, q\right), \quad 1 \leq q < \infty. \tag{1.1}$$

Let X be a Banach space and let $E_{m,n}(f, X)$ be the best approximation of a function $f(z_1, z_2) \in X$ by elements of the space P that consists of algebraic polynomials of degree $\leq m + n$ in two complex variables:

$$E_{m,n}(f, X) = \inf\{\|f - p\|_x; p \in P\}. \tag{1.2}$$

To the best of our knowledge, characterizations for the order and type of entire functions of two complex variables in Banach spaces have not been obtained so far. In this paper, we have made an attempt to bridge this gap.

Notation: For reducing the length of expressions we use the following notations in the main results.

$$B^{1/\kappa}\left[(n+1)\kappa+1;\kappa\left(\frac{1}{p}-\frac{1}{2}\right)\right]=B[n,p,2,\kappa]$$
$$B^{1/\kappa}\left[(m+1)\kappa+1;\kappa\left(\frac{1}{p}-\frac{1}{2}\right)\right]=B[m,p,2,\kappa]$$
$$B^{1/\kappa}\left[(n+1)\kappa+1;\kappa\left(\frac{1}{p}-\frac{1}{q}\right)\right]=B[n,p,q,\kappa]$$
$$B^{1/\kappa}\left[(m+1)\kappa+1;\kappa\left(\frac{1}{p}-\frac{1}{q}\right)\right]=B[m,p,q,\kappa].$$

2. MAIN RESULTS

Theorem 2.1. *Let $f(z_1,z_2)=\sum_{m,n=0}^{\infty}a_{mn}z_1^m z_2^n$, then the entire function $f(z_1,z_2)\in\mathbf{B}(p,q,\kappa)$ is of finite order ρ, if and only if*

$$\rho=\limsup_{m,n\to\infty}\frac{\ln(m^m n^n)}{-\ln E_{m,n}(f,\mathbf{B}(p,q,\kappa))}. \tag{2.1}$$

Proof. We prove the above result in two steps. First we consider the space $\mathbf{B}(p,q,\kappa)$, $q=2$, $0<p<2$ and $\kappa\geq 1$. Let $f(z_1,z_2)\in\mathbf{B}(p,q,\kappa)$ be of order ρ. From Theorem 1.1, for any $\epsilon>0$, there exists a natural number $n_0=n_0(\epsilon)$ such that

$$|a_{mn}|\leq m^{-m/\rho+\epsilon}n^{-n/\rho+\epsilon}\quad m,n>n_0. \tag{2.2}$$

We denote the partial sum of the Taylor series of a function $f(z_1,z_2)$ by

$$T_{m,n}(f,z_1,z_2)=\sum_{j_1=0}^{m}\sum_{j_2=0}^{n}a_{j_1j_2}z_1^{j_1}z_2^{j_2}.$$

We write

$$\begin{aligned} &E_{m,n}(f,\mathbf{B}(p,2,\kappa)) \\ &=\|f-T_{m,n}(f)\|_{p,2,\kappa} \\ &=\left\{\int_0^1\int_0^1\{(1-r_1)(1-r_2)\}^{\kappa(1/p-1/2)-1}\left(\sum_{j_1}\sum_{j_2}r_1^{2j_1}r_2^{2j_2}|a_{j_1j_2}|^2\right)^{\frac{\kappa}{2}}dr_1dr_2\right\}^{\frac{1}{\kappa}}, \end{aligned} \tag{2.3}$$

where

$$\sum_{j_1}\sum_{j_2}r_1^{2j_1}r_2^{2j_2}|a_{j_1j_2}|^2=S_1+S_2+\sum_{j_1=m+1}^{\infty}\sum_{j_2=n+1}^{\infty}r_1^{2j_1}r_2^{2j_2}|a_{j_1j_2}|^2,$$

$$S_1=\sum_{j_1=0}^{m}\sum_{j_2=n+1}^{\infty}r_1^{2j_1}r_2^{2j_2}|a_{j_1j_2}|^2\quad\text{and}\quad S_2=\sum_{j_1=m+1}^{\infty}\sum_{j_2=0}^{n}r_1^{2j_1}r_2^{2j_2}|a_{j_1j_2}|^2.$$

Since S_1,S_2 are bounded and $r_1,r_2<1$, therefore the above expression (2.3) becomes

$$E_{m,n}(f,\mathbf{B}(p,2,\kappa))\leq C\left\{\int_0^1\{(1-r)^{\kappa(1/p-1/2)-1}\}r^{(s+1)\kappa}dr\right\}\left\{\sum_{j_1=m+1}^{\infty}\sum_{j_2=n+1}^{\infty}|a_{j_1j_2}|^2\right\}^{\frac{1}{2}},$$

where

$$\left\{\int_0^1 \{(1-r)^{\kappa(1/p-1/2)-1}\}r^{(s+1)\kappa}dr\right\}$$
$$= \left\{\int_0^1 \{(1-r_1)^{\kappa(1/p-1/2)-1}\}r_1^{(m+1)\kappa}dr_1\right\}$$
$$\times \left\{\int_0^1 \{(1-r_2)\}^{\kappa(1/p-1/2)-1}r_2^{(n+1)\kappa}dr_2\right\}.$$

Therefore

$$E_{m,n}(f,\mathbf{B}(p,2,\kappa)) \leq CB[m,p,2,\kappa]B[n,p,2,\kappa]\left\{\sum_{j_1=m+1}^{\infty}\sum_{j_2=n+1}^{\infty}|a_{j_1j_2}|^2\right\}^{\frac{1}{2}}, \tag{2.4}$$

where C is a constant and $B(a,b)$ $(a,b>0)$ denotes the beta function.

By using (2.2), we have

$$\sum_{j_1=m+1}^{\infty}\sum_{j_2=n+1}^{\infty}|a_{j_1j_2}|^2 \leq \sum_{j_1=m+1}^{\infty}\sum_{j_2=n+1}^{\infty} j_1^{-\frac{2j_1}{\rho+\epsilon}} j_2^{-\frac{2j_2}{\rho+\epsilon}}$$
$$\leq \sum_{j_1=m+1}^{\infty} j_1^{-\frac{2j_1}{\rho+\epsilon}} \sum_{j_2=n+1}^{\infty} j_2^{-\frac{2j_2}{\rho+\epsilon}}$$
$$\leq O(1)(m+1)^{-2(m+1)/\rho+\epsilon}(n+1)^{-2(n+1)/\rho+\epsilon}.$$

Using the above inequality in (2.4), we have

$$E_{m,n}(f,\mathbf{B}(p,2,\kappa)) \leq CB[m,p,2,\kappa]B[n,p,2,\kappa](m+1)^{-(m+1)/\rho+\epsilon}(n+1)^{-(n+1)/\rho+\epsilon}.$$
$$\Rightarrow \rho+\epsilon \geq \frac{\ln\left[(m+1)^{(m+1)}(n+1)^{(n+1)}\right]}{-\ln\{E_{m,n}(f,\mathbf{B}(p,2,\kappa))\}+\ln\{B[m,p,2,\kappa]\}+\ln\{B[n,p,2,\kappa]\}}.$$

Now

$$B\left[(n+1)\kappa+1;\kappa\left(\frac{1}{p}-\frac{1}{2}\right)\right] = \frac{\Gamma((n+1)\kappa+1)\Gamma\left(\kappa\left(\frac{1}{p}-\frac{1}{2}\right)\right)}{\Gamma\left(\left(n+\frac{1}{2}+\frac{1}{p}\right)\kappa+1\right)}.$$

Hence

$$B\left[(n+1)\kappa+1;\kappa\left(\frac{1}{p}-\frac{1}{2}\right)\right] \simeq \frac{e^{-[(n+1)\kappa+1]}[(n+1)\kappa+1]^{(n+1)\kappa+3/2}\Gamma\left(\frac{1}{p}-\frac{1}{2}\right)}{e^{[(n+1/2+1/p)\kappa+1]}[(n+\frac{1}{2}+\frac{1}{p})\kappa+1]^{(n+1/2+1/p)\kappa+3/2}}.$$

Thus

$$\left\{B\left[(n+1)\kappa+1;\kappa\left(\frac{1}{p}-\frac{1}{2}\right)\right]\right\}^{\frac{1}{(n+1)}} \cong 1. \tag{2.5}$$

Now proceeding to limits, we obtain

$$\rho \geq \limsup_{m,n\to\infty}\frac{\ln(m^m n^n)}{-\ln\{E_{m,n}(f,\mathbf{B}(p,2,\kappa))\}}. \tag{2.6}$$

For the reverse inequality, since from the right hand side of the inequality (2.4), we have

$$|a_{m+1n+1}|B[m,p,2,\kappa]B[n,p,2,\kappa] \leq E_{m,n}(f,\mathbf{B}(p,2,\kappa)), \tag{2.7}$$

we have

$$\frac{\ln(m^m n^n)}{-\ln E_{m,n}(f,\mathbf{B}(p,2,\kappa))} \geq \frac{\ln(m^m n^n)}{-\ln|a_{m+1n+1}| + \ln\{B[m,p,2,\kappa]\} + \ln\{B[n,p,2,\kappa]\}}.$$

Now proceeding to limits, we obtain

$$\limsup_{m,n\to\infty} \frac{\ln(m^m n^n)}{-\ln E_{m,n}(f,\mathbf{B}(p,2,\kappa))} \geq \rho. \tag{2.8}$$

From (2.6) and (2.8), we get the required result.

In the second step, for the general case $\mathbf{B}(p,q,\kappa)$, $q \neq 2$, we have

$$\begin{aligned} E_{m,n}&(f,\mathbf{B}(p,q,\kappa)) \\ &\leq \|f - T_{m,n}(f)\|_{p,q,\kappa} \\ &= \left\{\int_0^1\int_0^1 \{(1-r_1)(1-r_2)\}^{\kappa(1/p-1/q)-1}\left(\sum_{j_1}\sum_{j_2} r_1^{qj_1} r_2^{qj_2}|a_{j_1j_2}|^q\right)^{\frac{\kappa}{q}} dr_1 dr_2\right\}^{\frac{1}{\kappa}}, \end{aligned} \tag{2.9}$$

where

$$\sum_{j_1}\sum_{j_2} r_1^{2j_1} r_2^{2j_2}|a_{j_1j_2}|^2 = S_1 + S_2 + \sum_{j_1=m+1}^{\infty}\sum_{j_2=n+1}^{\infty} r_1^{2j_1} r_2^{2j_2}|a_{j_1j_2}|^2,$$

$$S_1 = \sum_{j_1=0}^{m}\sum_{j_2=n+1}^{\infty} r_1^{2j_1} r_2^{2j_2}|a_{j_1j_2}|^2 \quad \text{and} \quad S_2 = \sum_{j_1=m+1}^{\infty}\sum_{j_2=0}^{n} r_1^{2j_1} r_2^{2j_2}|a_{j_1j_2}|^2.$$

Since S_1, S_2 are bounded and $r_1, r_2 < 1$, therefore the above expression (2.9) becomes

$$E_{m,n}(f,\mathbf{B}(p,q,\kappa)) \leq C'\left\{\int_0^1 \{(1-r)^{\kappa(1/p-1/q)-1}\} r^{(s+1)\kappa} dr\right\}\left\{\sum_{j_1=m+1}^{\infty}\sum_{j_2=n+1}^{\infty} |a_{j_1j_2}|^q\right\}^{\frac{1}{q}},$$

where

$$\begin{aligned} &\left\{\int_0^1 \{(1-r)^{\kappa(1/p-1/q)-1}\} r^{(s+1)\kappa} dr\right\} \\ &\qquad = \left\{\int_0^1 \{(1-r_1)^{\kappa(1/p-1/q)-1}\} r_1^{(m+1)\kappa} dr_1\right\} \\ &\qquad\qquad \times \left\{\int_0^1 \{(1-r_2)\}^{\kappa(1/p-1/q)-1} r_2^{(n+1)\kappa} dr_2\right\}. \end{aligned}$$

Therefore

$$E_{m,n}(f,\mathbf{B}(p,q,\kappa)) \leq C' B[m,p,q,\kappa] B[n,p,q,\kappa]\left\{\sum_{j_1=m+1}^{\infty}\sum_{j_2=n+1}^{\infty} |a_{j_1j_2}|^q\right\}^{\frac{1}{q}}, \tag{2.10}$$

where C' is constant and $B[m,p,q,\kappa]$ is Euler's integral of the first kind. By using (2.2), we get

$$\begin{aligned} \sum_{j_1=m+1}^{\infty}\sum_{j_2=n+1}^{\infty} |a_{j_1j_2}|^q &\leq \sum_{j_1=m+1}^{\infty} j_1^{\frac{-qj_1}{(\rho+\epsilon)}} \sum_{j_2=n+1}^{\infty} j_2^{\frac{-qj_2}{(\rho+\epsilon)}} \\ &\leq O(1)(m+1)^{\frac{-q(m+1)}{(\rho+\epsilon)}}(n+1)^{\frac{-q(n+1)}{(\rho+\epsilon)}}. \end{aligned}$$

Using above inequality in (2.10), we get

$$E_{m,n}(f,\mathbf{B}(p,q,\kappa)) \leq C' B[m,p,q,\kappa]B[n,p,q,\kappa](m+1)^{-(m+1)/\rho+\epsilon}(n+1)^{-(n+1)/\rho+\epsilon}.$$

$$\Rightarrow \rho+\epsilon \geq \frac{\ln\left[(m+1)^{m+1}(n+1)^{n+1}\right]}{-\ln E_{m,n}(f,\mathbf{B}(p,q,\kappa)) + \ln\{B[m,p,q,\kappa]\} + \ln\{B[n,p,q,\kappa]\}}.$$

Now proceeding to limits, we obtain

$$\rho \geq \limsup_{m,n\to\infty} \frac{\ln(m^m n^n)}{-\ln E_{m,n}(f,\mathbf{B}(p,q,\kappa))}. \tag{2.11}$$

Let $0 < p < q < 2$, and $\kappa, q \geq 1$. Since

$$E_{m,n}(f,\mathbf{B}(p_1,q_1,\kappa_1)) \leq 2^{1/q-1/q_1}\left[\kappa\left(\frac{1}{p}-\frac{1}{q}\right)\right]^{\frac{1}{\kappa}-\frac{1}{\kappa_1}} E_{m,n}(f,\mathbf{B}(p,q,\kappa)),$$

where $p_1 = p, q_1 = 2$ and $\kappa_1 = \kappa$, and the condition (2.1) is already proved for the space $\mathbf{B}(p,2,\kappa)$, we get

$$\limsup_{m,n\to\infty} \frac{\ln(m^m n^n)}{-\ln E_{m,n}(f,\mathbf{B}(p,q,\kappa))} \geq \limsup_{m,n\to\infty} \frac{\ln(m^m n^n)}{-\ln E_{m,n}(f,\mathbf{B}(p,2,\kappa))} = \rho. \tag{2.12}$$

Now let $0 < p \leq 2 < q$. Since

$$M_2(f,r_1,r_2) \leq M_q(f,r_1,r_2), \quad 0 < r_1, r_2 < 1,$$

therefore

$$\begin{aligned} E_{m,n}(f,\mathbf{B}(p,q,\kappa)) &\geq \left\{\int_0^1\int_0^1 \{(1-r_1)(1-r_2)\}^{\kappa(1/p-1/q)-1} Q\,dr_1 dr_2\right\}^{\frac{1}{\kappa}} \\ &\geq |a_{m+1n+1}| B[m,p,q,\kappa]B[n,p,q,\kappa], \end{aligned} \tag{2.13}$$

where $Q = \inf\left[M_2^{\kappa}(f-p;r_1,r_2) : p \in P\right]$. Hence we have

$$\frac{\ln(m^m n^n)}{-\ln E_{m,n}(f,\mathbf{B}(p,q,\kappa))} \geq \frac{\ln(m^m n^n)}{-\ln|a_{m+1n+1}| + \ln\{B[m,p,q,\kappa]\} + \ln\{B[n,p,q,\kappa]\}}.$$

Now proceeding to limits, we obtain

$$\limsup_{m,n\to\infty} \frac{\ln(m^m n^n)}{-\ln E_{m,n}(f,\mathbf{B}(p,q,\kappa))} \geq \rho. \tag{2.14}$$

From (2.11) and (2.14), we get the required result. □

Now we prove

Theorem 2.2. *Let $f(z_1,z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n$, then the entire function $f(z_1,z_2) \in H_q$ is of finite order ρ, if and only if*

$$\rho = \limsup_{m,n\to\infty} \frac{\ln(m^m n^n)}{-\ln E_{m,n}(f,H_q)}. \tag{2.15}$$

Proof. Let $f(z_1,z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n \in H_q$ be an entire transcendental function. Since f is entire, we have

$$\lim_{m,n\to\infty} \sqrt[(m+n)]{|a_{mn}|} = 0, \tag{2.16}$$

and $f \in H_q$, therefore

$$M_q(f;r_1,r_2) < \infty,$$

and $f(z_1, z_2) \in \mathbf{B}(p, q, \kappa)$, $0 < p < q \leq \infty$; $q, \kappa \geq 1$. By (1.1) we obtain

$$E_{m,n}(f, \mathbf{B}(q/2, q, q)) \leq \varsigma_q E_{m,n}(f, H_q), \quad 1 \leq q < \infty, \tag{2.17}$$

where ς_q is a constant independent of m, n and f. In the case of space H_∞,

$$E_{m,n}(f, \mathbf{B}(p, \infty, \infty)) \leq E_{m,n}(f, H_\infty), \quad 0 < p < \infty. \tag{2.18}$$

From (2.17), we have

$$\begin{aligned} \xi(f) &= \limsup_{m,n\to\infty} \frac{\ln (m^m n^n)}{-\ln E_{m,n}(f, H_q)} \\ &\geq \limsup_{m,n\to\infty} \frac{\ln (m^m n^n)}{-\ln E_{m,n}(f, \mathbf{B}(q/2, q, q))} \\ &\geq \rho, \qquad 1 \leq q < \infty, \end{aligned} \tag{2.19}$$

and using estimate (2.18) we prove inequality (2.19) for the case $q = \infty$.

For the reverse inequality

$$\xi(f) \leq \rho, \tag{2.20}$$

since

$$E_{m,n}(f, H_q) \leq O(1) \sum_{j_1=m+1}^{\infty} \sum_{j_2=n+1}^{\infty} |a_{j_1 j_2}(f)|,$$

using (2.2), we have

$$\begin{aligned} E_{m,n}(f, H_q) &\leq O(1) \sum_{j_1=m+1}^{\infty} \sum_{j_2=n+1}^{\infty} j_1^{-\frac{j_1}{\rho+\epsilon}} j_2^{-\frac{j_2}{\rho+\epsilon}} \\ &\leq O(1) \sum_{j_1=m+1}^{\infty} j_1^{-\frac{j_1}{\rho+\epsilon}} \sum_{j_2=n+1}^{\infty} j_2^{-\frac{j_2}{\rho+\epsilon}} \\ &\leq O(1)(m+1)^{-(m+1)/\rho+\epsilon}(n+1)^{-(n+1)/\rho+\epsilon}. \end{aligned}$$

$$\Rightarrow \rho + \epsilon \geq \frac{\ln [(m+1)^{(m+1)}(n+1)^{(n+1)}]}{-\ln [E_{m,n}(f, H_q)]}.$$

Now proceeding to limits and since ϵ is arbitrary, then we will get (2.20). From (2.19) and (2.20) we will obtain the required result.

Now we prove sufficiency. Assume that the condition (2.15) is satisfied. Then it follows that $\ln [1/E_{m,n}(f, H_q)]^{1/(m+n)} \to \infty$ as $m, n \to \infty$.

This yields

$$\lim_{m,n\to\infty} \sqrt[(m+n)]{E_{m,n}(f, H_q)} = 0.$$

This relation and the estimate $|a_{m+1n+1}(f)| \leq E_{m,n}(f, H_q)$ yield the relation (2.16). This means that $f(z_1, z_2) \in H_q$ is an entire transcendental function. □

Now we prove

Theorem 2.3. *Let $f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n$, then the entire function $f(z_1, z_2) \in \mathbf{B}(p, q, \kappa)$ of finite order ρ, is of type τ if and only if*

$$\tau = \frac{1}{e\rho} \limsup_{m,n\to\infty} \{m^m n^n E^{\rho}_{m,n}(f, \mathbf{B}(p, q, \kappa))\}^{\frac{1}{m+n}}. \tag{2.21}$$

Proof. We prove the above result in two steps.

First we consider the space $\mathbf{B}(p,q,\kappa)$, $q=2$, $0<p<2$ and $\kappa\geq 1$. Let $f(z)\in\mathbf{B}(p,q,\kappa)$ be of order ρ. From Theorem 1.2, for any $\epsilon>0$, there exists a natural number $n_0=n_0(\epsilon)$ such that

$$|a_{mn}|\leq m^{-m/\rho}n^{-n/\rho}[e\rho(\tau+\epsilon)]^{\frac{m+n}{\rho}}. \tag{2.22}$$

We denote the partial sum of the Taylor series of a function $f(z_1,z_2)$ by

$$T_{m,n}(f,z_1,z_2)=\sum_{j_1=0}^{m}\sum_{j_2=0}^{n}a_{j_1j_2}z_1^{j_1}z_2^{j_2},$$

we write

$$\begin{aligned} &E_{m,n}(f,\mathbf{B}(p,2,\kappa)) \\ &=\|f-T_{m,n}(f)\|_{p,2,\kappa} \\ &=\left\{\int_0^1\int_0^1\{(1-r_1)(1-r_2)\}^{\kappa(1/p-1/2)-1}\left(\sum_{j_1}\sum_{j_2}r_1^{2j_1}r_2^{2j_2}|a_{j_1j_2}|^2\right)^{\frac{\kappa}{2}}dr_1dr_2\right\}^{\frac{1}{\kappa}}, \end{aligned} \tag{2.23}$$

where

$$\sum_{j_1}\sum_{j_2}r_1^{2j_1}r_2^{2j_2}|a_{j_1j_2}|^2=S_1+S_2+\sum_{j_1=m+1}^{\infty}\sum_{j_2=n+1}^{\infty}r_1^{2j_1}r_2^{2j_2}|a_{j_1j_2}|^2,$$

$$S_1=\sum_{j_1=0}^{m}\sum_{j_2=n+1}^{\infty}r_1^{2j_1}r_2^{2j_2}|a_{j_1j_2}|^2 \quad\text{and}\quad S_2=\sum_{j_1=m+1}^{\infty}\sum_{j_2=0}^{n}r_1^{2j_1}r_2^{2j_2}|a_{j_1j_2}|^2.$$

Since S_1,S_2 are bounded, and $r_1,r_2<1$ therefore the above expression (2.23) becomes

$$E_{m,n}(f,\mathbf{B}(p,2,\kappa))\leq DB[m,p,2,\kappa]B[n,p,2,\kappa]\left\{\sum_{j_1=m+1}^{\infty}\sum_{j_2=n+1}^{\infty}|a_{j_1j_2}|^2\right\}^{\frac{1}{2}}, \tag{2.24}$$

where D is a constant and $B(a,b)$ $(a,b>0)$ denotes the beta function. By using (2.22), we have

$$\begin{aligned} \sum_{j_1=m+1}^{\infty}\sum_{j_2=n+1}^{\infty}|a_{j_1j_2}|^2 &\leq \sum_{j_1=m+1}^{\infty}\sum_{j_2=n+1}^{\infty}j_1^{-\frac{2j_1}{\rho}}j_2^{-\frac{2j_2}{\rho}}[e\rho(\tau+\epsilon)]^{\frac{2(j_1+j_2)}{\rho}} \\ &\leq \sum_{j_1=m+1}^{\infty}j_1^{-\frac{2j_1}{\rho}}[e\rho(\tau+\epsilon)]^{\frac{2j_1}{\rho}}\sum_{j_2=n+1}^{\infty}j_2^{-\frac{2j_2}{\rho+\epsilon}}[e\rho(\tau+\epsilon)]^{\frac{2j_2}{\rho}} \\ &\leq O(1)(m+1)^{-2(m+1)/\rho}(n+1)^{-2(n+1)/\rho}[e\rho(\tau+\epsilon)]^{\frac{2(m+n+2)}{\rho}}. \end{aligned}$$

Using the above inequality in (2.24), we get

$$E^{\rho}_{m,n}(f,\mathbf{B}(p,2,\kappa))\leq D^{\rho}B^{\rho}[m,p,2,\kappa]B^{\rho}[n,p,2,\kappa]Y[e\rho(\tau+\epsilon)]^{(m+n+2)},$$

where $Y=(m+1)^{-(m+1)}(n+1)^{-(n+1)}$.

Now proceeding to limits and since ϵ is arbitrary, we have

$$\frac{1}{e\rho}\limsup_{m,n\to\infty}\{m^mn^nE^{\rho}_{m,n}(f,\mathbf{B}(p,2,\kappa))\}^{\frac{1}{m+n}}\leq\tau. \tag{2.25}$$

For the reverse inequality, since from the right hand side of (2.24),

$$|a_{m+1n+1}|B[m,p,2,\kappa]B[n,p,2,\kappa]\leq E_{m,n}(f,\mathbf{B}(p,2,\kappa))$$

we have

$$m^{m/(m+n)}n^{n/(m+n)}|a_{m+1n+1}|^{\rho/(m+n)}B^{\frac{\rho}{(m+n)}}[m,p,2,\kappa]B^{\frac{\rho}{(m+n)}}[n,p,2,\kappa] \\ \leq \{E^{\rho}_{m,n}m^m n^n\}^{1/(m+n)}.$$

Now proceeding to limits, we obtain

$$\tau \leq \frac{1}{e\rho}\limsup_{m,n\to\infty}\{m^m n^n E^{\rho}_{m,n}(f,\mathbf{B}(p,2,\kappa))\}^{\frac{1}{m+n}}. \tag{2.26}$$

From (2.25) and (2.26), we get the required result.

In the second step, for the general case $\mathbf{B}(p,q,\kappa), q\neq 2$, we have

$$\begin{aligned} &E_{m,n}(f,\mathbf{B}(p,q,\kappa)) \\ &\leq \|f-T_{m,n}(f)\|_{p,q,\kappa} \\ &= \left\{\int_0^1\int_0^1\{(1-r_1)(1-r_2)\}^{\kappa(1/p-1/q)-1}\left(\sum_{j_1}\sum_{j_2} r_1^{qj_1}r_2^{qj_2}|a_{j_1j_2}|^q\right)^{\frac{\kappa}{q}}dr_1dr_2\right\}^{\frac{1}{\kappa}}, \end{aligned} \tag{2.27}$$

where

$$\sum_{j_1}\sum_{j_2} r_1^{2j_1}r_2^{2j_2}|a_{j_1j_2}|^2 = S_1+S_2+\sum_{j_1=m+1}^{\infty}\sum_{j_2=n+1}^{\infty} r_1^{2j_1}r_2^{2j_2}|a_{j_1j_2}|^2,$$

$$S_1=\sum_{j_1=0}^{m}\sum_{j_2=n+1}^{\infty} r_1^{2j_1}r_2^{2j_2}|a_{j_1j_2}|^2 \quad\text{and}\quad S_2=\sum_{j_1=m+1}^{\infty}\sum_{j_2=0}^{n} r_1^{2j_1}r_2^{2j_2}|a_{j_1j_2}|^2.$$

Since S_1, S_2 are bounded, therefore the above expression (2.27) becomes

$$E_{m,n}(f,\mathbf{B}(p,q,\kappa)) \leq G\left\{\int_0^1\{(1-r)^{\kappa(1/p-1/q)-1}\}r^{(s+1)\kappa}dr\right\}\left\{\sum_{j_1=m+1}^{\infty}\sum_{j_2=n+1}^{\infty}|a_{j_1j_2}|^q\right\}^{\frac{1}{q}},$$

where

$$\begin{aligned} &\left\{\int_0^1\{(1-r)^{\kappa(1/p-1/q)-1}\}r^{(s+1)\kappa}dr\right\} \\ &\qquad = \left\{\int_0^1\{(1-r_1)^{\kappa(1/p-1/q)-1}\}r_1^{(m+1)\kappa}dr_1\right\} \\ &\qquad\qquad \times\left\{\int_0^1\{(1-r_2)\}^{\kappa(1/p-1/q)-1}r_2^{(n+1)\kappa}dr_2\right\}. \end{aligned}$$

Since $r_1, r_2 < 1$, therefore we have

$$E_{m,n}(f,\mathbf{B}(p,q,\kappa)) \leq GB[m,p,q,\kappa]B[n,p,q,\kappa]\left\{\sum_{j_1=m+1}^{\infty}\sum_{j_2=n+1}^{\infty}|a_{j_1j_2}|^q\right\}^{\frac{1}{q}}, \tag{2.28}$$

where G is constant and $B[m,p,q,\kappa]$ is Euler's integral of the first kind. Using (2.22), we get

$$\begin{aligned}\sum_{j_1=m+1}^{\infty}\sum_{j_2=n+1}^{\infty}|a_{j_1j_2}|^q &\le \sum_{j_1=m+1}^{\infty}\sum_{j_2=n+1}^{\infty} j_1^{-\frac{qj_1}{\rho}} j_2^{-\frac{qj_2}{\rho}}[e\rho(\tau+\epsilon)]^{\frac{q(j_1+j_2)}{\rho}}\\ &\le \sum_{j_1=m+1}^{\infty} j_1^{-\frac{qj_1}{\rho}}[e\rho(\tau+\epsilon)]^{\frac{qj_1}{\rho}} \sum_{j_2=n+1}^{\infty} j_2^{-\frac{qj_2}{\rho+\epsilon}}[e\rho(\tau+\epsilon)]^{\frac{qj_2}{\rho}}\\ &\le O(1)(m+1)^{-q(m+1)/\rho}(n+1)^{-q(n+1)/\rho}[e\rho(\tau+\epsilon)]^{\frac{q(m+n+2)}{\rho}}.\end{aligned}$$

Using the above inequality in (2.28), we get

$$E^{\rho}_{m,n}(f,\mathbf{B}(p,q,\kappa)) \le G^{\rho}B^{\rho}[m,p,q,\kappa]B^{\rho}[n,p,q,\kappa]Y[e\rho(\tau+\epsilon)]^{(m+n+2)},$$

where $Y=(m+1)^{-(m+1)}(n+1)^{-(n+1)}$. Now proceeding to limits, since ϵ is arbitrary, we have

$$\frac{1}{e\rho}\limsup_{m,n\to\infty}\{m^m n^n E^{\rho}_{mn}(f,\mathbf{B}(p,q,\kappa))\}^{\frac{1}{m+n}} \le \tau. \tag{2.29}$$

Let $0<p<q<2$, and $\kappa,q\ge 1$. Since

$$E_{m,n}(f,\mathbf{B}(p_1,q_1,\kappa_1)) \le 2^{1/q-1/q_1}[\kappa(1/p-1/q)]^{1/\kappa-1/\kappa_1}E_{m,n}(f,\mathbf{B}(p,q,\kappa)),$$

where $p_1=p$, $q_1=2$ and $\kappa_1=\kappa$, and the condition (2.21) has already been proved for the space $\mathbf{B}(p,2,\kappa)$, we get

$$\begin{aligned}&\limsup_{m,n\to\infty}\{m^m n^n E^{\rho}_{m,n}(f,\mathbf{B}(p,q,\kappa))\}^{\frac{1}{m+n}}\\ &\qquad\qquad \ge \limsup_{m,n\to\infty}\{m^m n^n E^{\rho}_{m,n}(f,\mathbf{B}(p,2,\kappa))\}^{\frac{1}{m+n}} = \tau.\end{aligned}$$

Now let $0<p\le 2<q$. Since, in this case we have

$$M_2(f,r_1,r_2)\le M_q(f,r_1,r_2),\quad 0<r_1,r_2<1,$$

therefore

$$\begin{aligned}\limsup_{m,n\to\infty}\{m^m n^n E^{\rho}_{m,n}(f,\mathbf{B}(p,q,\kappa))\}^{\frac{1}{m+n}} &\ge \limsup_{m,n\to\infty}\{m^m n^n |a_{mn}|^{\rho}\}^{\frac{1}{m+n}}\\ &= e\rho\tau.\end{aligned} \tag{2.30}$$

From (2.29) and (2.30), we get the required result. □

Lastly we prove

Theorem 2.4. *Let $f(z_1,z_2)=\sum_{m,n=0}^{\infty}a_{mn}z_1^m z_2^n$, then the entire function $f(z_1,z_2)\in H_q$ having finite order ρ is of type τ if and only if*

$$\tau=\frac{1}{e\rho}\limsup_{m,n\to\infty}\{m^m n^n E^{\rho}_{m,n}(f,H_q)\}^{\frac{1}{m+n}}. \tag{2.31}$$

Proof. Since $f(z_1,z_2)=\sum_{m,n=0}^{\infty}a_{mn}z_1^m z_2^n$ is an entire transcendental function, we have

$$\lim_{m,n\to\infty}\sqrt[m+n]{|a_{mn}|}=0. \tag{2.32}$$

Therefore $f(z_1, z_2) \in \mathbf{B}(p, q, \kappa)$, $0 < p < q \leq \infty$; $q, \kappa \geq 1$. We have

$$\xi(f) = \frac{1}{e\rho} \limsup_{m,n\to\infty} \{m^m n^n E^\rho_{m,n}(f, H_q)\}^{\frac{1}{m+n}} \tag{2.33}$$
$$\geq \frac{1}{e\rho} \limsup_{m,n\to\infty} \left\{m^m n^n E^\rho_{m,n}\left(f, \mathbf{B}\left(\frac{q}{2}, q, q\right)\right)\right\}^{\frac{1}{m+n}} = \tau$$

for $1 \leq q < \infty$. Using the estimate (2.18) we prove inequality (2.33) in the case $q = \infty$. For the reverse inequality

$$\xi(f) \leq \tau, \tag{2.34}$$

we have

$$E_{m,n}(f, H_q) \leq \sum_{j_1=m+1}^{\infty} \sum_{j_2=n+1}^{\infty} |a_{j_1 j_2}(f)|.$$

Using (2.22), we get

$$E^\rho_{m,n}(f, H_q) \leq O(1)(m+1)^{-(m+1)}(n+1)^{-(n+1)}[e\rho(\tau+\epsilon)]^{(m+n+2)}$$
$$\Rightarrow \tau + \epsilon \geq \frac{1}{e\rho}\{(m+1)^{(m+1)}(n+1)^{(n+1)} E^\rho_{m,n}(f, H_q)\}^{\frac{1}{(m+n+2)}}.$$

Now proceeding to limits, since ϵ is arbitrary, we get

$$\tau \geq \frac{1}{e\rho} \limsup_{m,n\to\infty} \{m^m n^n E^\rho_{m,n}(f, H_q)\}^{\frac{1}{m+n}}. \tag{2.35}$$

From (2.33) and (2.35), we obtain the required result.

Now we prove sufficiency. Assume that the condition (2.31) is satisfied. Then it follows that $\{E^\rho_{m,n}(f, H_q)\}^{1/(m+n)} \to 0$ as $m, n \to \infty$. This yields

$$\lim_{m,n\to\infty} \sqrt[(m+n)]{E_{m,n}(f, H_q)} = 0.$$

This relation and the estimate $|a_{m+1n+1}(f)| \leq E_{m,n}(f, H_q)$ yield the inequality (2.32). This implies that $f(z_1, z_2) \in H_q$ is an entire transcendental function. □

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