Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 6, Issue 2, Article 45, 2005

ON THE REFINED HEISENBERG-WEYL TYPE INEQUALITY

JOHN MICHAEL RASSIAS

NATIONAL AND CAPODISTRIAN UNIVERSITY OF ATHENS
PEDAGOGICAL DEPARTMENT EE
SECTION OF MATHEMATICS,4, AGAMEMNONOS STR.
AGHIA PARASKEVI
ATHENS 15342, GREECE

jrassias@primedu.uoa.gr
URL: http://www.primedu.uoa.gr/∼jrassias/

Received 11 January, 2005; accepted 17 March, 2005
Communicated by S. Saitoh

ABSTRACT. The well-known *second moment Heisenberg-Weyl inequality (or uncertainty relation)* states: Assume that $f : \mathbb{R} \to \mathbb{C}$ is a complex valued function of a random real variable x such that $f \in L^2(\mathbb{R})$, where $\mathbb{R} = (-\infty, \infty)$. Then the product of the second moment of the random real x for $|f|^2$ and the second moment of the random real ξ for $\left|\widehat{f}\right|^2$ is at least $E_{\mathbb{R},|f|^2} \Big/ 4\pi$, where $\widehat{f}$ is the Fourier transform of f, $\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-2i\pi\xi x} f(x)\, dx$ and $f(x) = \int_{\mathbb{R}} e^{2i\pi\xi x} \hat{f}(\xi)\, d\xi$, and $E_{\mathbb{R},|f|^2} = \int_{\mathbb{R}} |f(x)|^2\, dx$. This uncertainty relation is well-known in classical quantum mechanics. In 2004, the author generalized the afore-mentioned result to *the higher order moments for $L^2(\mathbb{R})$ functions f*. In this paper, a refined form of the generalized Heisenberg-Weyl *type* inequality is established.

Key words and phrases: Heisenberg-Weyl Type Inequality, Uncertainty Principle, Gram determinant.

2000 *Mathematics Subject Classification.* 26, 33, 42, 60, 62.

1. INTRODUCTION

The serious question of certainty in science was high-lighted by Heisenberg, in 1927, via his "uncertainty principle" [2]. He demonstrated, for instance, the impossibility of specifying simultaneously the position and the speed (or the momentum) of an electron within an atom. In 1933, according to Wiener [7] "*a pair of transforms cannot both be very small*." This uncertainty principle was stated in 1925 by Wiener, according to Wiener's autobiography [8, p. 105–107], in a lecture in Göttingen. The following result of the *Heisenberg-Weyl Inequality* is credited to Pauli according to Weyl [6, p. 77, p. 393–394]. In 1928, according to Pauli [6] " *the less the uncertainty in $|f|^2$, the greater the uncertainty in $\left|\widehat{f}\right|^2$, and conversely*." This result does not actually appear in Heisenberg's seminal paper [2] (in 1927).

ISSN (electronic): 1443-5756

013-05

In 1998, Burke Hubbard [1] wrote a remarkable book on wavelets. According to her, most people first learn the Heisenberg uncertainty principle in connection with quantum mechanics, but it is also a central statement of information processing. The following second order moment Heisenberg-Weyl inequality provides a precise quantitative formulation of the above-mentioned uncertainty principle.

1.1. **Second Moment Heisenberg-Weyl Inequality (**[1], [4], [5]**).** *For any* $f \in L^2(\mathbb{R})$, $f : \mathbb{R} \to \mathbb{C}$, *such that*

$$\|f\|_{2,\mathbb{R}}^2 = \int_{\mathbb{R}} |f(x)|^2 \, dx = E_{\mathbb{R},|f|^2},$$

any fixed but arbitrary constants $x_m, \xi_m \in \mathbb{R}$, *and for the second order moments*

$$(\mu_2)_{\mathbb{R},|f|^2} = \sigma_{\mathbb{R},|f|^2}^2 = \int_{\mathbb{R}} (x - x_m)^2 |f(x)|^2 \, dx$$

and

$$(\mu_2)_{\mathbb{R},|\widehat{f}|^2} = \sigma_{\mathbb{R},|\widehat{f}|^2}^2 = \int_{\mathbb{R}} (\xi - \xi_m)^2 \left|\widehat{f}(\xi)\right|^2 d\xi,$$

the second order moment Heisenberg-Weyl inequality

$$(H_1) \qquad \sigma_{\mathbb{R},|f|^2}^2 \cdot \sigma_{\mathbb{R},|\widehat{f}|^2}^2 \geq \frac{\|f\|_{2,\mathbb{R}}^4}{16\pi^2},$$

holds. Equality holds in (H_1) *if and only if the generalized Gaussians*

$$f(x) = c_o \exp(2\pi i x \xi_m) \exp\left(-c(x - x_m)^2\right)$$

hold for some constants $c_o \in \mathbb{C}$ *and* $c > 0$.

The Heisenberg-Weyl inequality in *spectral analysis* says that the product of the effective duration Δx and the effective bandwidth $\Delta\xi$ of a signal cannot be less than the value $1/4\pi$, where $\Delta x^2 = \sigma_{\mathbb{R},|f|^2}^2 \big/ E_{\mathbb{R},|f|^2}$ and $\Delta\xi^2 = \sigma_{\mathbb{R},|\widehat{f}|^2}^2 \big/ E_{\mathbb{R},|f|^2}$ with $f : \mathbb{R} \to \mathbb{C}$, $\widehat{f} : \mathbb{R} \to \mathbb{C}$ defined as in (H_1), and

$$\text{(PPR)} \qquad E_{\mathbb{R},|f|^2} = \int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} \left|\widehat{f}(\xi)\right|^2 d\xi = E_{\mathbb{R},|\widehat{f}|^2}$$

according to the Plancherel-Parseval-Rayleigh identity.

1.2. **Fourth Moment Heisenberg-Weyl Inequality (**[4, pp. 26–27]**).** *For any* $f \in L^2(\mathbb{R})$, $f : \mathbb{R} \to \mathbb{C}$, *such that*

$$\|f\|_{2,\mathbb{R}}^2 = \int_{\mathbb{R}} |f(x)|^2 \, dx = E_{\mathbb{R},|f|^2},$$

any fixed but arbitrary constants $x_m, \xi_m \in \mathbb{R}$, *and for the fourth order moments*

$$(\mu_4)_{\mathbb{R},|f|^2} = \int_{\mathbb{R}} (x - x_m)^4 |f(x)|^2 \, dx$$

and

$$(\mu_4)_{\mathbb{R},|\widehat{f}|^2} = \int_{\mathbb{R}} (\xi - \xi_m)^4 \left|\widehat{f}(\xi)\right|^2 d\xi,$$

the fourth order moment Heisenberg-Weyl inequality

$$(H_2) \qquad (\mu_4)_{\mathbb{R},|f|^2} \cdot (\mu_4)_{\mathbb{R},|\widehat{f}|^2} \geq \frac{1}{64\pi^4} E_{2,\mathbb{R},f}^2,$$

holds, where

$$E_{2,\mathbb{R},f} = 2\int_{\mathbb{R}} \left[\left(1 - 4\pi^2\xi_m^2 x_\delta^2\right) |f(x)|^2 - x_\delta^2 |f'(x)|^2 - 4\pi\xi_m x_\delta^2 \operatorname{Im}\left(f(x)\overline{f'(x)}\right)\right] dx,$$

with $x_\delta = x - x_m$, $\xi_\delta = \xi - \xi_m$, $\operatorname{Im}(\cdot)$ *is the imaginary part of* $(\cdot)$, *and* $|E_{2,\mathbb{R},f}| < \infty$.

The "inequality" (H_2) holds, unless $f(x) = 0$.

We note that if the ordinary differential equation of second order

$$\text{(ODE)} \qquad f''_\alpha(x) = -2c_2 x_\delta^2 f_\alpha(x)$$

holds, with $\alpha = -2\pi\xi_m i$, $f_\alpha(x) = e^{\alpha x} f(x)$, and a constant $c_2 = \frac{1}{2}k_2^2 > 0$, $k_2 \in \mathbb{R}$ and $k_2 \neq 0$, then "equality" in (H_2) seems to occur. However, the solution of this differential equation (ODE), given by the function

$$f(x) = \sqrt{|x_\delta|} e^{2\pi i x \xi_m} \left[c_{20} J_{-1/4}\left(\frac{1}{2}|k_2| x_\delta^2\right) + c_{21} J_{1/4}\left(\frac{1}{2}|k_2| x_\delta^2\right)\right],$$

in terms of the Bessel functions $J_{\pm 1/4}$ of the first kind of orders $\pm 1/4$, leads to a contradiction, because this $f \notin L^2(\mathbb{R})$. Furthermore, a limiting argument is required for this problem. For the proof of this inequality see [4]. It is *open* to investigate cases, where the integrand on the right-hand side of the integral of $E_{2,\mathbb{R},f}$ will be nonnegative. For instance, for $x_m = \xi_m = 0$, this integrand is:= $|f(x)|^2 - x^2|f'(x)|^2$ (≥ 0).

In 2004, we [4] generalized the Heisenberg-Pauli-Weyl inequality in $\mathbb{R} = (-\infty, \infty)$. In this paper, a refined form of this generalized *Heisenberg-Weyl type inequality* is established in $I = [0, \infty)$. Afterwards, an open problem is proposed on some pertinent extremum principle. However, the above-mentioned Fourier transform is considered in $\mathbb{R}$, while our results in this paper are restricted to $I = [0, \infty)$. Futhermore, the corresponding inequality is investigated in $\mathbb{R}$, as well. Our second moment Heisenberg-Weyl type inequality and the fourth moment Heisenberg-Weyl type inequality are of the following forms (R_i), $(i = 1, 2)$.

1.3. **Second Moment Heisenberg-Weyl Type Inequality ([4]).** *For any* $f \in L^2(I)$, $I = [0, \infty)$, $f : I \to \mathbb{C}$, *such that* $\|f\|_{2,I}^2 = \int_I |f(x)|^2 dx = E_{I,|f|^2}$, *any fixed but arbitrary constant* $x_m \in \mathbb{R}$, *and for the second order moment*

$$(\mu_2)_{I,|f|^2} = \sigma_{I,|f|^2}^2 = \int_I (x - x_m)^2 |f(x)|^2 dx,$$

the second order moment Heisenberg-Weyl type inequality

$$(R_1) \qquad (\mu_2)_{I,|f|^2} \cdot \|f'\|_{2,I}^2 \geq \frac{1}{4} E_{1,I,f}^2 = \frac{1}{4}\left[-\int_I |f(x)|^2 dx\right]^2,$$

holds, where $|E_{1,I,f}| < \infty$. *Equality holds in (R_1) if and only if the Gaussians* $f(x) = c_o \exp\left(-c(x - x_m)^2\right)$ *hold for some constants* $c_o \in \mathbb{C}$ *and* $c > 0$.

We note that this inequality (R_1) *still holds* if we replace the interval of integration I with $\mathbb{R}$, *without* any other change.

1.4. **Fourth Moment Heisenberg-Weyl Type Inequality ([4]).** *For any* $f \in L^2(I)$, $I = [0, \infty)$, $f : I \to \mathbb{C}$, *such that* $\|f\|_{2,I}^2 = \int_I |f(x)|^2 dx = E_{I,|f|^2}$, *any fixed but arbitrary constant* $x_m \in \mathbb{R}$, *and for the fourth order moment*

$$(\mu_4)_{I,|f|^2} = \int_I (x - x_m)^4 |f(x)|^2 dx,$$

the fourth order moment Heisenberg – Weyl type inequality

$$(R_2) \qquad (\mu_4)_{I,|f|^2} \cdot \|f''\|_{2,I}^2 \geq \frac{1}{4} E_{2,I,f}^2 = \left[\int_I \left[|f(x)|^2 \, dx - x_\delta^2 \, |f'(x)|^2 \right] dx \right]^2$$

holds, where $x_\delta = x - x_m$, *and* $|E_{2,I,f}| < \infty$.

The "inequality" (R_2) holds, unless $f(x) = 0$.

We note that this inequality (R_2) *still holds* if we replace the interval of integration I with $\mathbb{R}$, *without* any other change except that one on the following condition (2.1), where $x \to \infty$ has to be substituted with $|x| \to \infty$.

We omit the proofs of the inequalities (R_i) $(i = 1, 2)$ *as special cases of* the corresponding proof of the following general *Theorem 2.1* (with $A = 0$) of this paper. Furthermore, we state our following four *pertinent propositions*. Their proofs are identical or analogous to the proofs of the corresponding propositions of [4].

Proposition 1.1 (Pascal type combinatorial identity, [4]). *If* $0 \leq \left[\frac{k}{2}\right]$ *is the greatest integer* $\leq \frac{k}{2}$, *then*

$$\text{(C)} \qquad \frac{k}{k-i}\binom{k-i}{i} + \frac{k-1}{k-i}\binom{k-i}{i-1} = \frac{k+1}{k-i+1}\binom{k-i+1}{i},$$

holds for any fixed but arbitrary $k \in \mathbb{N} = \{1, 2, \ldots\}$, *and* $0 \leq i \leq \left[\frac{k}{2}\right]$ *for* $i \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ *such that* $\binom{k}{-1} = 0$.

Proposition 1.2 (Generalized differential identity, [4]). *If* $f : I \to \mathbb{C}$ *is a complex valued function of a real variable* x, $I = [0, \infty)$, $0 \leq \left[\frac{k}{2}\right]$ *is the greatest integer* $\leq \frac{k}{2}$, $f^{(j)} = \frac{d^j}{dx^j} f$, *and* $\overline{(\cdot)}$ *is the conjugate of* $(\cdot)$, *then*

$$(*) \qquad f(x)\, \overline{f^{(k)}}(x) + f^{(k)}(x)\, \bar{f}(x) = \sum_{i=0}^{\left[\frac{k}{2}\right]} (-1)^i \frac{k}{k-i} \binom{k-i}{i} \frac{d^{k-2i}}{dx^{k-2i}} \left| f^{(i)}(x) \right|^2,$$

holds for any fixed but arbitrary $k \in \mathbb{N} = \{1, 2, \ldots\}$, *such that* $0 \leq i \leq \left[\frac{k}{2}\right]$ *for* $i \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

We note that the proof of (*) requires the application of the new identity (C). Furthermore, we note that the above *differential identity (*) still holds* if we replace the interval of integration I with $\mathbb{R}$, *without* any other change.

Proposition 1.3 (P^{th}-derivative of product, [4]). *If* $f_i : I \to \mathbb{C}$ $(i = 1, 2)$ *are two complex valued functions of a real variable* x, *then the* p^{th}*-derivative of the product* $f_1 f_2$ *is given, in terms of the lower derivatives* $f_1^{(m)}$, $f_2^{(p-m)}$ *by*

$$(1.1) \qquad (f_1 f_2)^{(p)} = \sum_{m=0}^{p} \binom{p}{m} f_1^{(m)} f_2^{(p-m)}$$

for any fixed but arbitrary $p \in \mathbb{N}_0$.

Proposition 1.4 (Generalized integral identity, [4]). *If* $f : I \to \mathbb{C}$ *is a complex valued function of a real variable* x, $I = [0, \infty)$, *and* $h : I \to \mathbb{R}$ *is a real valued function of* x, *as well as,* w, $w_p : I \to \mathbb{R}$ *are two real valued functions of* x, *such that* $w_p(x) = (x - x_m)^p w(x)$ *for any fixed but arbitrary constant* $x_m \in \mathbb{R}$ *and* $v = p - 2q$, $0 \leq q \leq \left[\frac{p}{2}\right]$, *then*

i)

$$(1.2) \qquad \int w_p(x)\, h^{(v)}(x)\, dx = \sum_{r=0}^{v-1} (-1)^r w_p^{(r)}(x)\, h^{(v-r-1)}(x) + (-1)^v \int w_p^{(v)}(x)\, h(x)\, dx$$

holds for any fixed but arbitrary $p \in \mathbb{N}_0$ *and* $v \in \mathbb{N}$*, and all* $r : r = 0, 1, 2, \ldots, v-1$*, as well as the integral identity*

ii)
$$\int_I w_p(x)\, h^{(v)}(x)\, dx = (-1)^v \int_I w_p^{(v)}(x)\, h(x)\, dx$$
holds if the limiting condition

iii)
$$\sum_{r=0}^{v-1} (-1)^r \lim_{x\to\infty} w_p^{(r)}(x)\, h^{(v-r-1)}(x) = 0,$$
holds, and if all of these integrals exist.

We note that the proof of (1.2) requires the application of the differential identity (1.1). Furthermore, we note that the above *integral identity* ii) *still holds* if we replace the interval of integration I with $\mathbb{R}$, *without* any other change except that on the above *limiting condition* iii), where $x \to \infty$ has to be substituted with $|x| \to \infty$.

2. REFINED HEISENBERG-WEYL TYPE INEQUALITY

We assume that $f : I \to \mathbb{C}$ is a complex valued function of a real variable x, and $w : I \to \mathbb{R}$ a real valued weight function of x, as well as x_m any fixed but arbitrary real constant. Also we denote

$$(\mu_{2p})_{w,I,|f|^2} = \int_I w^2(x)\,(x-x_m)^{2p}\,|f(x)|^2\, dx$$

the $2p^{th}$ weighted moment of x for $|f|^2$ with weight function $w : I \to \mathbb{R}$. Besides we denote

$$C_q = (-1)^q \frac{p}{p-q} \binom{p-q}{q},$$

if $0 \le q \le \left[\frac{p}{2}\right]$ (= the greatest integer $\le \frac{p}{2}$),

$$I_{ql} = (-1)^{p-2q} \int_I w_p^{(p-2q)}(x) \left|f^{(l)}(x)\right|^2 dx,$$

if $0 \le l \le q \le \left[\frac{p}{2}\right]$, and $w_p = (x-x_m)^p\, w$. We assume that all these integrals exist. Finally we denote $D_q = \sum_{l=0}^{q} I_{ql}$, if $|D_q| < \infty$ holds for $0 \le q \le \left[\frac{p}{2}\right]$, and

$$E_{p,I,f} = \sum_{q=0}^{[p/2]} C_q D_q,$$

if $|E_{p,I,f}| < \infty$ holds for $p \in \mathbb{N}$. In addition, we assume *the condition*:

$$\sum_{r=0}^{p-2q-1} (-1)^r \lim_{x\to\infty} w_p^{(r)}(x) \left(\left|f^{(l)}(x)\right|^2\right)^{(p-2q-r-1)} = 0, \tag{2.1}$$

for $0 \le l \le q \le \left[\frac{p}{2}\right]$. Furthermore,

$$|E^*_{p,I,f}| = \sqrt{E^2_{p,I,f} + 4A^2}, \tag{2.2}$$

where $A = \|u\|\, x_0 - \|v\|\, y_0$, with L^2-norm $\|\cdot\|^2 = \int_I |\cdot|^2$, inner product $(|u|, |v|) = \int_I |u|\,|v|$, and

$$u = w(x) x_\delta^p f(x), \quad v = f^{(p)}(x); \quad x_0 = \int_I |v(x)h(x)| dx, \quad y_0 = \int_I |u(x)h(x)| dx,$$

as well as

$$h(x) = \frac{1}{\sqrt{\sigma}} \sqrt[4]{\frac{2}{\pi}} e^{-\frac{1}{4}\left(\frac{x-\mu}{\sigma}\right)^2},$$

or

$$(H_I) \qquad h(x) = \sqrt{2} \frac{1}{\sqrt[4]{n\pi}} \sqrt{\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}} \cdot \frac{1}{\left(1+\frac{x^2}{n}\right)^{\frac{n+1}{4}}},$$

where μ is the mean, σ the standard deviation, and $n \in \mathbb{N}$, and

$$\|h(x)\|^2 = \int_I |h(x)|^2 dx = 1.$$

Theorem 2.1. *If (2.1) holds and $f \in L^2(\mathbb{R})$, then*

$$(R_p^*) \qquad \sqrt[2p]{(\mu_{2p})_{w,I,|f|^2}} \sqrt[p]{\|f^{(p)}\|_{2,I}} \geq \frac{1}{\sqrt[p]{2}} \sqrt[p]{\left|E_{p,I,f}^*\right|},$$

holds for any fixed but arbitrary $p \in \mathbb{N}$.

Equality holds in (R_p^*) iff $v(x) = -2c_p u(x)$ holds for constants $c_p > 0$, and any fixed but arbitrary $p \in \mathbb{N}$; $c_p = k_p^2/2 > 0$, $k_p \in \mathbb{R}$ and $k_p \neq 0$, $p \in \mathbb{N}$, and $A = 0$, or $h(x) = c_{1p}u(x) + c_{2p}v(x)$ and $x_0 = 0$, or $y_0 = 0$, where c_{ip} $(i = 1, 2)$ are constants and $A^2 > 0$.

We note that this inequality (R_p^*) *still holds* if we replace the interval of integration I with $\mathbb{R}$, *without* any other change except that one on the above condition (2.1), where $x \to \infty$ has to be substituted with $|x| \to \infty$, and the choice of h from (H_I) must be replaced with

$$h(x) = \frac{1}{\sqrt[4]{2\pi}\sqrt{\sigma}} e^{-\frac{1}{4}\left(\frac{x-\mu}{\sigma}\right)^2},$$

or

$$(H_{\mathbb{R}}) \qquad h(x) = \frac{1}{\sqrt[4]{n\pi}} \sqrt{\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}} \cdot \frac{1}{\left(1+\frac{x^2}{n}\right)^{\frac{n+1}{4}}},$$

where μ is the mean, σ the standard deviation, and $n \in \mathbb{N}$.

Proof. In fact, one gets

$$\begin{aligned}
(2.3) \qquad M_p^* &= M_p - A^2 \\
&= (\mu_{2p})_{w,I,|f|^2} \cdot \left\|f^{(p)}\right\|_{2,I}^2 - A^2 \\
&= \left(\int_I w^2(x)(x-x_m)^{2p}|f(x)|^2 dx\right) \cdot \left(\int_I \left|f^{(p)}(x)\right|^2 dx\right) - A^2 \\
(2.4) \qquad &= \|u\|^2 \|v\|^2 - A^2
\end{aligned}$$

with $u = w(x)x_\delta^p f_(x),\ v = f^{(p)}(x)$, where $x_\delta = x - x_m$.

From (2.3) – (2.4), *the Cauchy-Schwarz inequality* $(|u|, |v|) \leq \|u\| \|v\|$ and the non-negativeness of the following *Gram determinant* [3] or

$$(2.5) \qquad 0 \leq \begin{vmatrix} \|u\|^2 & (|u|, |v|) & y_0 \\ (|v|, |u|) & \|v\|^2 & x_0 \\ y_0 & x_0 & 1 \end{vmatrix}$$

$$= \|u\|^2 \|v\|^2 - (|u|, |v|)^2 - \left[\|u\|^2 x_0^2 - 2(|u|, |v|)x_0 y_0 + \|v\|^2 y_0^2\right],$$

$$0 \leq \|u\|^2 \|v\|^2 - (|u|, |v|)^2 - A^2$$

with

$$A = \|u\| x_0 - \|v\| y_0,$$
$$x_0 = \int_I |v(x)h(x)|dx,$$
$$y_0 = \int_I |u(x)h(x)|dx,$$
$$\|h(x)\|^2 = \int_I |h(x)|^2 dx = 1,$$

we find

(2.6) $$M_p^* \geq (|u|, |v|)^2 = \left(\int_I |u|\, |v|\right)^2 = \left(\int_I \left|w_p(x) f(x) f^{(p)}(x)\right| dx\right)^2,$$

where $w_p = (x - x_m)^p w$. In general, if $\|h\| \neq 0$, then one gets

$$(u, v)^2 \leq \|u\|^2 \|v\|^2 - R^2,$$

where

$$R = A/\|h\| = \|u\| x - \|v\| y,$$

such that $x = x_0/\|h\|,\ y = y_0/\|h\|$.

In this case, A *has to be replaced by* R in all the pertinent relations of this paper.

From (2.6) and *the complex inequality,*

$$|ab| \geq \frac{1}{2}\left(a\bar{b} + \bar{a}b\right)$$

with $a = w_p(x) f(x)$, $b = f^{(p)}(x)$, we get

(2.7) $$M_p^* = \left[\frac{1}{2}\int_I w_p(x)(f(x)\overline{f^{(p)}(x)} + f^{(p)}(x)\overline{f(x)})dx\right]^2.$$

From (2.7) and the generalized differential identity (*), one finds

(2.8) $$M_p^* \geq \frac{1}{2^2}\left[\int_I w_p(x)\left(\sum_{q=0}^{[p/2]} C_q \frac{d^{p-2q}}{dx^{p-2q}} \left|f^{(q)}(x)\right|^2\right) dx\right]^2.$$

From the generalized integral identity (1.2), the condition (2.1), and that all the integrals exist, one gets

$$\int_I w_p(x) \frac{d^{p-2q}}{dx^{p-2q}} \left|f^{(l)}(x)\right|^2 dx = (-1)^{p-2q} \int_I w_p^{(p-2q)}(x) \left|f^{(l)}(x)\right|^2 dx = I_{ql}.$$

Thus we find

$$M_p^* \geq \frac{1}{2^2}\left[\sum_{q=0}^{[p/2]} C_q \left(\sum_{l=0}^{q} I_{ql}\right)\right]^2 = \frac{1}{2^2} E_{p,I,f}^2,$$

where $E_{p,I,f} = \sum_{q=0}^{[p/2]} C_q D_q$, if $|E_{p,I,f}| < \infty$ holds, or *the refined moment uncertainty formula*

$$\sqrt[2p]{M_p} \geq \frac{1}{\sqrt[p]{2}} \sqrt[p]{\left|E_{p,I,f}^*\right|} \qquad \left(\geq \frac{1}{\sqrt[p]{2}} \sqrt[p]{|E_{p,I,f}|}\right),$$

where $M_p = M_p^* + A^2$.

We note that the corresponding Gram matrix to the above Gram determinant is positive definite if and only if the above Gram determinant is positive if and only if u, v, h are linearly independent. In addition, the equality in (2.5) holds if and only if h is a linear combination of linearly independent u and v and $u = 0$ or $v = 0$, completing the proof of the above theorem. □

3. APPLIED REFINED HEISENBERG-WEYL TYPE INEQUALITY

We apply the above Theorem 2.1 to the following simpler cases of the refined Heisenberg-Weyl type inequality.

3.1. Refined Second Moment Heisenberg-Weyl Type Inequality. For any $f \in L^2(I),\ I = [0,\infty),\ f : I \to \mathbb{C}$, such that $\|f\|_{2,I}^2 = \int_I |f(x)|^2 dx = E_{I,|f|^2}$, *any fixed but arbitrary constant* $x_m \in \mathbb{R}$*, and for the second order moment*

$$(\mu_2)_{I,|f|^2} = \sigma^2_{I,|f|^2} = \int_I (x - x_m)^2 |f(x)|^2 dx,$$

the second order moment Heisenberg-Weyl type inequality

$$(R_1^*) \qquad (\mu_2)_{I,|f|^2} \cdot \|f'\|_{2,I}^2 \geq \frac{1}{4}\left(E^*_{1,I,f}\right)^2 = \frac{1}{4}\left[\int_I |f(x)|^2 dx + 4A^2\right]^2,$$

holds, where $\left|E^*_{1,I,f}\right| < \infty$.

Equality holds in (R_1^*) *iff* $v(x) = -2c_1 u(x)$ *holds for constants* $c_1 > 0$*, and any fixed* $c_1 = k_1^2/2 > 0,\ k_1 \in \mathbb{R}$ *and* $k_1 \neq 0$*, and* $A = 0$*, or* $h(x) = c_{11}u(x) + c_{21}v(x)$ *and* $x_0 = 0$, or $y_0 = 0$*, where* c_{i1} $(i = 1, 2)$ *are constants and* $A^2 > 0$.

We note that this inequality (R_1^*) *still holds* if we replace the interval of integration I with $\mathbb{R}$, *without* any other change except that one on the choice of h, where (H_I) has to be replaced with ($H_{\mathbb{R}}$).

3.2. Refined Fourth Moment Heisenberg-Weyl Type Inequality. *For any* $f \in L^2(I)$*,* $I = [0,\infty)$*,* $f : I \to \mathbb{C}$*, such that* $\|f\|_{2,I}^2 = \int_I |f(x)|^2 dx = E_{I,|f|^2}$*, any fixed but arbitrary constant* $x_m \in \mathbb{R}$*, and for the fourth order moment*

$$(\mu_4)_{I,|f|^2} = \int_I (x - x_m)^4 |f(x)|^2 dx,$$

the fourth order moment Heisenberg-Weyl type inequality

$$(R_2^*) \qquad (\mu_4)_{I,|f|^2} \cdot \|f''\|_{2,I}^2 \geq \frac{1}{4}(E^*_{2,I,f})^2 = \frac{1}{4}\left[\int_I \left[|f(x)|^2 dx - x_\delta^2 |f'(x)|^2\right] dx + 4A^2\right]^2$$

holds, where $x_\delta = x - x_m$*, and* $\left|E^*_{2,I,f}\right| < \infty$.

Equality holds in (R_2^*) *iff* $v(x) = -2c_2 u(x)$ *holds for constants* $c_2 > 0$*, and any fixed but arbitrary* $c_2 = \frac{1}{2}k_2^2 > 0,\ k_2 \in \mathbb{R}$ *and* $k_2 \neq 0$*, and* $A = 0$*, or* $h(x) = c_{12}u(x) + c_{22}v(x)$ *and* $x_0 = 0$, or $y_0 = 0$*, where* c_{i2} $(i = 1, 2)$ *are constants and* $A^2 > 0$.

We note that this inequality (R_2^*) *still holds* if we replace the interval of integration I with $\mathbb{R}$, *without* any other change except that one on the above condition (2.1), where $x \to \infty$ has to be substituted with $|x| \to \infty$, and the choice of h, where (H_I) has to be replaced with ($H_{\mathbb{R}}$).

Remark 3.1. Take $w_p(x) = x^p$, and $w_p^{(p)}(x) = p!$ $(p = 1, 2, 3, 4, \ldots)$. Thus

$$E_{1,I,f} = -\int_I |f(x)|^2 \, dx = -E_{I,|f|^2},$$
$$E_{2,I,f} = 2\int_I \left[|f(x)|^2 - x^2 |f'(x)|^2\right] dx,$$
$$E_{3,I,f} = -3\int_I \left[2|f(x)|^2 - 3x^2 |f'(x)|^2\right] dx,$$
$$E_{4,I,f} = 2\int_I \left[12|f(x)|^2 - 24x^2 |f'(x)|^2 + x^4 |f''(x)|^2\right] dx,$$

respectively, if $|E_{p,I,f}| < \infty$ holds for $p = 1, 2, 3, 4$. Therefore

$$D_q = A_{qq} I_{qq} = I_{qq} = (-1)^{p-2q} \int_I w_p^{(p-2q)}(x) \left|f^{(q)}(x)\right|^2 dx,$$

if $|D_q| < \infty$, for $0 \le q \le \left[\frac{p}{2}\right]$.

Furthermore,

$$w_p^{(p-2q)}(x) = (x^p)^{(p-2q)} = p(p-1)\cdots(p-(p-2q)+1)\, x^{p-(p-2q)},$$

or

$$w_p^{(p-2q)}(x) = \frac{p!}{(p-(p-2q))!} x^{2q} = \frac{p!}{(2q)!} x^{2q}, \quad 0 \le q \le \left[\frac{p}{2}\right].$$

In addition

$$D_q = (-1)^{p-2q} \frac{p!}{(2q)!} \int_I x^{2q} \left|f^{(q)}(x)\right|^2 dx,$$

if $|D_q| < \infty$ holds for $0 \le q \le \left[\frac{p}{2}\right]$.

Therefore

$$E_{p,I,f} = \sum_{q=0}^{[p/2]} C_q D_q = \sum_{q=0}^{[p/2]} \left[(-1)^q \frac{p}{p-q} \binom{p-q}{q}\right] \left[(-1)^{p-2q} \frac{p!}{(2q)!} \int_I x^{2q} \left|f^{(q)}(x)\right|^2 dx\right],$$

or the formula

$$E_{p,I,f} = \int_I \sum_{q=0}^{[p/2]} (-1)^{p-q} \frac{p}{p-q} \cdot \frac{p!}{(2q)!} \binom{p-q}{q} x^{2q} \left|f^{(q)}(x)\right|^2 dx,$$

if $|E_{p,I,f}| < \infty$ holds for $0 \le q \le \left[\frac{p}{2}\right]$, when $w = 1$ and $x_m = 0$.

Let

$$(m_{2p})_{I,|f|^2} = \int_I x^{2p} |f(x)|^2 \, dx$$

be the $2p^{th}$ moment of x for $|f|^2$ about the origin $x_m = 0$.

Denote

$$\varepsilon_{p,q} = (-1)^{p-q} \frac{p}{p-q} \cdot \frac{p!}{(2q)!} \binom{p-q}{q},$$

for $p \in \mathbb{N}$ and $0 \le q \le \left[\frac{p}{2}\right]$.

Thus

$$E_{p,I,f} = \int_I \sum_{q=0}^{[p/2]} \varepsilon_{p,q} x^{2q} \left|f^{(q)}(x)\right|^2 dx, \quad \text{if} \quad |E_{p,I,f}| < \infty$$

holds for $0 \le q \le \left[\frac{p}{2}\right]$.

Corollary 3.2. *Assume that $f : I \to \mathbb{C}$ is a complex valued function of a real variable x, $w = 1$, $x_m = 0$. If $f \in L^2(I)$, then the following inequality*

$$(S_p) \qquad \sqrt[2p]{(m_{2p})_{I,|f|^2}}\sqrt[p]{\|f^{(p)}\|_{2,I}} \geq \frac{1}{\sqrt[p]{2}}\sqrt[p]{\left|\sum_{q=0}^{[p/2]}\varepsilon_{p,q}\,(m_{2q})_{I,\left|f^{(q)}\right|^2}\right|^2 + 4A^2},$$

holds for any fixed but arbitrary $p \in \mathbb{N}$ and $0 \leq q \leq \left[\frac{p}{2}\right]$, where

$$(m_{2q})_{I,\left|f^{(q)}\right|^2} = \int_I x^{2q}\left|f^{(q)}(x)\right|^2 dx$$

and A is analogous to the one in the above theorem.

Similar conditions are assumed for the *"equality"* in (S_p) with respect to those in the above theorem. We note that this inequality (S_p) *still holds* if we replace the interval of integration I with $\mathbb{R}$, *without* any other change except that one on the above condition (2.1), where $x \to \infty$ has to be substituted with $|x| \to \infty$, and the choice of h, where (H_I) has to be replaced with $(H_{\mathbb{R}})$.

Problem 1. Concerning our inequality (H_2) further investigation is needed for the case of the "equality". As a matter of fact, our function f is not in $L^2(\mathbb{R})$, leading the left-hand side to be infinite in that "equality". A limiting argument is required for this problem. On the other hand, why does not the corresponding "inequality" (H_2) attain an extremal in $L^2(\mathbb{R})$?

Here are some of our old results [4] related to the above problem. In particular, if we take into account these results contained in Section 9 on pp. 46-70 [4], where the Gaussian function and the Euler gamma function Γ are employed, then via Corollary 9.1 on pp 50-51 of [4] we conclude that "equality" in (H_p) of [4, p. 22], $p \in \mathbb{N} = \{1, 2, 3, \ldots\}$, holds only for $p = 1$. Furthermore, employing the above Gaussian function, we established the following *extremum principle* (via (9.33) on p. 51 [4]):

$$\text{(R)} \qquad R(p) \geq 1/2\pi, \quad p \in \mathbb{N}$$

for the corresponding "inequality" in (H_p) of [4, p. 22], $p \in \mathbb{N}$, where the constant $1/2\pi$ "on the right-hand side" is the best lower bound for $p \in \mathbb{N}$. Therefore "equality" in (H_p) of [4, p. 22], $p \in \mathbb{N}$ and $p \neq 1$, in Section 8.1 on pp 19-46 [4] cannot occur under the afore-mentioned well-known functions. On the other hand, there is a lower bound "on the right-hand side" of the corresponding "inequality" (H_2) if we employ the above Gaussian function, which bound equals to $\frac{1}{64\pi^4}E^2_{2,\mathbb{R},f} = \frac{1}{512\pi^3}\frac{|c_0|^4}{c}$, with c_0, c constants and $c_0 \in \mathbb{C}$, $c > 0$, because $E_{\mathbb{R},|f|^2} = |c_0|^2\sqrt{\frac{\pi}{2c}}$ and $E_{2,\mathbb{R},f} = \frac{1}{2}E_{\mathbb{R},|f|^2}$.

Analogous pertinent results are investigated via our Corollaries 9.2-9.6 on pp 53-68 [4].

REFERENCES

[1] B. BURKE HUBBARD, *The World According to Wavelets, the Story of a Mathematical Technique in the Making* (A.K. Peters, Natick, Massachusetts, 1998).

[2] W. HEISENBERG, Über den anschaulichen Inhalt der quantentheoretischen Kinematic und Mechanik, *Zeit. Physik*, 43, **172** (1927); *The Physical Principles of the Quantum Theory* (Dover, New York, 1949; The Univ. Chicago Press, 1930).

[3] G. MINGZHE, On the Heisenberg's inequality, *J. Math. Anal. Appl.*, **234** (1999), 727–734.

[4] J.M. RASSIAS, On the Heisenberg-Pauli-Weyl inequality, *J. Inequ. Pure & Appl. Math.*, **5** (2004), Art. 4. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=356`]

[5] J.M. RASSIAS, On the Heisenberg-Weyl inequality, *J. Inequ. Pure & Appl. Math*., **6** (2005), Art. 11. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=480`]

[6] H. WEYL, *Gruppentheorie und Quantenmechanik*, (S. Hirzel, Leipzig, 1928; and Dover edition, New York, 1950)

[7] N. WIENER, *The Fourier Integral and Certain of its Applications*, (Cambridge, 1933).

[8] N. WIENER, *I am a Mathematician*, (MIT Press, Cambridge, 1956).