

SUPERSTABILITY FOR GENERALIZED MODULE LEFT DERIVATIONS AND GENERALIZED MODULE DERIVATIONS ON A BANACH MODULE (II)

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Abstract: In this paper, we introduce and discuss the superstability of generalized module left derivations and generalized module derivations on a Banach module.

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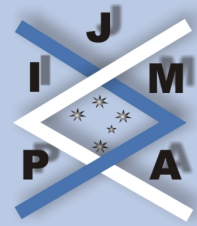
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1. Introduction

The study of stability problems was formulated by Ulam in [28] during a talk in 1940: “Under what conditions does there exist a homomorphism near an approximate homomorphism?” In the following year 1941, Hyers in [12] answered the question of Ulam for Banach spaces, which states that if $\varepsilon > 0$ and $f : X \rightarrow Y$ is a map with a normed space X and a Banach space Y such that

$$(1.1) \quad \|f(x + y) - f(x) - f(y)\| \leq \varepsilon,$$

for all x, y in X , then there exists a unique additive mapping $T : X \rightarrow Y$ such that

$$(1.2) \quad \|f(x) - T(x)\| \leq \varepsilon,$$

for all x in X . In addition, if the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed x in X , then the mapping T is real linear. This stability phenomenon is called the *Hyers-Ulam stability* of the additive functional equation $f(x + y) = f(x) + f(y)$. A generalized version of the theorem of Hyers for approximately additive mappings was given by Aoki in [1] and for approximate linear mappings was presented by Th. M. Rassias in [26] by considering the case when the left hand side of the inequality (1.1) is controlled by a sum of powers of norms [25]. The stability of approximate ring homomorphisms and additive mappings were discussed in [6, 7, 8, 10, 11, 13, 14, 21].

The stability result concerning derivations between operator algebras was first obtained by P. Semrl in [27]. Badora [5] and Moslehian [17, 18] discussed the Hyers-Ulam stability and the superstability of derivations. C. Baak and M. S. Moslehian [4] discussed the stability of J^* -homomorphisms. Miura et al. proved the Hyers-Ulam-Rassias stability and Bourgin-type superstability of derivations on Banach algebras in [16]. Various stability results on derivations and left derivations can be found in

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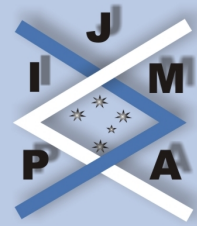
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[3, 19, 20, 2, 9]. More results on stability and superstability of homomorphisms, special functionals and equations can be found in J. M. Rassias' papers [22, 23, 24].

Recently, S.-Y. Kang and I.-S. Chang in [15] discussed the superstability of generalized left derivations and generalized derivations. In the present paper, we will discuss the superstability of generalized module left derivations and generalized module derivations on a Banach module.

To give our results, let us give some notations. Let \mathcal{A} be an algebra over the real or complex field \mathbb{F} and X be an \mathcal{A} -bimodule.

Definition 1.1. A mapping $d : \mathcal{A} \rightarrow \mathcal{A}$ is said to be module- X additive if

$$(1.3) \quad xd(a + b) = xd(a) + xd(b) \quad (a, b \in \mathcal{A}, x \in X).$$

A module- X additive mapping $d : \mathcal{A} \rightarrow \mathcal{A}$ is said to be a module- X left derivation (resp., module- X derivation) if the functional equation

$$(1.4) \quad xd(ab) = axd(b) + bxd(a) \quad (a, b \in \mathcal{A}, x \in X)$$

(resp.,

$$(1.5) \quad xd(ab) = axd(b) + d(a)xb \quad (a, b \in \mathcal{A}, x \in X))$$

holds.

Definition 1.2. A mapping $f : X \rightarrow X$ is said to be module- \mathcal{A} additive if

$$(1.6) \quad af(x_1 + x_2) = af(x_1) + af(x_2) \quad (x_1, x_2 \in X, a \in \mathcal{A}).$$

A module- \mathcal{A} additive mapping $f : X \rightarrow X$ is called a generalized module- \mathcal{A} left derivation (resp., generalized module- \mathcal{A} derivation) if there exists a module- X left derivation (resp., module- X derivation) $\delta : \mathcal{A} \rightarrow \mathcal{A}$ such that

$$(1.7) \quad af(bx) = abf(x) + ax\delta(b) \quad (x \in X, a, b \in \mathcal{A})$$



(resp.,

$$(1.8) \quad af(bx) = abf(x) + a\delta(b)x \quad (x \in X, a, b \in \mathcal{A}).$$

In addition, if the mappings f and δ are all linear, then the mapping f is called a linear generalized module- \mathcal{A} left derivation (resp., linear generalized module- \mathcal{A} derivation).

Remark 1. Let $\mathcal{A} = X$ and \mathcal{A} be one of the following cases:

- (a) a unital algebra;
- (b) a Banach algebra with an approximate unit.

Then module- \mathcal{A} left derivations, module- \mathcal{A} derivations, generalized module- \mathcal{A} left derivations and generalized module- \mathcal{A} derivations on \mathcal{A} become left derivations, derivations, generalized left derivations and generalized derivations on \mathcal{A} as discussed in [15].

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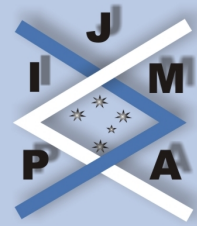
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2. Main Results

Theorem 2.1. Let \mathcal{A} be a Banach algebra, X a Banach \mathcal{A} -bimodule, k and l be integers greater than 1, and $\varphi : X \times X \times \mathcal{A} \times X \rightarrow [0, \infty)$ satisfy the following conditions:

$$(a) \lim_{n \rightarrow \infty} k^{-n} [\varphi(k^n x, k^n y, 0, 0) + \varphi(0, 0, k^n z, w)] = 0 \quad (x, y, w \in X, z \in \mathcal{A}).$$

$$(b) \lim_{n \rightarrow \infty} k^{-2n} \varphi(0, 0, k^n z, k^n w) = 0 \quad (z \in \mathcal{A}, w \in X).$$

$$(c) \tilde{\varphi}(x) := \sum_{n=0}^{\infty} k^{-n+1} \varphi(k^n x, 0, 0, 0) < \infty \quad (x \in X).$$

Suppose that $f : X \rightarrow X$ and $g : \mathcal{A} \rightarrow \mathcal{A}$ are mappings such that $f(0) = 0$, $\delta(z) := \lim_{n \rightarrow \infty} \frac{1}{k^n} g(k^n z)$ exists for all $z \in \mathcal{A}$ and

$$(2.1) \quad \left\| \Delta_{f,g}^1(x, y, z, w) \right\| \leq \varphi(x, y, z, w)$$

for all $x, y, w \in X$ and $z \in \mathcal{A}$ where

$$\Delta_{f,g}^1(x, y, z, w) = f\left(\frac{x}{k} + \frac{y}{l} + zw\right) + f\left(\frac{x}{k} - \frac{y}{l} + zw\right) - \frac{2f(x)}{k} - 2zf(w) - 2wg(z).$$

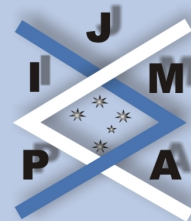
Then f is a generalized module- \mathcal{A} left derivation and g is a module- X left derivation.

Proof. By taking $w = z = 0$, we see from (2.1) that

$$(2.2) \quad \left\| f\left(\frac{x}{k} + \frac{y}{l}\right) + f\left(\frac{x}{k} - \frac{y}{l}\right) - \frac{2f(x)}{k} \right\| \leq \varphi(x, y, 0, 0)$$

for all $x, y \in X$. Letting $y = 0$ and replacing x by kx in (2.2), we get

$$(2.3) \quad \left\| f(x) - \frac{f(kx)}{k} \right\| \leq \frac{1}{2} \varphi(kx, 0, 0, 0)$$



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for all $x \in X$. Hence, for all $x \in X$, we have from (2.3) that

$$\begin{aligned} \left\| f(x) - \frac{f(k^2x)}{k^2} \right\| &\leq \left\| f(x) - \frac{f(kx)}{k} \right\| + \left\| \frac{f(kx)}{k} - \frac{f(k^2x)}{k^2} \right\| \\ &\leq \frac{1}{2}\varphi(kx, 0, 0, 0) + \frac{1}{2}k^{-1}\varphi(k^2x, 0, 0, 0). \end{aligned}$$

By induction, one can check that

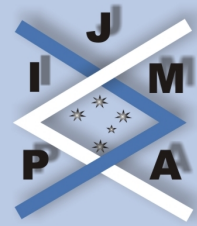
$$(2.4) \quad \left\| f(x) - \frac{f(k^n x)}{k^n} \right\| \leq \frac{1}{2} \sum_{j=1}^n k^{-j+1} \varphi(k^j x, 0, 0, 0)$$

for all x in X and $n = 1, 2, \dots$. Let $x \in X$ and $n > m$. Then by (2.4) and condition (c), we obtain that

$$\begin{aligned} \left\| \frac{f(k^n x)}{k^n} - \frac{f(k^m x)}{k^m} \right\| &= \frac{1}{k^m} \left\| \frac{f(k^{n-m} \cdot k^m x)}{k^{n-m}} - f(k^m x) \right\| \\ &\leq \frac{1}{k^m} \cdot \frac{1}{2} \sum_{j=1}^{n-m} k^{-j+1} \varphi(k^j \cdot k^m x, 0, 0, 0) \\ &\leq \frac{1}{2} \sum_{s=m}^{\infty} k^{-s+1} \varphi(k^s x, 0, 0, 0) \\ &\rightarrow 0 \quad (m \rightarrow \infty). \end{aligned}$$

This shows that the sequence $\left\{ \frac{f(k^n x)}{k^n} \right\}$ is a Cauchy sequence in the Banach \mathcal{A} -module X and therefore converges for all $x \in X$. Put $d(x) = \lim_{n \rightarrow \infty} \frac{f(k^n x)}{k^n}$ for every $x \in X$ and $f(0) = d(0) = 0$. By (2.4), we get

$$(2.5) \quad \|f(x) - d(x)\| \leq \frac{1}{2} \tilde{\varphi}(x) \quad (x \in X).$$



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Next, we show that the mapping d is additive. To do this, let us replace x, y by $k^n x, k^n y$ in (2.2), respectively. Then

$$\left\| \frac{1}{k^n} f \left(\frac{k^n x}{k} + \frac{k^n y}{l} \right) + \frac{1}{k^n} f \left(\frac{k^n x}{k} - \frac{k^n y}{l} \right) - \frac{1}{k} \cdot \frac{2f(k^n x)}{k^n} \right\| \leq k^{-n} \varphi(k^n x, k^n y, 0, 0)$$

for all $x, y \in X$. If we let $n \rightarrow \infty$ in the above inequality, then the condition (a) yields that

$$(2.6) \quad d \left(\frac{x}{k} + \frac{y}{l} \right) + d \left(\frac{x}{k} - \frac{y}{l} \right) = \frac{2}{k} d(x)$$

for all $x, y \in X$. Since $d(0) = 0$, taking $y = 0$ and $y = \frac{l}{k}x$, respectively, we see that $d \left(\frac{x}{k} \right) = \frac{d(x)}{k}$ and $d(2x) = 2d(x)$ for all $x \in X$, and then we obtain that $d(x+y) + d(x-y) = 2d(x)$ for all $x, y \in X$. Now, for all $u, v \in X$, put $x = \frac{k}{2}(u+v)$, $y = \frac{l}{2}(u-v)$. Then by (2.6), we get that

$$\begin{aligned} d(u) + d(v) &= d \left(\frac{x}{k} + \frac{y}{l} \right) + d \left(\frac{x}{k} - \frac{y}{l} \right) \\ &= \frac{2}{k} d(x) = \frac{2}{k} d \left(\frac{k}{2}(u+v) \right) = d(u+v). \end{aligned}$$

This shows that d is additive.

Now, we are going to prove that f is a generalized module- \mathcal{A} left derivation. Letting $x = y = 0$ in (2.1), we get

$$\|f(zw) + f(zw) - 2zf(w) - 2wg(z)\| \leq \varphi(0, 0, z, w),$$

that is

$$(2.7) \quad \|f(zw) - zf(w) - wg(z)\| \leq \frac{1}{2} \varphi(0, 0, z, w)$$



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for all $z \in \mathcal{A}$ and $w \in X$. By replacing z, w with $k^n z, k^n w$ in (2.7) respectively, we deduce that

$$(2.8) \quad \left\| \frac{1}{k^{2n}} f(k^{2n} zw) - z \frac{1}{k^n} f(k^n w) - w \frac{1}{k^n} g(k^n z) \right\| \leq \frac{1}{2} k^{-2n} \varphi(0, 0, k^n z, k^n w)$$

for all $z \in \mathcal{A}$ and $w \in X$. Letting $n \rightarrow \infty$, condition (b) yields that

$$(2.9) \quad d(zw) = zd(w) + w\delta(z)$$

for all $z \in \mathcal{A}$ and $w \in X$. Since d is additive, δ is module- X additive. Put $\Delta(z, w) = f(zw) - zf(w) - wg(z)$. Then by (2.7) we see from condition (a) that

$$k^{-n} \|\Delta(k^n z, w)\| \leq \frac{1}{2} k^{-n} \varphi(0, 0, k^n z, w) \rightarrow 0 \quad (n \rightarrow \infty)$$

for all $z \in \mathcal{A}$ and $w \in X$. Hence

$$\begin{aligned} d(zw) &= \lim_{n \rightarrow \infty} \frac{f(k^n z \cdot w)}{k^n} \\ &= \lim_{n \rightarrow \infty} \left(\frac{k^n z f(w) + wg(k^n z) + \Delta(k^n z, w)}{k^n} \right) \\ &= z f(w) + w\delta(z) \end{aligned}$$

for all $z \in \mathcal{A}$ and $w \in X$. It follows from (2.9) that $zf(w) = zd(w)$ for all $z \in \mathcal{A}$ and $w \in X$, and then $d(w) = f(w)$ for all $w \in X$. Since d is additive, f is module- \mathcal{A} additive. So, for all $a, b \in \mathcal{A}$ and $x \in X$ by (2.9),

$$af(bx) = ad(bx) = abf(x) + ax\delta(b)$$

and

$$\begin{aligned}
 x\delta(ab) &= d(abx) - abf(x) \\
 &= af(bx) + bx\delta(a) - abf(x) \\
 &= a(d(bx) - bf(x)) + bx\delta(a) \\
 &= ax\delta(b) + bx\delta(a).
 \end{aligned}$$

This shows that if δ is a module- X left derivation on \mathcal{A} , then f is a generalized module- \mathcal{A} left derivation on X .

Lastly, we prove that g is a module- X left derivation on \mathcal{A} . To do this, we compute from (2.7) that

$$\left\| \frac{f(k^n zw)}{k^n} - z \frac{f(k^n w)}{k^n} - wg(z) \right\| \leq \frac{1}{2} k^{-n} \varphi(0, 0, z, k^n w)$$

for all $z \in \mathcal{A}$ and all $w \in X$. By letting $n \rightarrow \infty$, we get from condition (a) that

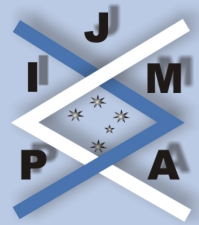
$$d(zw) = zd(w) + wg(z)$$

for all $z \in \mathcal{A}$ and all $w \in X$. Now, (2.9) implies that $wg(z) = w\delta(z)$ for all $z \in \mathcal{A}$ and all $w \in X$. Hence, g is a module- X left derivation on \mathcal{A} . This completes the proof. \square

Corollary 2.2. *Let \mathcal{A} be a Banach algebra, X a Banach \mathcal{A} -bimodule, $\varepsilon \geq 0$, $p, q, s, t \in [0, 1)$ and k and l be integers greater than 1. Suppose that $f : X \rightarrow X$ and $g : \mathcal{A} \rightarrow \mathcal{A}$ are mappings such that $f(0) = 0$, $\delta(z) := \lim_{n \rightarrow \infty} \frac{1}{k^n} g(k^n z)$ exists for all $z \in \mathcal{A}$ and*

$$(2.10) \quad \left\| \Delta_{f,g}^1(x, y, z, w) \right\| \leq \varepsilon (\|x\|^p + \|y\|^q + \|z\|^s \|w\|^t)$$

for all $x, y, w \in X$ and all $z \in \mathcal{A}$ ($0^0 := 1$). Then f is a generalized module- \mathcal{A} left derivation and g is a module- X left derivation.



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Proof. It is easy to check that the function

$$\varphi(x, y, z, w) = \varepsilon(\|x\|^p + \|y\|^q + \|z\|^s \|w\|^t)$$

satisfies conditions (a), (b) and (c) of Theorem 2.1. \square

Corollary 2.3. *Let \mathcal{A} be a Banach algebra with unit e , $\varepsilon \geq 0$, and k and l be integers greater than 1. Suppose that $f, g : \mathcal{A} \rightarrow \mathcal{A}$ are mappings with $f(0) = 0$ such that*

$$\|\Delta_{f,g}^1(x, y, z, w)\| \leq \varepsilon$$

for all $x, y, w, z \in \mathcal{A}$. Then f is a generalized left derivation and g is a left derivation.

Proof. By taking $w = e$ in (2.8), we see that the limit $\delta(z) := \lim_{n \rightarrow \infty} \frac{1}{k^n} g(k^n z)$ exists for all $z \in \mathcal{A}$. It follows from Corollary 2.2 and Remark 1 that f is a generalized left derivation and g is a left derivation. This completes the proof. \square

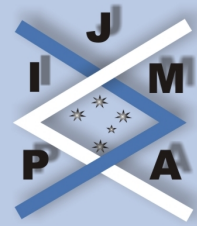
Lemma 2.4. *Let X, Y be complex vector spaces. Then a mapping $f : X \rightarrow Y$ is linear if and only if*

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

for all $x, y \in X$ and all $\alpha, \beta \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.

Proof. It suffices to prove the sufficiency. Suppose that $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ for all $x, y \in X$ and all $\alpha, \beta \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Then f is additive and $f(\alpha x) = \alpha f(x)$ for all $x \in X$ and all $\alpha \in \mathbb{T}$. Let α be any nonzero complex number. Take a positive integer n such that $|\alpha/n| < 2$. Take a real number θ such that $0 \leq a := e^{-i\theta} \alpha/n < 2$. Put $\beta = \arccos \frac{a}{2}$. Then $\alpha = n(e^{i(\beta+\theta)} + e^{-i(\beta-\theta)})$ and therefore

$$\begin{aligned} f(\alpha x) &= n f(e^{i(\beta+\theta)} x) + n f(e^{-i(\beta-\theta)} x) \\ &= n e^{i(\beta+\theta)} f(x) + n e^{-i(\beta-\theta)} f(x) = \alpha f(x) \end{aligned}$$



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for all $x \in X$. This shows that f is linear. The proof is completed. \square

Theorem 2.5. Let \mathcal{A} be a Banach algebra, X a Banach \mathcal{A} -bimodule, k and l be integers greater than 1, and $\varphi : X \times X \times \mathcal{A} \times X \rightarrow [0, \infty)$ satisfy the following conditions:

$$(a) \lim_{n \rightarrow \infty} k^{-n} [\varphi(k^n x, k^n y, 0, 0) + \varphi(0, 0, k^n z, w)] = 0 \quad (x, y, w \in X, z \in \mathcal{A}).$$

$$(b) \lim_{n \rightarrow \infty} k^{-2n} \varphi(0, 0, k^n z, k^n w) = 0 \quad (z \in \mathcal{A}, w \in X).$$

$$(c) \tilde{\varphi}(x) := \sum_{n=0}^{\infty} k^{-n+1} \varphi(k^n x, 0, 0, 0) < \infty \quad (x \in X).$$

Suppose that $f : X \rightarrow X$ and $g : \mathcal{A} \rightarrow \mathcal{A}$ are mappings such that $f(0) = 0$, $\delta(z) := \lim_{n \rightarrow \infty} \frac{1}{k^n} g(k^n z)$ exists for all $z \in \mathcal{A}$ and

$$(2.11) \quad \|\Delta_{f,g}^3(x, y, z, w, \alpha, \beta)\| \leq \varphi(x, y, z, w)$$

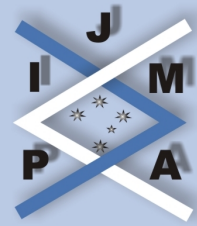
for all $x, y, w \in X$, $z \in \mathcal{A}$ and all $\alpha, \beta \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, where $\Delta_{f,g}^3(x, y, z, w, \alpha, \beta)$ stands for

$$f\left(\frac{\alpha x}{k} + \frac{\beta y}{l} + zw\right) + f\left(\frac{\alpha x}{k} - \frac{\beta y}{l} + zw\right) - \frac{2\alpha f(x)}{k} - 2zf(w) - 2wg(z).$$

Then f is a linear generalized module- \mathcal{A} left derivation and g is a linear module- X left derivation.

Proof. Clearly, the inequality (2.1) is satisfied. Hence, Theorem 2.1 and its proof show that f is a generalized left derivation and g is a left derivation on \mathcal{A} with

$$(2.12) \quad f(x) = \lim_{n \rightarrow \infty} \frac{f(k^n x)}{k^n}, \quad g(x) = f(x) - xf(e)$$



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for every $x \in X$. Taking $z = w = 0$ in (2.11) yields that

$$(2.13) \quad \left\| f\left(\frac{\alpha x}{k} + \frac{\beta y}{l}\right) + f\left(\frac{\alpha x}{k} - \frac{\beta y}{l}\right) - \frac{2\alpha f(x)}{k} \right\| \leq \varphi(x, y, 0, 0)$$

for all $x, y \in X$ and all $\alpha, \beta \in \mathbb{T}$. If we replace x and y with $k^n x$ and $k^n y$ in (2.13) respectively, then we see that

$$\begin{aligned} & \left\| \frac{1}{k^n} f\left(\frac{\alpha k^n x}{k} + \frac{\beta k^n y}{l}\right) + \frac{1}{k^n} f\left(\frac{\alpha k^n x}{k} - \frac{\beta k^n y}{l}\right) - \frac{1}{k^n} \frac{2\alpha f(k^n x)}{k} \right\| \\ & \leq k^{-n} \varphi(k^n x, k^n y, 0, 0) \\ & \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$ for all $x, y \in X$ and all $\alpha, \beta \in \mathbb{T}$. Hence,

$$(2.14) \quad f\left(\frac{\alpha x}{k} + \frac{\beta y}{l}\right) + f\left(\frac{\alpha x}{k} - \frac{\beta y}{l}\right) = \frac{2\alpha f(x)}{k}$$

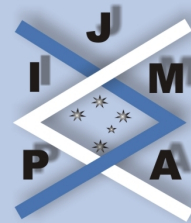
for all $x, y \in X$ and all $\alpha, \beta \in \mathbb{T}$. Since f is additive, taking $y = 0$ in (2.14) implies that

$$(2.15) \quad f(\alpha x) = \alpha f(x)$$

for all $x \in X$ and all $\alpha \in \mathbb{T}$. Lemma 2.4 yields that f is linear and so is g . Next, similar to the proof of Theorem 2.3 in [15], one can show that $g(\mathcal{A}) \subset \mathbf{Z}(\mathcal{A}) \cap \text{rad}(\mathcal{A})$. This completes the proof. \square

Corollary 2.6. *Let \mathcal{A} be a complex semi-prime Banach algebra with unit e , $\varepsilon \geq 0$, $p, q, s, t \in [0, 1)$ and k and l be integers greater than 1. Suppose that $f, g : \mathcal{A} \rightarrow \mathcal{A}$ are mappings with $f(0) = 0$ and satisfy following inequality:*

$$(2.16) \quad \left\| \Delta_{f,g}^3(x, y, z, w, \alpha, \beta) \right\| \leq \varepsilon (\|x\|^p + \|y\|^q + \|z\|^s \|w\|^t)$$



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for all $x, y, z, w \in \mathcal{A}$ and all $\alpha, \beta \in \mathbb{T}$ ($0^0 := 1$). Then f is a linear generalized left derivation and g is a linear left derivation which maps \mathcal{A} into the intersection of the center $Z(\mathcal{A})$ and the Jacobson radical $\text{rad}(\mathcal{A})$ of \mathcal{A} .

Proof. Since \mathcal{A} has a unit e , letting $w = e$ in (2.8) shows that the limit $\delta(z) := \lim_{n \rightarrow \infty} \frac{1}{k^n} g(k^n z)$ exists for all $z \in \mathcal{A}$. Thus, using Theorem 2.5 for $\varphi(x, y, z, w) = \varepsilon(\|x\|^p + \|y\|^q + \|z\|^s \|w\|^t)$ yields that f is a linear generalized left derivation and g is a linear left derivation since \mathcal{A} has a unit. Similar to the proof of Theorem 2.3 in [15], one can check that the mapping g maps \mathcal{A} into the intersection of the center $Z(\mathcal{A})$ and the Jacobson radical $\text{rad}(\mathcal{A})$ of \mathcal{A} . This completes the proof. \square

Corollary 2.7. *Let \mathcal{A} be a complex semiprime Banach algebra with unit e , $\varepsilon \geq 0$, k and l be integers greater than 1. Suppose that $f, g : \mathcal{A} \rightarrow \mathcal{A}$ are mappings with $f(0) = 0$ and satisfy the following inequality:*

$$\|\Delta_{f,g}^3(x, y, z, w, \alpha, \beta)\| \leq \varepsilon$$

for all $x, y, z, w \in \mathcal{A}$ and all $\alpha, \beta \in \mathbb{T}$. Then f is a linear generalized left derivation and g is a linear left derivation which maps \mathcal{A} into the intersection of the center $Z(\mathcal{A})$ and the Jacobson radical $\text{rad}(\mathcal{A})$ of \mathcal{A} .

Remark 2. Inequalities (2.10) and (2.16) are controlled by their right-hand sides by the “mixed sum-product of powers of norms”, introduced by J. M. Rassias (in 2007) and applied afterwards by K. Ravi et al. (2007-2008). Moreover, it is easy to check that the function

$$\varphi(x, y, z, w) = P\|x\|^p + Q\|y\|^q + S\|z\|^s + T\|w\|^t$$

satisfies conditions (a), (b) and (c) of Theorem 2.1 and Theorem 2.5, where $P, Q, T, S \in [0, \infty)$ and $p, q, s, t \in [0, 1)$ are all constants.

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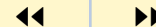
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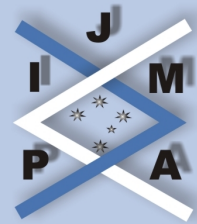
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