SUPERSTABILITY FOR GENERALIZED MODULE LEFT DERIVATIONS AND GENERALIZED MODULE DERIVATIONS ON A BANACH MODULE (II)

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Received: 12 January, 2009

Accepted: 12 May, 2009

Communicated by: S.S. Dragomir

2000 AMS Sub. Class.: Primary 39B52; Secondary 39B82.

Key words: Superstability, Generalized module left derivation, Generalized module derivation, Module left derivation, Module derivation, Banach module.

Abstract: In this paper, we introduce and discuss the superstability of generalized module left derivations and generalized module derivations on a Banach module.

Acknowledgements: This subject is supported by the NNSFs of China (No. 10571113, 10871224).

journal of inequalities in pure and applied mathematics
issn: 1443-5756

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journal of inequalities
in pure and applied
mathematics
issn: 1443-5756

1. Introduction

The study of stability problems was formulated by Ulam in [28] during a talk in 1940: "Under what conditions does there exist a homomorphism near an approximate homomorphism?" In the following year 1941, Hyers in [12] answered the question of Ulam for Banach spaces, which states that if $\varepsilon > 0$ and $f : X \to Y$ is a map with a normed space X and a Banach space Y such that

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon, \tag{1.1}$$

for all x, y in X, then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \varepsilon, \tag{1.2}$$

for all x in X. In addition, if the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed x in X, then the mapping T is real linear. This stability phenomenon is called the *Hyers-Ulam stability* of the additive functional equation $f(x + y) = f(x) + f(y)$. A generalized version of the theorem of Hyers for approximately additive mappings was given by Aoki in [1] and for approximate linear mappings was presented by Th. M. Rassias in [26] by considering the case when the left hand side of the inequality (1.1) is controlled by a sum of powers of norms [25]. The stability of approximate ring homomorphisms and additive mappings were discussed in [6, 7, 8, 10, 11, 13, 14, 21].

The stability result concerning derivations between operator algebras was first obtained by P. Semrl in [27]. Badora [5] and Moslehian [17, 18] discussed the Hyers-Ulam stability and the superstability of derivations. C. Baak and M. S. Moslehian [4] discussed the stability of J^*-homomorphisms. Miura et al. proved the Hyers-Ulam-Rassias stability and Bourgin-type superstability of derivations on Banach algebras in [16]. Various stability results on derivations and left derivations can be found in

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[3, 19, 20, 2, 9]. More results on stability and superstability of homomorphisms, special functionals and equations can be found in J. M. Rassias' papers [22, 23, 24].

Recently, S.-Y. Kang and I.-S. Chang in [15] discussed the superstability of generalized left derivations and generalized derivations. In the present paper, we will discuss the superstability of generalized module left derivations and generalized module derivations on a Banach module.

To give our results, let us give some notations. Let $\mathscr{A}$ be an algebra over the real or complex field $\mathbb{F}$ and X be an $\mathscr{A}$-bimodule.

Definition 1.1. *A mapping $d : \mathscr{A} \to \mathscr{A}$ is said to be module-X additive if*

$$xd(a+b) = xd(a) + xd(b) \quad (a, b \in \mathscr{A}, x \in X). \tag{1.3}$$

A module-X additive mapping $d : \mathscr{A} \to \mathscr{A}$ is said to be a module-X left derivation (resp., module-X derivation) if the functional equation

$$xd(ab) = axd(b) + bxd(a) \quad (a, b \in \mathscr{A}, x \in X) \tag{1.4}$$

(resp.,

$$xd(ab) = axd(b) + d(a)xb \quad (a, b \in \mathscr{A}, x \in X)) \tag{1.5}$$

holds.

Definition 1.2. *A mapping $f : X \to X$ is said to be module-$\mathscr{A}$ additive if*

$$af(x_1 + x_2) = af(x_1) + af(x_2) \quad (x_1, x_2 \in X, a \in \mathscr{A}). \tag{1.6}$$

A module-$\mathscr{A}$ additive mapping $f : X \to X$ is called a generalized module-$\mathscr{A}$ left derivation (resp., generalized module-$\mathscr{A}$ derivation) if there exists a module-X left derivation (resp., module-X derivation) $\delta : \mathscr{A} \to \mathscr{A}$ such that

$$af(bx) = abf(x) + ax\delta(b) \quad (x \in X, a, b \in \mathscr{A}) \tag{1.7}$$

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(resp.,

$$af(bx) = abf(x) + a\delta(b)x \quad (x \in X, a, b \in \mathscr{A})). \tag{1.8}$$

In addition, if the mappings f and δ are all linear, then the mapping f is called a linear generalized module-$\mathscr{A}$ left derivation (resp., linear generalized module-$\mathscr{A}$ derivation).

Remark 1. Let $\mathscr{A} = X$ and $\mathscr{A}$ be one of the following cases:

(a) a unital algebra;

(b) a Banach algebra with an approximate unit.

Then module-$\mathscr{A}$ left derivations, module-$\mathscr{A}$ derivations, generalized module-$\mathscr{A}$ left derivations and generalized module-$\mathscr{A}$ derivations on $\mathscr{A}$ become left derivations, derivations, generalized left derivations and generalized derivations on $\mathscr{A}$ as discussed in [15].

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2. Main Results

Theorem 2.1. *Let $\mathscr{A}$ be a Banach algebra, X a Banach $\mathscr{A}$-bimodule, k and l be integers greater than 1, and $\varphi : X \times X \times \mathscr{A} \times X \to [0, \infty)$ satisfy the following conditions:*

(a) $\lim\limits_{n\to\infty} k^{-n}[\varphi(k^n x, k^n y, 0, 0) + \varphi(0, 0, k^n z, w)] = 0 \ (x, y, w \in X, z \in \mathscr{A})$.

(b) $\lim\limits_{n\to\infty} k^{-2n}\varphi(0, 0, k^n z, k^n w) = 0 \ (z \in \mathscr{A}, w \in X)$.

(c) $\tilde{\varphi}(x) := \sum_{n=0}^{\infty} k^{-n+1}\varphi(k^n x, 0, 0, 0) < \infty \ (x \in X)$.

Suppose that $f : X \to X$ and $g : \mathscr{A} \to \mathscr{A}$ are mappings such that $f(0) = 0$, $\delta(z) := \lim\limits_{n\to\infty} \frac{1}{k^n} g(k^n z)$ exists for all $z \in \mathscr{A}$ and

$$\left\|\Delta^1_{f,g}(x, y, z, w)\right\| \le \varphi(x, y, z, w) \tag{2.1}$$

for all $x, y, w \in X$ and $z \in \mathscr{A}$ where

$$\Delta^1_{f,g}(x, y, z, w) = f\left(\frac{x}{k} + \frac{y}{l} + zw\right) + f\left(\frac{x}{k} - \frac{y}{l} + zw\right) - \frac{2f(x)}{k} - 2zf(w) - 2wg(z).$$

Then f is a generalized module-$\mathscr{A}$ left derivation and g is a module-X left derivation.

Proof. By taking $w = z = 0$, we see from (2.1) that

$$\left\| f\left(\frac{x}{k} + \frac{y}{l}\right) + f\left(\frac{x}{k} - \frac{y}{l}\right) - \frac{2f(x)}{k} \right\| \le \varphi(x, y, 0, 0) \tag{2.2}$$

for all $x, y \in X$. Letting $y = 0$ and replacing x by kx in (2.2), we get

$$\left\| f(x) - \frac{f(kx)}{k} \right\| \le \frac{1}{2}\varphi(kx, 0, 0, 0) \tag{2.3}$$

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for all $x \in X$. Hence, for all $x \in X$, we have from (2.3) that

$$\begin{aligned}\left\|f(x)-\frac{f(k^2x)}{k^2}\right\| &\le \left\|f(x)-\frac{f(kx)}{k}\right\| + \left\|\frac{f(kx)}{k}-\frac{f(k^2x)}{k^2}\right\| \\ &\le \frac{1}{2}\varphi(kx,0,0,0)+\frac{1}{2}k^{-1}\varphi(k^2x,0,0,0).\end{aligned}$$

By induction, one can check that

$$\left\|f(x)-\frac{f(k^nx)}{k^n}\right\| \le \frac{1}{2}\sum_{j=1}^{n}k^{-j+1}\varphi(k^jx,0,0,0) \tag{2.4}$$

for all x in X and $n = 1, 2, \dots$. Let $x \in X$ and $n > m$. Then by (2.4) and condition (c), we obtain that

$$\begin{aligned}\left\|\frac{f(k^nx)}{k^n}-\frac{f(k^mx)}{k^m}\right\| &= \frac{1}{k^m}\left\|\frac{f(k^{n-m}\cdot k^mx)}{k^{n-m}}-f(k^mx)\right\| \\ &\le \frac{1}{k^m}\cdot\frac{1}{2}\sum_{j=1}^{n-m}k^{-j+1}\varphi(k^j\cdot k^mx,0,0,0) \\ &\le \frac{1}{2}\sum_{s=m}^{\infty}k^{-s+1}\varphi(k^sx,0,0,0) \\ &\to 0\ (m\to\infty).\end{aligned}$$

This shows that the sequence $\left\{\frac{f(k^nx)}{k^n}\right\}$ is a Cauchy sequence in the Banach $\mathscr{A}$-module X and therefore converges for all $x \in X$. Put $d(x) = \lim_{n\to\infty}\frac{f(k^nx)}{k^n}$ for every $x \in X$ and $f(0) = d(0) = 0$. By (2.4), we get

$$\|f(x)-d(x)\| \le \frac{1}{2}\tilde{\varphi}(x) \quad (x \in X). \tag{2.5}$$

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Next, we show that the mapping d is additive. To do this, let us replace x, y by $k^n x, k^n y$ in (2.2), respectively. Then

$$\left\| \frac{1}{k^n} f\left(\frac{k^n x}{k} + \frac{k^n y}{l}\right) + \frac{1}{k^n} f\left(\frac{k^n x}{k} - \frac{k^n y}{l}\right) - \frac{1}{k} \cdot \frac{2f(k^n x)}{k^n} \right\| \le k^{-n} \varphi(k^n x, k^n y, 0, 0)$$

for all $x, y \in X$. If we let $n \to \infty$ in the above inequality, then the condition (a) yields that

$$d\left(\frac{x}{k} + \frac{y}{l}\right) + d\left(\frac{x}{k} - \frac{y}{l}\right) = \frac{2}{k} d(x) \tag{2.6}$$

for all $x, y \in X$. Since $d(0) = 0$, taking $y = 0$ and $y = \frac{l}{k} x$, respectively, we see that $d\left(\frac{x}{k}\right) = \frac{d(x)}{k}$ and $d(2x) = 2d(x)$ for all $x \in X$, and then we obtain that $d(x+y) + d(x-y) = 2d(x)$ for all $x, y \in X$. Now, for all $u, v \in X$, put $x = \frac{k}{2}(u+v)$, $y = \frac{l}{2}(u - v)$. Then by (2.6), we get that

$$\begin{aligned} d(u) + d(v) &= d\left(\frac{x}{k} + \frac{y}{l}\right) + d\left(\frac{x}{k} - \frac{y}{l}\right) \\ &= \frac{2}{k} d(x) = \frac{2}{k} d\left(\frac{k}{2}(u+v)\right) = d(u+v). \end{aligned}$$

This shows that d is additive.

Now, we are going to prove that f is a generalized module-$\mathscr{A}$ left derivation. Letting $x = y = 0$ in (2.1), we get

$$\|f(zw) + f(zw) - 2zf(w) - 2wg(z)\| \le \varphi(0, 0, z, w),$$

that is

$$\|f(zw) - zf(w) - wg(z)\| \le \frac{1}{2} \varphi(0, 0, z, w) \tag{2.7}$$

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for all $z \in \mathscr{A}$ and $w \in X$. By replacing z, w with $k^n z, k^n w$ in (2.7) respectively, we deduce that

$$\left\| \frac{1}{k^{2n}} f\left(k^{2n} zw\right) - z\frac{1}{k^n} f(k^n w) - w\frac{1}{k^n} g(k^n z) \right\| \leq \frac{1}{2} k^{-2n} \varphi(0, 0, k^n z, k^n w) \tag{2.8}$$

for all $z \in \mathscr{A}$ and $w \in X$. Letting $n \to \infty$, condition (b) yields that

$$d(zw) = zd(w) + w\delta(z) \tag{2.9}$$

for all $z \in \mathscr{A}$ and $w \in X$. Since d is additive, δ is module-X additive. Put $\Delta(z, w) = f(zw) - zf(w) - wg(z)$. Then by (2.7) we see from condition (a) that

$$k^{-n} \|\Delta(k^n z, w)\| \leq \frac{1}{2} k^{-n} \varphi(0, 0, k^n z, w) \to 0 \quad (n \to \infty)$$

for all $z \in \mathscr{A}$ and $w \in X$. Hence

$$\begin{aligned} d(zw) &= \lim_{n\to\infty} \frac{f(k^n z \cdot w)}{k^n} \\ &= \lim_{n\to\infty} \left(\frac{k^n z f(w) + wg(k^n z) + \Delta(k^n z, w)}{k^n} \right) \\ &= zf(w) + w\delta(z) \end{aligned}$$

for all $z \in \mathscr{A}$ and $w \in X$. It follows from (2.9) that $zf(w) = zd(w)$ for all $z \in \mathscr{A}$ and $w \in X$, and then $d(w) = f(w)$ for all $w \in X$. Since d is additive, f is module-$\mathscr{A}$ additive. So, for all $a, b \in \mathscr{A}$ and $x \in X$ by (2.9),

$$af(bx) = ad(bx) = abf(x) + ax\delta(b)$$

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and

$$\begin{aligned} x\delta(ab) &= d(abx) - abf(x) \\ &= af(bx) + bx\delta(a) - abf(x) \\ &= a(d(bx) - bf(x)) + bx\delta(a) \\ &= ax\delta(b) + bx\delta(a). \end{aligned}$$

This shows that if δ is a module-X left derivation on $\mathscr{A}$, then f is a generalized module-$\mathscr{A}$ left derivation on X.

Lastly, we prove that g is a module-X left derivation on $\mathscr{A}$. To do this, we compute from (2.7) that

$$\left\| \frac{f(k^n zw)}{k^n} - z\frac{f(k^n w)}{k^n} - wg(z) \right\| \le \frac{1}{2} k^{-n} \varphi(0, 0, z, k^n w)$$

for all $z \in \mathscr{A}$ and all $w \in X$. By letting $n \to \infty$, we get from condition (a) that

$$d(zw) = zd(w) + wg(z)$$

for all $z \in \mathscr{A}$ and all $w \in X$. Now, (2.9) implies that $wg(z) = w\delta(z)$ for all $z \in \mathscr{A}$ and all $w \in X$. Hence, g is a module-X left derivation on $\mathscr{A}$. This completes the proof. $\square$

Corollary 2.2. *Let $\mathscr{A}$ be a Banach algebra, X a Banach $\mathscr{A}$-bimodule, $\varepsilon \ge 0$, $p, q, s, t \in [0, 1)$ and k and l be integers greater than 1. Suppose that $f : X \to X$ and $g : \mathscr{A} \to \mathscr{A}$ are mappings such that $f(0) = 0$, $\delta(z) := \lim_{n\to\infty} \frac{1}{k^n} g(k^n z)$ exists for all $z \in \mathscr{A}$ and*

$$\left\| \Delta^1_{f,g}(x, y, z, w) \right\| \le \varepsilon(\|x\|^p + \|y\|^q + \|z\|^s \|w\|^t) \tag{2.10}$$

for all $x, y, w \in X$ and all $z \in \mathscr{A}$ $(0^0 := 1)$. Then f is a generalized module-$\mathscr{A}$ left derivation and g is a module-X left derivation.

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Proof. It is easy to check that the function

$$\varphi(x,y,z,w)=\varepsilon(\|x\|^p+\|y\|^q+\|z\|^s\|w\|^t)$$

satisfies conditions (a), (b) and (c) of Theorem 2.1. □

Corollary 2.3. *Let $\mathscr{A}$ be a Banach algebra with unit e, $\varepsilon \geq 0$, and k and l be integers greater than 1. Suppose that $f,g:\mathscr{A}\to\mathscr{A}$ are mappings with $f(0)=0$ such that*

$$\left\|\Delta^1_{f,g}(x,y,z,w)\right\|\leq\varepsilon$$

for all $x,y,w,z\in\mathscr{A}$. Then f is a generalized left derivation and g is a left derivation.

Proof. By taking $w=e$ in (2.8), we see that the limit $\delta(z):=\lim\limits_{n\to\infty}\frac{1}{k^n}g(k^nz)$ exists for all $z\in\mathscr{A}$. It follows from Corollary 2.2 and Remark 1 that f is a generalized left derivation and g is a left derivation. This completes the proof. □

Lemma 2.4. *Let X,Y be complex vector spaces. Then a mapping $f:X\to Y$ is linear if and only if*

$$f(\alpha x+\beta y)=\alpha f(x)+\beta f(y)$$

for all $x,y\in X$ and all $\alpha,\beta\in\mathbb{T}:=\{z\in\mathbb{C}:|z|=1\}$.

Proof. It suffices to prove the sufficiency. Suppose that $f(\alpha x+\beta y)=\alpha f(x)+\beta f(y)$ for all $x,y\in X$ and all $\alpha,\beta\in\mathbb{T}:=\{z\in\mathbb{C}:|z|=1\}$. Then f is additive and $f(\alpha x)=\alpha f(x)$ for all $x\in X$ and all $\alpha\in\mathbb{T}$. Let α be any nonzero complex number. Take a positive integer n such that $|\alpha/n|<2$. Take a real number θ such that $0\leq a:=e^{-i\theta}\alpha/n<2$. Put $\beta=\arccos\frac{a}{2}$. Then $\alpha=n(e^{i(\beta+\theta)}+e^{-i(\beta-\theta)})$ and therefore

$$\begin{aligned}f(\alpha x)&=nf(e^{i(\beta+\theta)}x)+nf(e^{-i(\beta-\theta)}x)\\&=ne^{i(\beta+\theta)}f(x)+ne^{-i(\beta-\theta)}f(x)=\alpha f(x)\end{aligned}$$

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for all $x \in X$. This shows that f is linear. The proof is completed. $\square$

Theorem 2.5. *Let $\mathscr{A}$ be a Banach algebra, X a Banach $\mathscr{A}$-bimodule, k and l be integers greater than 1, and $\varphi : X \times X \times \mathscr{A} \times X \to [0, \infty)$ satisfy the following conditions:*

(a) $\lim_{n\to\infty} k^{-n}[\varphi(k^n x, k^n y, 0, 0) + \varphi(0, 0, k^n z, w)] = 0 \quad (x, y, w \in X, z \in \mathscr{A}).$

(b) $\lim_{n\to\infty} k^{-2n}\varphi(0, 0, k^n z, k^n w) = 0 \quad (z \in \mathscr{A}, w \in X).$

(c) $\tilde{\varphi}(x) := \sum_{n=0}^{\infty} k^{-n+1}\varphi(k^n x, 0, 0, 0) < \infty \quad (x \in X).$

Suppose that $f : X \to X$ and $g : \mathscr{A} \to \mathscr{A}$ are mappings such that $f(0) = 0$, $\delta(z) := \lim_{n\to\infty} \frac{1}{k^n} g(k^n z)$ exists for all $z \in \mathscr{A}$ and

$$\left\|\Delta^3_{f,g}(x, y, z, w, \alpha, \beta)\right\| \leq \varphi(x, y, z, w) \tag{2.11}$$

for all $x, y, w \in X$, $z \in \mathscr{A}$ and all $\alpha, \beta \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, where $\Delta^3_{f,g}(x, y, z, w, \alpha, \beta)$ stands for

$$f\left(\frac{\alpha x}{k} + \frac{\beta y}{l} + zw\right) + f\left(\frac{\alpha x}{k} - \frac{\beta y}{l} + zw\right) - \frac{2\alpha f(x)}{k} - 2zf(w) - 2wg(z).$$

Then f is a linear generalized module-$\mathscr{A}$ left derivation and g is a linear module-X left derivation.

Proof. Clearly, the inequality (2.1) is satisfied. Hence, Theorem 2.1 and its proof show that f is a generalized left derivation and g is a left derivation on $\mathscr{A}$ with

$$f(x) = \lim_{n\to\infty} \frac{f(k^n x)}{k^n}, \qquad g(x) = f(x) - xf(e) \tag{2.12}$$

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for every $x \in X$. Taking $z = w = 0$ in (2.11) yields that

$$\left\| f\left(\frac{\alpha x}{k} + \frac{\beta y}{l}\right) + f\left(\frac{\alpha x}{k} - \frac{\beta y}{l}\right) - \frac{2\alpha f(x)}{k}\right\| \leq \varphi(x, y, 0, 0) \tag{2.13}$$

for all $x, y \in X$ and all $\alpha, \beta \in \mathbb{T}$. If we replace x and y with $k^n x$ and $k^n y$ in (2.13) respectively, then we see that

$$\begin{aligned}&\left\| \frac{1}{k^n} f\left(\frac{\alpha k^n x}{k} + \frac{\beta k^n y}{l}\right) + \frac{1}{k^n} f\left(\frac{\alpha k^n x}{k} - \frac{\beta k^n y}{l}\right) - \frac{1}{k^n}\frac{2\alpha f(k^n x)}{k}\right\| \\ &\leq k^{-n}\varphi(k^n x, k^n y, 0, 0) \\ &\to 0\end{aligned}$$

as $n \to \infty$ for all $x, y \in X$ and all $\alpha, \beta \in \mathbb{T}$. Hence,

$$f\left(\frac{\alpha x}{k} + \frac{\beta y}{l}\right) + f\left(\frac{\alpha x}{k} - \frac{\beta y}{l}\right) = \frac{2\alpha f(x)}{k} \tag{2.14}$$

for all $x, y \in X$ and all $\alpha, \beta \in \mathbb{T}$. Since f is additive, taking $y = 0$ in (2.14) implies that

$$f(\alpha x) = \alpha f(x) \tag{2.15}$$

for all $x \in X$ and all $\alpha \in \mathbb{T}$. Lemma 2.4 yields that f is linear and so is g. Next, similar to the proof of Theorem 2.3 in [15], one can show that $g(\mathscr{A}) \subset \mathrm{Z}(\mathscr{A}) \cap \mathrm{rad}(\mathscr{A})$. This completes the proof. $\square$

Corollary 2.6. *Let $\mathscr{A}$ be a complex semi-prime Banach algebra with unit e, $\varepsilon \geq 0$, $p, q, s, t \in [0, 1)$ and k and l be integers greater than 1. Suppose that $f, g : \mathscr{A} \to \mathscr{A}$ are mappings with $f(0) = 0$ and satisfy following inequality:*

$$\left\|\Delta^3_{f,g}(x, y, z, w, \alpha, \beta)\right\| \leq \varepsilon(\|x\|^p + \|y\|^q + \|z\|^s\|w\|^t) \tag{2.16}$$

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for all $x, y, z, w \in \mathscr{A}$ and all $\alpha, \beta \in \mathbb{T}$ $(0^0 := 1)$. Then f is a linear generalized left derivation and g is a linear left derivation which maps $\mathscr{A}$ into the intersection of the center $Z(\mathscr{A})$ and the Jacobson radical rad$(\mathscr{A})$ of $\mathscr{A}$.

Proof. Since $\mathscr{A}$ has a unit e, letting $w = e$ in (2.8) shows that the limit $\delta(z) := \lim_{n\to\infty} \frac{1}{k^n} g(k^n z)$ exists for all $z \in \mathscr{A}$. Thus, using Theorem 2.5 for $\varphi(x, y, z, w) = \varepsilon(\|x\|^p + \|y\|^q + \|z\|^s\|w\|^t)$ yields that f is a linear generalized left derivation and g is a linear left derivation since $\mathscr{A}$ has a unit. Similar to the proof of Theorem 2.3 in [15], one can check that the mapping g maps $\mathscr{A}$ into the intersection of the center $Z(\mathscr{A})$ and the Jacobson radical rad$(\mathscr{A})$ of $\mathscr{A}$. This completes the proof. □

Corollary 2.7. *Let $\mathscr{A}$ be a complex semiprime Banach algebra with unit e, $\varepsilon \geq 0$, k and l be integers greater than 1. Suppose that $f, g : \mathscr{A} \to \mathscr{A}$ are mappings with $f(0) = 0$ and satisfy the following inequality:*

$$\left\|\Delta^3_{f,g}(x, y, z, w, \alpha, \beta)\right\| \leq \varepsilon$$

for all $x, y, z, w \in \mathscr{A}$ and all $\alpha, \beta \in \mathbb{T}$. Then f is a linear generalized left derivation and g is a linear left derivation which maps $\mathscr{A}$ into the intersection of the center $Z(\mathscr{A})$ and the Jacobson radical rad$(\mathscr{A})$ of $\mathscr{A}$.

Remark 2. Inequalities (2.10) and (2.16) are controlled by their right-hand sides by the “mixed sum-product of powers of norms", introduced by J. M. Rassias (in 2007) and applied afterwards by K. Ravi et al. (2007-2008). Moreover, it is easy to check that the function

$$\varphi(x, y, z, w) = P\|x\|^p + Q\|y\|^q + S\|z\|^s + T\|w\|^t$$

satisfies conditions (a), (b) and (c) of Theorem 2.1 and Theorem 2.5, where $P, Q, T, S \in [0, \infty)$ and $p, q, s, t \in [0, 1)$ are all constants.

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vol. 10, iss. 3, art. 85, 2009

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