Journal of Inequalities in Pure and Applied Mathematics

A NEW LOOK AT NEWTON'S INEQUALITIES

CONSTANTIN P. NICULESCU

University of Craiova
Department of Mathematics
Craiova 1100
Romania
EMail: tempus@oltenia.ro

volume 1, issue 2, article 17, 2000.

Received 17 November, 1999; accepted 14 April, 2000.

Communicated by: A. Lupaş

Abstract
Contents
Home Page
Go Back
Close
Quit

ISSN (electronic): 1443-5756
002-00

Abstract

New families of inequalities involving the elementary symmetric functions are built as a consequence that all zeros of certain real polynomials are real numbers.

2000 Mathematics Subject Classification: 26C10, 26D05, 20B35
Key words: Elementary Symmetric Functions, (sub)Resultant, Sturm Sequence.

This paper has been completed during a visit at Hebrew University in Jerusalem. The author would like to express his gratitude to Prof. Benjamin Weiss for a number of valuable remarks, particularly for calling his attention to the book [1] by R. Benedetti and J.-J. Risler.

Contents

1. Introduction

The *elementary symmetric functions* of n variables are defined by

$$\begin{aligned} e_0(x_1, x_2, \ldots, x_n) &= 1 \\ e_1(x_1, x_2, \ldots, x_n) &= x_1 + x_2 + \cdots + x_n \\ e_2(x_1, x_2, \ldots, x_n) &= \sum_{i<j} x_i x_j \\ &\vdots \\ e_n(x_1, x_2, \ldots, x_n) &= x_1 x_2 \ldots x_n. \end{aligned}$$

The different e_k, being of different degrees, are not comparable. However, they are connected by nonlinear inequalities. To state them, it is more convenient to consider their averages,

$$E_k(x_1, x_2, \ldots, x_n) = \frac{e_k(x_1, x_2, \ldots, x_n)}{\binom{n}{k}}$$

and to write E_k for $E_k(x_1, x_2, \ldots, x_n)$ in order to avoid excessively long formulae.

Theorem 1.1. *(Newton [17] and Maclaurin [14]). Let $\mathcal{F}$ be an $n-$tuple of non-negative numbers. Then:*

$$E_k^2(\mathcal{F}) > E_{k-1}(\mathcal{F}) \cdot E_{k+1}(\mathcal{F}), \quad 1 \le k \le n-1 \tag{1.1}$$

unless all entries of $\mathcal{F}$ coincide;

$$E_1(\mathcal{F}) > E_2^{1/2}(\mathcal{F}) > \cdots > E_n^{1/n}(\mathcal{F}) \tag{1.2}$$

unless all entries of $\mathcal{F}$ coincide.

Actually, the Newton inequalities (1.1) work for $n-$tuples of real, not necessarily positive elements. An analytic proof along Maclaurin's ideas will be presented below. In Section 2 we shall indicate an alternative argument, based on mathematical induction, which yields more Newton type inequalities in an interpolatory scheme.

The inequalities (1.2) can be deduced from (1.1) since

$$(E_0E_2)(E_1E_3)^2(E_2E_4)^3\dots(E_{k-1}E_{k+1})^k < E_1^2E_2^4E_3^6\dots E_k^{2k}$$

gives $E_{k+1}^k < E_k^{k+1}$ or, equivalently,

$$E_k^{1/k} > E_{k+1}^{1/(k+1)}.$$

Among the inequalities noticed above, the most notable is of course the $AM-GM$ inequality:

$$\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^n \geq x_1x_2\cdots x_n,$$

for every $x_1,x_2,\dots,x_n \geq 0$. A hundred years after Maclaurin, Cauchy [5] gave his beautiful inductive argument. Notice that the $AM-GM$ inequality was known to Euclid [7] in the special case where $n=2$.

Remark 1.2. Newton's inequalities were intended to solve the problem of counting the number of imaginary roots of an algebraic equation. In Chap. 2 of part 2 of *Arithmetica Universalis*, entitled *De Forma Æquationis*, Newton made (without any proof) the following statement: *Given an equation with real coefficients,*

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0 \quad (a_0 \neq 0),$$

the number of its imaginary roots cannot be less than the number of changes of sign that occur in the sequence

$$a_0^2, \quad \left(\frac{a_1}{\binom{n}{1}}\right)^2 - \frac{a_2}{\binom{n}{2}} \cdot \frac{a_0}{\binom{n}{0}}, \ \ldots, \ \left(\frac{a_{n-1}}{\binom{n}{n-1}}\right)^2 - \frac{a_n}{\binom{n}{n}} \cdot \frac{a_{n-2}}{\binom{n}{n-2}}, \qquad a_n^2.$$

Accordingly, if all the roots are real, then all the entries in the above sequence must be non-negative (a fact which yields Newton's inequalities).

Trying to understand Newton's argument, Maclaurin [14] gave a direct proof of the inequalities (1.1) and (1.2), but the Newton counting problem remained open until 1865, when J. Sylvester [23], [24] succeeded in proving a remarkable general result.

Quite surprisingly, it is Real Algebraic Geometry (not Analysis) which gives us the best understanding of Newton's inequalities. The basic fact (discovered by J. Sylvester [22] in 1853) concerns the *semi-algebraic character* of the set of all real polynomials with all roots real.

Theorem 1.3. *For each natural number $n \geq 2$ there exists a set of at most $n-1$ polynomials with integer coefficients,*

$$R_{n,1}(x_1, \ldots, x_n), \ \ldots\ , \ R_{n,k(n)}(x_1, \ldots, x_n), \tag{1.3}$$

such that the monic real polynomials of order n,

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_n,$$

which have only real roots are precisely those for which

$$R_{n,1}(a_1, \ldots, a_n) \geq 0, \ \ldots\ , \ R_{n,k(n)}(a_1, \ldots, a_n) \geq 0\,.$$

The above result can be seen as a generalization of the well known fact that *the roots of a quadratic polynomial $x^2 + a_1 x + a_2$ are real if and only if its discriminant*

$$D_2(1, a_1, a_2) = a_1^2 - 4a_2, \tag{1.4}$$

is non-negative .

Theorem 1.3 is built on the Sturm method of counting real roots, taking into account that only the leading coefficients enter the play. It turns out that they are nothing but the principal subresultant coefficients (with convenient signs added), which are determinants extracted from the Sylvester matrix.

For evident reasons, we shall call a family $(R_{n,k})_k^{k(n)}$ a *discriminating family* (of order *n*). For the convenience of the reader, a summary of the Sylvester algorithm will be presented in the Appendix at the end of this paper.

In Sylvester's approach, $R_{n,1}(a_1, \ldots, a_n)$ equals the *discriminant* D_n of the polynomial $P(x) = x^n + a_1 x^{n-1} + \cdots + a_n$ i.e.,

$$D_n = D_n(1, a_1, \ldots, a_n) = \prod_{1 \leq i < j \leq n} (x_i - x_j)^2\,,$$

where $x_1, \ldots, x_n$ are the roots of $P(x)$; D_n is a polynomial (of weight $n^2 - n$) in $\mathbb{Z}[a_1, \ldots, a_n]$ as being a symmetric and homogeneous polynomial (of degree $n^2 - n$) in $\mathbb{Z}[x_1, \ldots, x_n]$. See, for details, [1] or [13]. Unfortunately, at present no compact formula for D_n is known. According to [21], the number of non-zero coefficients in the expression for the discriminant increases rapidly with the degree; e.g., D_9 has 26095 terms!

For $n \in \{2, 3\}$ one can indicate discriminating families consisting of just a single polynomial, the corresponding discriminant. An inspection of the argument given by L. Euler to solve in radicals the quartic equations allows us to write down a discriminating family for $n = 4$. See Section 4 below, where the computation was performed by using the Maple version incorporated in Scientific WorkPlace 2.5.

Due to the celebrated result on the impossibility of solving in radicals the arbitrary algebraic equations of order $n \geq 5$, we cannot pursue the idea of using resolvants in the general case.

Remark 1.4. Having a discriminating family for $n = N$, we can easily indicate such a family for each $k \in \{1, \ldots, N\}$. The trick is to replace a $P(x)$ of degree k by $x^{N-k}P(x)$, which is of degree N.

Also, having a discriminating family $(R_{n,k})_{k=1}^{k(n)}$ for some $n \geq 2$, we can decide which monic real polynomials $P(x) = x^n + a_1 x^{n-1} + \cdots + a_n$ have only non-negative roots. They are precisely those for which

$$-a_1 \geq 0, \; \ldots \;, (-1)^n a_n \geq 0$$

and

$$R_{n,1}(a_1, \ldots, a_n) \geq 0, \; \ldots \;, R_{n,k(n)}(a_1, \ldots, a_n) \geq 0.$$

Notice that under the above circumstances, $x < 0$ yields $P(x) \neq 0$.

The Newton inequalities (1.1) were proved in [11] following Maclaurin's argument. The basic ingredient is the following lemma, a consequence of repeated application of the Rolle Theorem, which we give here under the formulation of J. Sylvester [24]:

Lemma 1.5. *If*

$$F(x,y) = c_0 x^m + c_1 x^{m-1} y + \cdots + c_m y^m$$

is a homogeneous function of the nth degree in x and y which has all its roots x/y real, then the same is true for all non-identical 0 equations $\frac{\partial^{i+j} F}{\partial x^i \partial y^j} = 0$, obtained from it by partial differentiation with respect to x and y. Further, if E is one of these equations, and it has a multiple root α, then α is also a root, of multiplicity one higher, of the equation from which E is derived by differentiation.

Any polynomial of the nth degree, with real roots, can be represented as

$$E_0 x^n - \binom{n}{1} E_1 x^{n-1} + \binom{n}{2} E_1 x^{n-2} - \cdots + (-1)^n E_n$$

and we shall apply Lemma 1.5 to the associated homogeneous polynomial

$$F(x,y) = E_0 x^n - \binom{n}{1} E_1 x^{n-1} y + \binom{n}{2} E_1 x^{n-2} y^2 - \cdots + (-1)^n E_n y^n.$$

Considering the case of the derivatives $\frac{\partial^{n-2} F}{\partial x^k \partial y^{n-2-k}}$ (for $k = 0, \ldots, n-2$) we arrive to the fact that all the quadratic polynomials

$$E_{k-1} x^2 - 2E_k xy + E_{k+1} y^2$$

for $k=0,\dots,n-2$ also have real roots. Consequently, the Newton inequalities express precisely this fact in the language of discriminants. For this reason we shall refer to (1.1) as the *quadratic Newton inequalities*.

Stopping a step ahead, we get what S. Rosset [20] called the *cubic Newton inequalities*:

$$6E_kE_{k+1}E_{k+2}E_{k+3}+3E_{k+1}^2E_{k+2}^2 \geq 4E_kE_{k+2}^3+E_k^2E_{k+3}^2+4E_{k+1}^3E_{k+3} \tag{1.5}$$

for $k=0,\dots,n-3$. They are motivated by the well known fact that a cubic real polynomial

$$x^3+a_1x^2+a_2x+a_3$$

have only real roots if and only if its discriminant

$$\begin{aligned} D_3 &= D_3(1,a_1,a_2,a_3) \\ &= 18a_1a_2a_3+a_1^2a_2^2-27a_3^2-4a_2^3-4a_1^3a_3 \end{aligned}$$

is non-negative. Consequently the equation

$$E_kx^3-3E_{k+1}x^2y+3E_{k+2}xy^2-E_{k+3}y^3=0$$

has all its roots x/y real if and only if (1.5) holds.

S. Rosset [20] derived the inequalities (1.5) by an inductive argument and noticed that they are strictly stronger than (1.1). In fact, (1.5) can be rewritten as

$$4(E_{k+1}E_{k+3}-E_{k+2}^2)(E_kE_{k+2}-E_{k+1}^2) \geq (E_{k+1}E_{k+2}-E_kE_{k+3})^2$$

which yields (1.1). That (1.1) does not imply (1.5), see the case of the cubic polynomial,

$$x^3 - 8.9x^2 + 26x - 24 = 0,$$

whose roots are

$$x_1 = 1.8587, \quad x_2 = 3.5207 - 0.71933i, \quad x_3 = 3.5207 + 0.71933i.$$

As concerns the Newton inequalities (N_n) of order $n \geq 2$ (when applied to strings of $m \geq n$ elements), they consist of at most $n-1$ sets of relations, the first one being

$$D_n \left(1, (-1)^1 \binom{n}{1} \frac{E_{k+1}}{E_k}, (-1)^2 \binom{n}{2} \frac{E_{k+2}}{E_k}, \dots, (-1)^n \binom{n}{n} \frac{E_{k+n}}{E_k}\right) \geq 0$$

for $k \in \{0, \dots, m-n\}$.

Notice that each of these inequalities is homogeneous (e.g., the above ones consists of terms of weight $n^2 - n$) and the sum of all coefficients (in the left hand side) is 0.

2. An inductive approach of the quadratic Newton inequalities

Our argument will yield directly the log concavity of the functions $k \to E_k$:

Theorem 2.1. *Suppose that* $\alpha, \beta \in \mathbb{R}_+$ *and* $j,\ k \in \mathbb{N}$ *are numbers such that*

$$\alpha + \beta = 1 \quad \textit{and} \quad j\alpha + k\beta \in \{0, \dots, n\}.$$

Then

$$E_{j\alpha+k\beta}(\mathcal{F}) \geq E_j^{\alpha}(\mathcal{F}) \cdot E_k^{\beta}(\mathcal{F}),$$

for every $n-$*tuple* $\mathcal{F}$ *of non-negative real numbers. Moreover, equality occurs if and only if all entries of* $\mathcal{F}$ *are equal.*

The proof will be done by induction on the length of $\mathcal{F}$, following a technique due to S. Rosset [20].

According to Rolle's theorem, if all roots of a polynomial $P \in \mathbb{R}[X]$ are real (respectively, real and distinct), then the same is true for its derivative P'. Given an $n-$tuple $\mathcal{F} = (x_1, \dots, x_n)$, we shall attach to it the polynomial

$$P_{\mathcal{F}}(x) = (x - x_1) \dots (x - x_n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} E_k(x_1, \dots, x_n)\, x^{n-k}.$$

The $(n-1)$ $-$tuple $\mathcal{F}' = \{y_1, \dots, y_{n-1}\}$, consisting of all roots of the derivative of $P_{\mathcal{F}}(x)$ will be called the *derived* $n-$*tuple* of $\mathcal{F}$. Because

$$(x - y_1) \dots (x - y_{n-1}) = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} E_k(y_1, \dots, y_{n-1})\, x^{n-k}$$

and

$$\begin{aligned}
(x-y_1)\dots(x-y_{n-1}) &= \frac{1}{n}\cdot\frac{dP_{\mathcal{F}}}{dx}(x)\\
&= \sum_{k=0}^{n}(-1)^k\,\frac{n-k}{n}\binom{n}{k}E_k(x_1,\dots,x_n)\,x^{n-k-1}\\
&= \sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}E_k(x_1,\dots,x_n)\,x^{n-1-k}.
\end{aligned}$$

We are led to the following result, which enables us to reduce the number of variables when dealing with symmetric functions.

Lemma 2.2. $E_j(\mathcal{F}) = E_j(\mathcal{F}')$ for every $j \in \{0,\dots,|\mathcal{F}|-1\}$.

Another simple remark is as follows.

Lemma 2.3. *Suppose that $\mathcal{F}$ is an $n-$tuple of real numbers and $0 \notin \mathcal{F}$. Put $\mathcal{F}^{-1} = \{1/a \,|\, a \in \mathcal{F}\}$. Then*

$$E_j(\mathcal{F}^{-1}) = E_{n-j}(\mathcal{F})\,/\,E_n(\mathcal{F})$$

for every $j \in \{0,\dots,n\}$.

We move now to the proof of Theorem 2.1.

Proof of Theorem 2.1. For $|\mathcal{F}| = 2$ we have to prove just one inequality namely,

$$x_1x_2 \le \left(\frac{x_1+x_2}{2}\right)^2,$$

valid for every $x_1, x_2 \in \mathbb{R}$. Clearly the equality occurs if and only if $x_1 = x_2$.

Suppose now that the assertion of Theorem 2.1 holds for all $k-$tuples with $k \leq n-1$. Let $\mathcal{F}$ be an $n-$tuple of non-negative numbers $(n \geq 3)$ and let $j, k \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}_+ \setminus \{0\}$ be numbers such that

$$\alpha + \beta = 1 \quad \text{and} \quad j\alpha + k\beta \in \{0, \ldots, n\}.$$

According to Lemma 2.2 (and to our induction hypothesis), we have

$$E_{j\alpha+k\beta}(\mathcal{F}) \geq E_j^{\alpha}(\mathcal{F}) \cdot E_k^{\beta}(\mathcal{F}),$$

except for the case where $j < k = n$ or $k < j = n$. Suppose, for example, that $j < k = n$; then necessarily $j\alpha + n\beta < n$. We have to show that

$$E_{j\alpha+n\beta}(\mathcal{F}) \geq E_j^{\alpha}(\mathcal{F}) \cdot E_n^{\beta}(\mathcal{F}).$$

If $0 \in \mathcal{F}$, then $E_n(\mathcal{F}) = 0$, and the inequality is clear. The equality occurs if and only if $E_{j\alpha+n\beta}(\mathcal{F}') = E_{j\alpha+n\beta}(\mathcal{F}) = 0$, i.e. (according to our induction hypothesis), when all entries of $\mathcal{F}$ coincide.

If $0 \notin \mathcal{F}$, then by Lemma 2.3 we have to prove that

$$E_{n-j\alpha-n\beta}(\mathcal{F}^{-1}) \geq E_{n-j}^{\alpha}(\mathcal{F}^{-1}),$$

or equivalently (see Lemma 2.2),

$$E_{n-j\alpha-n\beta}\left((\mathcal{F}^{-1})'\right) \geq E_{n-j}^{\alpha}\left((\mathcal{F}^{-1})'\right).$$

The latter is true by virtue of our induction hypothesis. $\square$

Remark 2.4. The argument above covers the Newton inequalities even for $n-$tuples of real (not necessarily positive) elements.

The general problem of comparing monomials in $E_1, \ldots, E_n$ was completely solved by G. Hardy, J.E. Littlewood and G. Pólya in [11], Theorem 77, page 64:

Theorem 2.5. *Let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ be non-negative numbers. Then*

$$E_1^{\alpha_1}(\mathcal{F}) \cdot \cdots \cdot E_n^{\alpha_n}(\mathcal{F}) \leq E_1^{\beta_1}(\mathcal{F}) \cdot \cdots \cdot E_n^{\beta_n}(\mathcal{F})$$

for every $n-$tuple $\mathcal{F}$ of positive numbers if, and only if,

$$\alpha_m + 2\alpha_{m+1} + \cdots + (n-m+1)\alpha_n \geq \beta_m + 2\beta_{m+1} + \cdots + (n-m+1)\beta_n$$

for $1 \leq m \leq n$, with equality when $m = 1$.

An alternative proof, also based on the Newton inequalities (1.1) is given in [15], p. 93, where the final conclusion is derived by a technique from the majorization theory.

3. The Quartic Newton Inequalities

While the cubic Newton inequalities can be read off directly from Cardano's formulae, the quartic case is a bit more complicated but still alongside the methods of solving the algebraic equations in radicals. The argument of the following lemma can be traced back to L. Euler.

Lemma 3.1. *The roots of a quartic real polynomial*

$$y^4 + py^2 + qy + r$$

are real if, and only if, the roots of the cubic polynomial

$$z^3 + \frac{p}{2}\,z^2 + \frac{p^2 - 4r}{16}\,z - \frac{q^2}{64}$$

are non-negative .

Proof. In fact, letting $y = u + v + t$ we infer that

$$\begin{aligned} u^2 + v^2 + t^2 &= -p/2 \\ u^2v^2 + v^2t^2 + t^2u^2 &= \left(p^2 - 4r\right)/16 \\ u^2v^2t^2 &= q^2/64 \end{aligned}$$

i.e., u^2, v^2, t^2 are the roots of the cubic polynomial $z^3 + \frac{p}{2}\,z^2 + \frac{p^2-4r}{16}\,z - \frac{q^2}{64}$. The conclusion of the statement is now obvious. □

Lemma 3.2. *The roots of the cubic polynomial*

$$Q(z) = z^3 + \frac{p}{2}\,z^2 + \frac{p^2 - 4r}{16}\,z - \frac{q^2}{64}$$

are non-negative if, and only if,

$$p \leq 0, \; p^2 - 4r \geq 0 \text{ and } D_3(1, p/2, (p^2 - 4r)/16, -q^2/64) \geq 0.$$

Proof. It was already noticed that the roots of $Q(z)$ are real if and only if its discriminant is non-negative. Then the necessity of $p \leq 0$ and $p^2 - 4r \geq 0$ is a consequence of Viète's relations. Their sufficiency is simply the remark that (under their presence) $P(z) < 0$ for $z < 0$. $\square$

In order to write down a discriminating family of order $n = 4$, we have to notice that the substitution

$$x = y - a_1/4$$

changes the general quartic equation

$$x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$$

to

$$y^4 + \left(a_2 - \frac{3a_1^2}{8}\right)y^2 + \left(\frac{a_1^3}{8} - \frac{a_1a_2}{2} + a_3\right)y - \frac{3a_1^4}{256} + \frac{a_1^2a_2}{16} - \frac{a_1a_3}{4} + a_4 = 0.$$

According to Lemma 3.1 and Lemma 3.2, a discriminating family for the quartic monic polynomials is

$$
\begin{aligned}
R_{4,1}(a_1,a_2,a_3,a_4) &= D_3\left(1,\left(a_2-\frac{3a_1^2}{8}\right)/2,\right.\\
&\qquad\left(\frac{3a_1^4}{16}-a_1^2a_2+a_1a_3+a_2^2-4a_4\right)/16,\\
&\qquad\left.-\left(\frac{a_1^3}{8}-\frac{a_1a_2}{2}+a_3\right)^2/64\right)\\
&= -\frac{27}{4096}a_1^4a_4^2-\frac{1}{1024}a_1^3a_3^3+\frac{9}{2048}a_1^3a_2a_4a_3-\frac{1}{1024}a_1^2a_2^3a_4\\
&\quad+\frac{9}{256}a_1^2a_2a_4^2-\frac{3}{2048}a_1^2a_3^2a_4+\frac{1}{4096}a_1^2a_2^2a_3^2+\frac{9}{2048}a_1a_2a_3^3\\
&\quad-\frac{5}{256}a_1a_2^2a_4a_3-\frac{3}{64}a_1a_3a_4^2-\frac{1}{1024}a_2^3a_3^2-\frac{1}{32}a_2^2a_4^2\\
&\quad+\frac{9}{256}a_2a_3^2a_4-\frac{27}{4096}a_3^4+\frac{1}{256}a_2^4a_4+\frac{1}{16}a_4^3\\
&= D_4(1,a_1,a_2,a_3,a_4)/4096\\
R_{4,2}(a_1,a_2,a_3,a_4) &= \frac{3a_1^4}{16}-a_1^2a_2+a_1a_3+a_2^2-4a_4\\
R_{4,3}(a_1,a_2,a_3,a_4) &= \frac{3a_1^2}{8}-a_2\,.
\end{aligned}
$$

The above three relations are written in this order to be in agreement with the general scheme of constructing discriminating families in terms of subresolvents. See the Appendix at the end of this paper.

The *quartic Newton inequalities* will represent the necessary and sufficient conditions under which all polynomials

$$E_k x^4 - 4E_{k+1}x^3 + 6E_{k+2}x^2 - 4E_{k+3}x + E_{k+4} \quad (k \in \{0, \ldots, n-4\})$$

have only real roots. They are

$$\begin{aligned} &R_{4,1}\left(-4E_{k+1}/E_k, 6E_{k+2}/E_k, -4E_{k+3}/E_k, E_{k+4}/E_k\right) \geq 0,\\ &R_{4,2}\left(-4E_{k+1}/E_k, 6E_{k+2}/E_k, -4E_{k+3}/E_k, E_{k+4}/E_k\right) \geq 0,\\ &R_{4,3}\left(-4E_{k+1}/E_k, 6E_{k+2}/E_k, -4E_{k+3}/E_k, E_{k+4}/E_k\right) \geq 0, \end{aligned} \tag{3.1}$$

or, expanding and then eliminating the denominators,

$$\begin{aligned} &-27\, E_k^2 E_{k+3}^4 + E_k^3 E_{k+4}^3 - 54\, E_k E_{k+2}^3 E_{k+3}^2 - 64\, E_{k+1}^3 E_{k+3}^3 - 18\, E_k^2 E_{k+2}^2 E_{k+4}^2\\ &+81\, E_k E_{k+2}^4 E_{k+4} - 27\, E_{k+1}^4 E_{k+4}^2 + 36\, E_{k+1}^2 E_{k+2}^2 E_{k+3}^2\\ &+108\, E_k E_{k+1} E_{k+2} E_{k+3}^3 + 108\, E_{k+1}^3 E_{k+2} E_{k+4} E_{k+3} - 54\, E_{k+1}^2 E_{k+2}^3 E_{k+4}\\ &-180\, E_k E_{k+1} E_{k+2}^2 E_{k+3} E_{k+4} + 54\, E_k^2 E_{k+2} E_{k+3}^2 E_{k+4} - 6 E_k E_{k+1}^2 E_{k+3}^2 E_{k+4}\\ &+54\, E_k E_{k+1}^2 E_{k+2} E_{k+4}^2 - 12\, E_k^2 E_{k+1} E_{k+3} E_{k+4}^2 \geq 0, \end{aligned}$$

$$9\, E_k^2 E_{k+2}^2 + 4\, E_k^2 E_{k+1} E_{k+3} - 24\, E_k E_{k+1}^2 E_{k+2} + 12\, E_{k+1}^4 - E_k^3 E_{k+4} \geq 0,$$

$$E_{k+1}^2 - E_k E_{k+2} \geq 0.$$

The explicit computation of the family (3.1) (as indicated above) was possible by using the Maple version incorporated in Scientific WorkPlace 2.5.

4. An Application to Blundon's Inequalities

Usually, the Newton inequalities can be used either to derive new inequalities, or to conclude that certain polynomials have complex roots. The above analysis on the cubic equations leads us to the following geometric result.

Lemma 4.1. *Consider the cubic equation*

$$x^3 + a_1 x^2 + a_2 x + a_3 = 0$$

with real coefficients. Then its roots $x_1,\ x_2,\ x_3$ *are the side lengths of a* (*nondegenerate*) *triangle if, and only if, the following three conditions are verified:*

i) $18a_1a_2a_3 + a_1^2a_2^2 - 27a_3^2 - 4a_2^3 - 4a_1^3a_3 > 0,$

ii) $-a_1 > 0, \quad a_2 > 0, \quad -a_3 > 0,$

iii) $a_1^3 - 4a_1a_2 + 8a_3 > 0.$

Proof. According to Remark 1.4 above and the discussion after Lemma 1.5, the conjunction i) & ii) is equivalent to the positivity of the roots of the given equation. Then $x_1,\ x_2,\ x_3$ are the side lengths of a (nondegenerate) triangle if, and only if,

$$x_1 + x_2 - x_3 > 0, \quad x_2 + x_3 - x_1 > 0, \quad x_3 + x_1 - x_2 > 0,$$

i.e., if, and only if,

$$(x_1 + x_2 - x_3)(x_2 + x_3 - x_1)(x_3 + x_1 - x_2) > 0.$$

Or, an easy computation shows that the last product equals $a_1^3 - 4a_1a_2 + 8a_3$. □

Corollary 4.2. *(W.J. Blundon [2]). Three positive numbers p, R and r are respectively the semiperimeter, the circumcircle radius and the incircle radius of a triangle if, and only if, the following inequalities are verified:*

$$2R^2 + 10Rr - r^2 - 2(R-2r)\sqrt{R(R-2r)} \le p^2 \le 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R(R-2r)}. \tag{4.1}$$

Proof. The Necessity. As is well known, the side lengths are the roots of the cubic equation

$$x^3 - 2px^2 + (p^2 + r^2 + 4Rr)x - 4pRr = 0. \tag{4.2}$$

In this case the condition i) in Lemma 4.1 leads us to

$$p^4 - 2\left(2R^2 + 10Rr - r^2\right)p^2 + 64rR^3 + 48r^2R^2 + 12r^3R + r^4 \le 0$$

i.e.,

$$\left(p^2 - 2R^2 - 10Rr + r^2\right)^2 \le 4R\left(R - 2r\right)^3$$

which implies both Euler's inequality $R \ge 2r$ and W.J. Blundon's inequalities.

The Sufficiency. We have to verify that the equation (4.2) fulfils the hypothesis of Lemma 4.1 above. The conditions i) and ii) are clear. As concerns iii), we have

$$\begin{aligned} a_1^3 - 4a_1a_2 + 8a_3 &= -8p^3 + 8p(p^2 + r^2 + 4Rr) - 32pRr \\ &= 8pr^2 > 0. \end{aligned}$$

□

The problem of the equality in the Blundon inequalities was settled by A. Lupaş [12]: precisely, the equality occurs in the left-side hand inequality if, and only if, the triangle is either equilateral or isosceles, having the basis greater than the congruent sides. In the right-side hand inequality, the equality occurs if, and only if, the triangle is either equilateral or isosceles, with the basis less than the congruent sides.

5. Other forms of Newton Inequalities

If $P(x) = x^n + a_1 x^{n-1} + \cdots + a_n$ is a real polynomial whose roots are real, then the same is true for

$$x^n + a_1 \binom{n}{1} x^{n-1} + \cdots + \binom{n}{n} a_n .$$

This result appears as Problem 719 in [8]. The interested reader will find there not only the details of proof, but also a large generalization. See also [19], Part V of vol. II. As a consequence, we get the following.

Proposition 5.1. *All the Newton inequalities satisfied by the functions E_k are also satisfied by the functions e_k.*

Particularly,

$$a_k^2 > a_{k-1} a_{k+1} \quad \text{for all } k = 1, \ldots, n-1$$

unless all roots of $P(x)$ are equal; we made here the convention $a_0 = 1$. This latter fact was first noticed by L'Abbé Gua de Malves in 1741. Its proof in the case of $n-$tuples of non-negative numbers is quite simple and appears as the first step in deriving the Newton inequalities in [11], page 52.

From the Taylor's expansion of a polynomial,

$$\begin{aligned} P(x) &= \sum_{k=0}^{n} \frac{1}{k!} \cdot \frac{d^k P}{dx^k}(a)\,(x-a)^k \\ &= \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \left[k! \cdot \frac{d^{n-k} P}{dx^{n-k}}(a) \right] (x-a)^{n-k} \end{aligned}$$

we infer immediately the *differential form* of Newton's inequalities:

$$\left(\frac{d^kP}{dx^k}\right)^2 > \frac{n-k+1}{n-k}\cdot\frac{d^{k+1}P}{dx^{k+1}}\cdot\frac{d^{k-1}P}{dx^{k-1}} \quad \text{for all } k=1,\dots,n-1, \tag{5.1}$$

except when all roots of P coincide. In fact, even more is true.

Proposition 5.2. *Suppose that $P(x)$ is a real polynomial with real roots and* $\deg P = n$. *Then all the Newton inequalities satisfied by the functions E_k remain valid by replacing them with the functions*

$$(-1)^k\ k!\cdot\frac{d^{n-k}P}{dx^{n-k}}\,.$$

In the simplest case, (5.1) gives us

$$P'^{\,2} \geq \frac{n}{n-1}\,PP'',$$

which is stronger than what most textbooks offer,

$$\left(\frac{P'}{P}\right)' = \frac{P''P-P'^{\,2}}{P^2} < 0.$$

One can also formulate a *finite difference analogue* of the Newton inequalities.

Proposition 5.3. *Let*

$$P(x) = c_0 + \sum_{k=1}^{n} c_k x(x-1)\dots(x-k+1)$$

be a real polynomial whose roots are real. Then all the Newton inequalities satisfied by the functions E_k are also satisfied by the functions

$$C_k = (-1)^{n-k}\, c_{n-k} \,/\, \binom{n}{k} c_n.$$

Proof. In fact, according to a result due to F. Brenti [3], Theorem 2.4.2, if

$$P(x) = c_0 + \sum_{k=1}^{n} c_k x(x-1)\dots(x-k+1)$$

is a real polynomial whose roots are real, then all roots of

$$Q(x) = c_0 + \sum_{k=1}^{n} c_k x^k$$

are real and simple. Actually Brenti discussed only the case where all roots of $P(x)$ are non-positive, but his argument also works in the general case of real roots. □

The inequalities of Newton have a companion for matrices, due to the well known connection between the entries of a matrix $A = (a_{ij})_{i,j} \in M_n(\mathbb{R})$ and its eigenvalues. In fact, by defining *the symmetric functions of* A as

$$\begin{aligned}
E_0(A) &= 1, \\
E_1(A) &= \frac{1}{n}\sum a_{jj}\,, \\
E_2(A) &= \frac{2}{n(n-1)}\sum_{j<k}\begin{vmatrix} a_{jj} & a_{jk} \\ a_{kj} & a_{kk}\end{vmatrix}, \\
&\ \vdots \\
E_n(A) &= \det A,
\end{aligned}$$

one can show that

$$E_k(A) = E_k(\sigma(A)) \qquad \text{for every} \quad k \in \{1,\dots,n\},$$

where $\sigma(A)$ denotes the *spectrum* of A, i.e., the set of all its eigenvalues. In particular, Theorem 1.1 allows us to retrieve the following well known generalization of the $AM - GM$ inequality: *If* $A \in M_n(\mathbb{R})$ *is a symmetric matrix, then*

$$\left(\frac{\text{Trace } A}{n}\right)^n > \det A,$$

unless A *is a multiple of the identity* I.

Remark 5.4. There is still another possibility to look at the Newton inequalities in the noncommutative framework, suggested by the following analogue of $E_1^2 \geq E_2$. For self-adjoint elements $A_1,\dots,A_n$ in a $C^\star-$algebra $\mathfrak{A}$ we have

$$\left(\frac{1}{n}\sum_{k=1}^{n} A_k\right)^n \geq \frac{1}{n(n-1)}\sum_{j\neq k} A_j A_k.$$

However, even the form of the $AM - GM$ inequality in this context seems to be open; see [9] for a new approach on this matter.

Possibly, the recent paper [10] of Gelfand et al. on symmetric functions of variables in a noncommutative ring, will eventually yield a better understanding of the whole problem on the analogues of Newton's inequalities.

All inequalities in terms of symmetric functions can be equally expressed in terms of *Newton's sums*:

$$\begin{aligned} s_0(x_1, x_2, \dots, x_n) &= n \\ s_k(x_1, x_2, \dots, x_n) &= x_1^k + x_2^k + \cdots + x_n^k \quad \text{for } k \geq 1. \end{aligned}$$

In fact, if $k \leq n$, then we have the triangular system of equations

$$\begin{aligned} s_1 - e_1 &= 0 \\ s_2 - e_1 s_1 + 2e_2 &= 0 \\ &\vdots \\ s_k - e_1 s_{k-1} + \cdots + (-1)^k k e_k &= 0 \end{aligned}$$

and if $k \geq n$ we have

$$s_k - e_1 s_{k-1} + \cdots + (-1)^n e_n s_{k-n} = 0.$$

A sample of what can be obtained this way is the following inequality, noticed in [6], pp. 179 and 187: *For* $a, b, c, d \in \mathbb{R}$,

$$\left(\frac{a^2 + b^2 + c^2 + d^2}{4}\right)^3 > \left(\frac{abc + abd + acd + bcd}{4}\right)^2$$

unless $a = b = c = d$. In fact, we have to prove that $(4E_1^2 - 3E_2)^3 > E_3^2$ (unless $a = b = c = d$) . Or, according to Newton's inequalities,

$$(4E_1^2 - 3E_2)^3 > E_2^3 > E_3^2$$

unless $a = b = c = d$.

6. Appendix. Sylvester's Algorithm for Finding a Discriminating Family

The set of all points $(a_1, \ldots, a_n)$ of $\mathbb{R}^n$ where the polynomial

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_n$$

has exactly n real roots can be described as the set of solutions of a suitable set of polynomial inequalities with integer coefficients,

$$R_{n,1}(a_1, \ldots, a_n) \geq 0, \ \ldots \ , \ R_{n,k(n)}(a_1, \ldots, a_n) \geq 0.$$

This is a consequence of Sylvester's theory on subresultants, briefly presented in what follows:

The *Sylvester matrix* attached to P and P' (= the derivative of P) is the matrix M_0 of dimension $(2n-1) \times (2n-1)$, defined by

$$M_0 = \begin{pmatrix} a_n & a_{n-1} & a_{n-2} & \ldots & a_2 & a_1 & 1 & \ldots & 0 \\ 0 & a_n & a_{n-1} & \ldots & a_3 & a_2 & a_1 & \ldots & 0 \\ \vdots & & & & & & & & \\ 0 & 0 & 0 & \ldots & a_n & a_{n-1} & a_{n-2} & \ldots & 1 \\ a_{n-1} & 2a_{n-2} & 3a_{n-3} & \ldots & (n-1)a_1 & n & 0 & & \\ 0 & a_{n-1} & 2a_{n-2} & \ldots & (n-2)a_2 & (n-1)a_1 & n & & \\ \vdots & & & & & & & & \\ 0 & 0 & 0 & \ldots & 0 & a_{n-1} & 2a_{n-2} & \ldots & n \end{pmatrix}.$$

Its determinant,

$$r_0 = \det M_0$$

is called the *resultant* of P and P'. Because the leading coefficient of P is 1, we also have

$$r_0 = D_n(1, a_1, \ldots, a_n).$$

For each $j \in \{1, \ldots, n-1\}$ we consider the matrix M_j of dimension $(2n-1-2j) \times (2n-1-2j)$, obtained by removing from M_0

- the last j columns
- the rows with indices from $(n-1) - j + 1$ to $n-1$
- the last j rows.

Then the *subresultant of order* j is the determinant r_j of the $(2n-1-2j) \times (2n-1-2j)$ submatrix of M_j obtained of M_j by including all its rows, the last $2n-1-2j-1$ columns and the column of index $j+1$. Clearly, all subresultants are polynomials in $a_1, \ldots, a_n$. Viewed this way, they constitute a discriminating family of order n. In fact, the dominant coefficients of the Sturm sequence of P and P' are precisely their subresultants (with convenient signs added). This fact is proved in a number of monographs such as that of R. Benedetti and J.-J. Risler [1].

Example 6.1. Let $P(x) = x^3 + a_1x^2 + a_2x + a_3$. Then:

$$\begin{aligned} r_0 &= \det \begin{pmatrix} a_3 & a_2 & a_1 & 1 & 0 \\ 0 & a_3 & a_2 & a_1 & 1 \\ a_2 & 2a_1 & 3 & 0 & 0 \\ 0 & a_2 & 2a_1 & 3 & 0 \\ 0 & 0 & a_2 & 2a_1 & 3 \end{pmatrix} \\ &= 27a_3^2 - 18a_1a_2a_3 + 4a_3a_1^3 + 4a_2^3 - a_2^2a_1^2 \end{aligned}$$

$$
\begin{aligned}
r_1 &= \det \begin{pmatrix} a_2 & a_1 & 1 \\ 2a_1 & 3 & 0 \\ a_2 & 2a_1 & 3 \end{pmatrix} = 6a_2 - 2a_1^2 \\
r_2 &= \det(3) = 3.
\end{aligned}
$$

The number of the real roots of $P(x)$ is given by the Sturm sequence attached to $P(x)$, when restricted to the leading coefficients,

$$
x^3, \quad 3x^2, \quad -\left(6a_2 - 2a_1^2\right)x, \quad -\left(27a_3^2 - 18a_1a_2a_3 + 4a_3a_1^3 + 4a_2^3 - a_2^2a_1^2\right).
$$

Accordingly, in order to assure that $P(x)$ has 3 real roots, we have to impose that

$$
V(-\infty) - V(\infty) = 3,
$$

where $V(-\infty)$ and $V(\infty)$ denote the numbers of sign changes at $-\infty$ and respectively at ∞. That forces

$$
a_1^2 - 3a_2 \geq 0 \quad \text{and} \quad 18a_1a_2a_3 + a_1^2a_2^2 - 27a_3^2 - 4a_2^3 - 4a_1^3a_3 \geq 0.
$$

However, as noticed in the Introduction, the first inequality is a consequence of the second one.

References

[1] R. BENEDETTI AND J.-J. RISLER, *Real algebraic and semi-algebraic sets*, Hermann, Editeurs des Sciences et des Arts, Paris, 1990.

[2] W.J. BLUNDON, Inequalities associated with the triangle, *Canad. Math. Bull.*, **8** (1965), 615–626.

[3] F. BRENTI, Unimodal, log-concave and Polya frequence sequences in combinatorics, *Memoirs of Amer. Math. Soc.*, **81**(413) (1989).

[4] F. CAJORI, *A History of Mathematics*, Chelsea Publ. Co., New York, 3rd Ed., 1980.

[5] A.L. CAUCHY, *Cours d'analyse de l'Ecole Royale Polytechnique*, 1ère partie, *Analyse algébrique*, Paris, 1821. See also, *Œuvres complètes*, IIe série, VII.

[6] A. ENGEL, *Problem-Solving Strategies*, Springer-Verlag, 1998.

[7] EUCLID, *The thirteen books of Euclid's Elements* (translated by Sir Thomas Heath), Cambridge, 1908.

[8] D. FADDEÉV AND I. SOMINSKI, *Receuil d'exercices d'algèbre supérieure*, Ed. Mir, Moscou, 1980.

[9] T. FURUTA AND M. YANAGIDA, Generalized Means and Convexity of Inversion for Positive Operators, *The Amer. Math. Month.*, **105** (1998), 258–259.

[10] I.M. GELFAND, D. KROB, A. LASCOUX, B. LECLERC, V.S. RETAKH AND J.-Y. THIBON, Noncommutative Symmetric Functions, *Adv. in Math.*, **112** (1995), 218–348.

[11] G. HARDY, J.E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, Cambridge Mathematical Library, 2nd ed., 1952.

[12] A. LUPAŞ, On some geometric inequalities, *Revista de matematică a elevilor din Timişoara*, **XV** (1) (1984), 21–23. (Romanian)

[13] I.G. MACDONALD, *Symmetric Functions and Hall Polynomials*, Oxford University Press, 1979.

[14] C. MACLAURIN, A second letter to Martin Folkes, Esq.; concerning the roots of equations, with the demonstration of other rules in algebra, *Phil. Transactions*, **36** (1729), 59–96.

[15] A.W. MARSHALL AND I. OLKIN, *Inequalities: Theory of Majorization and its Applications*, Academic Press, 1979.

[16] J. McKAY, On Computing Discriminants, *Amer. Math. Month.*, **94** (1987), 523–527.

[17] I. NEWTON, *Arithmetica universalis*: *sive de compositione et resolutione arithmetica liber*, 1707.

[18] G. PÓLYA, *Mathematics and Plausible Reasoning*, vols. I & II, Princeton Univ. Press., Princeton, New Jersey, 1954.

[19] G. PÓLYA AND G. SZEGÖ, *Problems and Theorems in Analysis*, vols. I and II, Springer-Verlag, Berlin, Heidelberg, 1976.

[20] S. ROSSET, Normalized Symmetric Functions, Newton's Inequalities and a New Set of Stronger Inequalities, *Amer. Math. Soc.*, **96** (1989), 815–820.

[21] T. SASAKI, Y. KANADA AND S. WATANABE, Calculation of discriminants of higher degree equations, *Tokyo J. Math.*, **4** (1981), 493–503.

[22] J. SYLVESTER, On a theory of syzgetic relations of two rational integral functions comprising an appplication to the theory of Sturm's functions and that of the greatest algebraical common measure, *Phil. Trans. of the Royal Soc. of London*, **CXLIII** (1853), 407–548. See also *The Collected Mathematical Papers of James Joseph Sylvester*, vol. I (1837-1853), 429–586, Cambridge Univ. Press, 1904.

[23] J. SYLVESTER, On Newton's Rule for the discovery of imaginary roots of equations, *Proc. of the Royal Society of London*, **XIV** (1865), 268–270. See also *The Collected Mathematical Papers of James Joeseph Sylvester*, vol. II (1854–1873), 493–494, Cambridge Univ. Press, 1908.

[24] J. SYLVESTER, On an elementary proof and generalization of Sir Isaac Newton's hitherto undemonstrated rule for discovery of imaginary roots, *Proc. of the London Math. Soc.*, **1** (1865–1866), 1–16. See also *The Collected Mathematical Papers of James Joseph Sylvester*, vol. II (1854–1873), pp. 498–513, Cambridge Univ. Press, 1908.