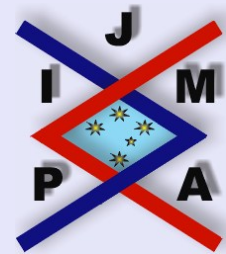# Journal of Inequalities in Pure and Applied Mathematics

ON MULTIPLICATIVELY PERFECT NUMBERS

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Abstract

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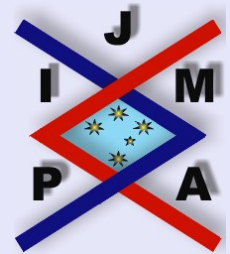
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Abstract

We study multiplicatively perfect, superperfect and analogous numbers . Connection to various arithmetic functions is pointed out. New concepts, inequalities and asymptotic evaluations are introduced.

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1. Introduction

It is well-known that a number n is said to be perfect if the **sum** of aliquot divisors of n is equal to n. By introducing the function σ (sum of divisors), this can be written equivalently as

$$(1.1) \qquad \sigma(n) = 2n.$$

The Euclid-Euler theorem gives the form of even perfect numbers: $n = 2^k p$, where $p = 2^{k+1} - 1$ is prime ("Mersenne prime"). No odd perfect numbers are known. The number n is said to be super-perfect if

$$(1.2) \qquad \sigma(\sigma(n)) = 2n.$$

The Suryanarayana-Kanold theorem [16], [4] gives the general form of even super-perfect numbers: $n = 2^k$, where $2^{k+1} - 1 = p$ is a prime. No odd super-perfect numbers are known. For new proofs of these results, see [10], [11]. Many open problems are stated e.g. in [1], [10].
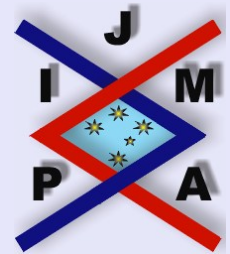
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2. m-**Perfect Numbers**

Let $T(n)$ denote the **product** of all divisors of n. There are many numbers n with the property $T(n) = n^2$, but none satisfying $T(T(n)) = n^2$. Let us call the number $n > 1$ **multiplicatively perfect** (or, for short, m-perfect) if

$$(2.1) \qquad\qquad\qquad T(n) = n^2,$$

and **multiplicatively super-perfect** (m-super-perfect), if

$$(2.2) \qquad\qquad\qquad T(T(n)) = n^2.$$

To begin with, we prove the following little result:

Theorem 2.1. *All m-perfect numbers n have one of the following forms: $n = p_1 p_2$ or $n = p_1^3$, where p_1, p_2 are arbitrary, distinct primes. There are no m-super-perfect numbers.*

Proof. Firstly, we note that if $d_1, d_2, \ldots, d_s$ are all divisors of n, then
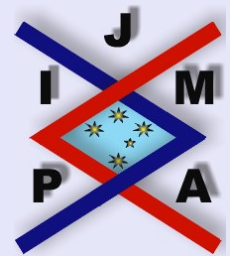
$$\{d_1, d_2, \ldots, d_s\} = \left\{ \frac{n}{d_1}, \frac{n}{d_2}, \ldots, \frac{n}{d_s} \right\},$$

implying that

$$d_1 d_2 \ldots d_s = \frac{n}{d_1} \cdot \frac{n}{d_2} \ldots \frac{n}{d_s},$$

i.e.

$$(2.3) \qquad\qquad\qquad T(n) = n^{s/2},$$

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where $s = d(n)$ denotes the number of (distinct) divisors of n.

Let $n = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ be the prime factorisation of $n > 1$. It is well-known that $d(n) = (\alpha_1 + 1) \ldots (\alpha_r + 1)$, so equation (2.1) combined with (2.3) gives

$$(2.4) \qquad (\alpha_1 + 1) \ldots (\alpha_r + 1) = 4.$$

Since $\alpha_i + 1 > 1$, for $r \geq 2$ we can have only $\alpha_1 + 1 = 2$, $\alpha_2 + 1 = 2$, implying $\alpha_1 = \alpha_2 = 1$, i.e. $n = p_1 p_2$. For $r = 1$ we have $\alpha_1 + 1 = 4$, i.e. $\alpha_1 = 3$, giving $n = p^3$. There are no other solutions $n > 1$ ($n = 1$ is a trivial solution) of equation (2.1).

On the other hand, let us remark that for $n \geq 2$ one has $d(n) \geq 2$, so
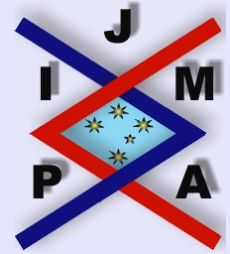
$$(2.5) \qquad T(n) \geq n$$

with equality only for $n = $ prime. If $n \neq$ prime, then it is immediate that $d(n) \geq 3$, giving

$$(2.6) \qquad T(n) \geq n^{3/2} \quad \text{for} \quad n \neq \text{prime}.$$

Now, relations (2.5) and (2.6) together give

$$(2.7) \qquad T(T(n)) \geq n^{9/4}, \quad n \neq \text{prime}.$$

Thus, by $9/4 > 2$, there are no non-trivial (i.e. $n \neq 1$) m-super-perfect numbers.

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In fact, we have found that the equation

$$(2.8) \qquad T(T(n)) = n^a, \quad a \in \left(1, \frac{9}{4}\right)$$

has no nontrivial solutions. $\qquad \square$

Note. According to the referee the notion of "m-perfect numbers", as well as Theorem 2.1 appears in [3].

Corollary 2.2. $n = 6$ is the only perfect number, which is also m-perfect.

Indeed, n cannot be odd, since by a result of Sylvester, an odd perfect number must have at least five prime divisors. If n is even, then $n = 2^k p = p_1 p_2 \Leftrightarrow k = 1$, when $2 = p_1$ and $2^2 - 1 = 3$, when $3 = p_2$. Thus $n = 2 \cdot 3 = 6$.
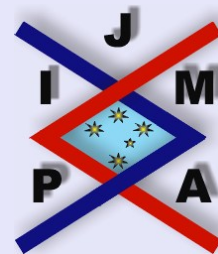
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3. k-m-**Perfect Numbers**

In a similar manner, one can define k-m-perfect numbers by

$$(3.1) \qquad T(n) = n^k$$

where $k \geq 2$ is given. Since the equation $(\alpha_1 + 1) \ldots (\alpha_r + 1) = 2k$ has a finite number of solutions, the general form of k-multiply perfect numbers can be determined. We collect certain particular cases in the following.

Theorem 3.1. *1) All tri-m-perfect numbers have the forms $n = p_1 p_2^2$ or $n = p_1^5$;*

2) All 4-m-perfect numbers have the forms $n = p_1 p_2^3$ or $n = p_1 p_2 p_3$ or $n = p_1^7$;

3) All 5-m-perfect numbers have the forms $n = p_1 p_2^4$ or $n = p_1^9$;

4) All 6-m-perfect numbers have the forms $n = p_1 p_2 p_3^2$, $n = p_1 p_2^5$, $n = p_1^{11}$;
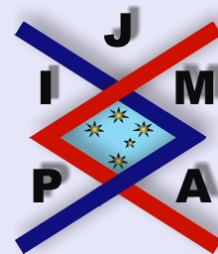
5) All 7-m-perfect numbers have the forms $n = p_1 p_2^6$, or $n = p_1^{13}$;

6) All 8-m-perfect numbers have the forms $n = p_1 p_2 p_3 p_4$ or $n = p_1 p_2 p_3^3$ or $n = p_1^3 p_2^3$, $n = p_1^{15}$;

7) All 9-m-perfect numbers have the forms $n = p_1 p_2^2 p_3^2$, $n = p_1 p_2^8$, $n = p_1^{17}$;

8) All 10-m-perfect numbers have the forms $n = p_1 p_2 p_3^4$, $n = p_1 p_2^9$, $n = p_1^{19}$, etc.

(Here p_i denote certain distinct primes.)

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Proof. We prove only the case 6). By relation (2.3) we must solve the equation

$$(3.2) \qquad (\alpha_1 + 1) \ldots (\alpha_r + 1) = 16$$

in α_r and r. It is easy to see that the following four cases are possible:

i) $\alpha_1 + 1 = 2, \alpha_2 + 1 = 2, \alpha_3 + 1 = 2, \alpha_4 + 1 = 2$;

ii) $\alpha_1 + 1 = 2, \alpha_2 + 1 = 2, \alpha_3 + 1 = 4, \alpha_4 + 1 = 4$;

iii) $\alpha_1 + 1 = 4, \alpha_2 + 1 = 4$;

iv) $\alpha_1 + 1 = 16$.

This gives the general forms of all 8-m-perfect numbers, namely ($\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$) $n = p_1 p_2 p_3 p_4$; ($\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 3$) $n = p_1 p_2 p_3^3$; ($\alpha_1 = 3, \alpha_2 = 3$) $n = p_1^3 p_2^3$; ($\alpha_1 = 15$) $n = p_1^{15}$. $\square$

Corollary 3.2. *1) $n = 28$ is the single perfect and tri-perfect number.*

2) There are no perfect and 4-perfect numbers;

3) $n = 496$ is the only perfect number which is 5-m-perfect;

4) There are no perfect numbers which are 6-m-perfect;

5) $n = 8128$ is the only perfect number which is 7-m-perfect.

In fact, we have:

Theorem 3.3. *Let p be a prime, with $2^p - 1$ prime too (i.e. $2^p - 1$ is a Mersenne prime). Then $2^{p-1}(2^p - 1)$ is the only perfect number, which is p-m-perfect.*
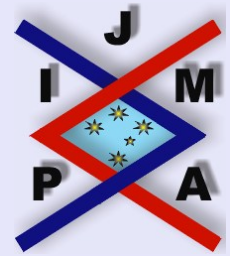
Proof. By writing $(\alpha_1 + 1)\ldots(\alpha_r + 1) = 2p$ (p prime), the following cases are only possible:

   i) $\alpha_1 + 1 = 2$, $\alpha_2 + 1 = p$;

   ii) $\alpha_1 + 1 = 2p$.

Then $n = p_1 p_2^{p-1}$ or $n = p_1^{2p-1}$ are the general forms of p-m-perfect numbers. By the Euclid-Euler theorem $p_1 p_2^{p-1} = 2^{p-1}(2^p - 1)$ iff $p_2 = 2$ and $p_1 = 2^p - 1$ is prime. $\qquad\square$

Remark 3.1. *For $p < 10000$ the following Mersenne primes are known; namely for $p = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941$. It is an open problem to show the existence of infinitely many Mersenne primes ([1]).*
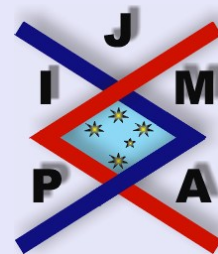
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4.   Some Results for k-m-Perfect Numbers

As we have seen, the equation (2.2), i.e. $T(T(n)) = n^2$ has no nontrivial solutions. A similar problem arises for the equation

$$(4.1) \qquad\qquad T(T(n)) = n^k; \quad n > 1$$

($k \geq 2$, fixed). By (2.3) we can see that this is equivalent to

$$(4.2) \qquad\qquad \frac{d(n)d(T(n))}{4} = k.$$

Let $n = p_1^{\alpha_1} \ldots p_r^{\alpha_r} > 1$ be the canonical representation of n. By $d(n) = (\alpha_1 + 1) \ldots (\alpha_r + 1)$, and (2.3) we have

$$T(n) = p_1^{\alpha_1(\alpha_1+1)\ldots(\alpha_r+1)/2} \ldots p_r^{\alpha_r(\alpha_1+1)\ldots(\alpha_r+1)/2},$$
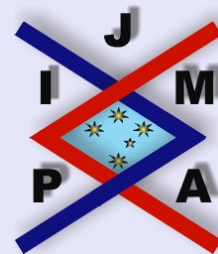
so (4.2) becomes equivalent to

$$(4.3) \quad (\alpha_1 + 1) \ldots (\alpha_r + 1) \left[\frac{\alpha_1(\alpha_1 + 1) \ldots (\alpha_r + 1)}{2} + 1 \right]$$
$$\ldots \left[\frac{\alpha_r(\alpha_1 + 1) \ldots (\alpha_r + 1)}{2} + 1 \right] = 4k,$$

and this, clearly has at most a finite number of solutions.

Theorem 4.1.   *1) Equation (4.1) is not solvable for $k = 4, 5, 6$;*

2) For $k = 3$ the general solutions are $n = p_1^2$;

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3) For $k = 7$ the solutions are $n = p_1^3$;

4) For $k = 9$ the solutions are: $n = p_1 p_2$ ($p_1 \neq p_2$ primes).

Proof. For $k = 4, 5, 6$, from (4.3) we must solve the corresponding equations for 16, 20, 24. It is a simple exercise to verify these impossibilities. For $k = 3$ we have the single equality $12 = 3 \cdot 4$, when $\alpha_1 = 2$, $\frac{\alpha_1(\alpha_1+1)}{2} + 1 = 4$. For $k = 7$, $\alpha_1 = 3$ by $\frac{3 \cdot 4}{2} + 1 = 7$ and $4 \cdot 7 = 28$. For $k = 9$ we have $2 \cdot 2 \cdot 3 \cdot 3 = 36$ and $\alpha_1 = \alpha_2 = 1$. $\qquad \square$

Corollary 4.2. *$n = 6$ is the single perfect number which is also 9-super-m-perfect.*

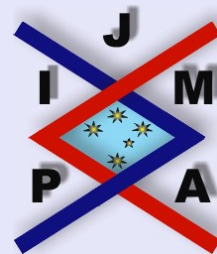Indeed, $p_1 p_2 = 2 \cdot (2^2 - 1) = 2 \cdot 3 = 6$ by Theorem 4.1 and the Euclid-Euler theorem.

Remark 4.1. *By relation (2.6), by consecutive iteration we can deduce*

$$\underbrace{T(T(\ldots T(n) \ldots))}_{k} \geq n^{3^k/2^k}$$

for $n \neq$ prime. Since $3^k > 2^k \cdot k$ for all $k \geq 1$ (induction: $3^{k+1} = 3 \cdot 3^k > 3 \cdot 2^k \cdot k > 2 \cdot 2^k(k+1) = 2^{k+1}(k+1)$) we can obtain the following generalization of equation (2.2):

$$\underbrace{T(T(\ldots T(n) \ldots))}_{k} = n^k$$

has no nontrivial solutions.

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5. Other Results

By relation (2.3) we have

$$\frac{\log T(n)}{\log n} = \frac{d(n)}{2}.$$
(5.1)

Clearly, this implies

$$\liminf_{n\to\infty} \frac{\log T(n)}{\log n} = 1, \quad \limsup_{n\to\infty} \frac{\log T(n)}{\log n} = +\infty$$

(take e.g. $n = p$ (prime); $n = 2^k$ ($k \in \mathbb{N}$)). Since $2 \le d(n) \le 2\sqrt{n}$ (see e.g. [13]) for $n \ge 2$ we get
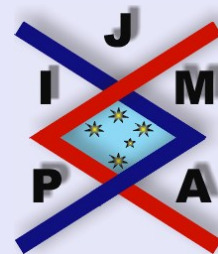
$$1 \le \frac{\log T(n)}{\log n} \le \sqrt{n}.$$

By $2^{\omega(n)} \le d(n) \le 2^{\Omega(n)}$ (see e.g. [12]) we can deduce:

$$2^{\omega(n)-1} \le \frac{\log T(n)}{\log n} \le 2^{\Omega(n)-1} \quad (n \ge 2).$$

Since it is known by a theorem of Hardy and Ramanujan [2] that the normal order of magnitude of $\omega(n)$ and $\Omega(n)$ is $\log\log n$, the above double inequality implies that:
the normal order of magnitude of

$$\log\log T(n) - \log\log n \, is \, (\log 2)(\log\log n - 1).$$
(5.2)

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By a theorem of Wiegert ([17]) we have

$$\limsup_{n \to \infty} \frac{\log d(n) \log \log n}{\log n} = \log 2,$$

so by (5.1) we get:

(5.3) $$\limsup_{n \to \infty} \frac{(\log \log T(n))(\log \log n)}{\log n} = \log 2.$$

In fact, by a result of Nicolas and Robin ([7]), for $n \geq 3$ one has

$$\frac{\log d(n)}{\log 2} \leq c \frac{\log n}{\log \log n} \quad (c \approx 1,5379\dots),$$

we can obtain the following inequality:

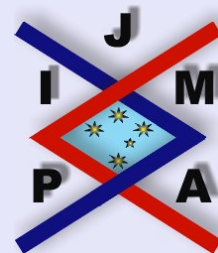(5.4) $$\log \log T(n) \leq \log \log n + \frac{k \log n}{\log \log n} - \log 2,$$

where $k = c \log 2$ and $n \geq 3$. This gives

(5.5) $$\lim_{n \to \infty} \frac{\log \log T(n)}{f(n)} = 0$$

for any positive function with $\frac{\log n}{f(n) \log \log n} \to 0 \ (n \to \infty)$.

By $\varphi(n) d(n) \geq n$ (see [14]) and $\varphi(n) d^2(n) \leq n^2$ for $n \neq 4$ (see [8]) we get

$$\frac{n}{\varphi(n)} \leq d(n) \leq \frac{n}{\sqrt{\varphi(n)}} \quad \text{for} \quad n > 4,$$

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and this, by (5.1) yields

$$(5.6) \qquad \frac{n}{2\varphi(n)} \leq \frac{\log T(n)}{\log n} \leq \frac{n}{2\sqrt{\varphi(n)}}.$$

Here φ is the usual Euler totient function.

Hence, the arithmetic function T is connected to the other classical arithmetic functions.

By $\sqrt{n} \leq \frac{\sigma(n)}{d(n)} \leq \frac{n+1}{2}$ (see [14], [5], [6]), we get

$$(5.7) \qquad \frac{\sigma(n)}{n+1} \leq \frac{\log T(n)}{\log n} \leq \frac{\sigma(n)}{2\sqrt{n}}.$$

For infinitely many primes p we have

$$d(p-1) > \exp\left(c\frac{\log p}{\log\log p}\right)$$
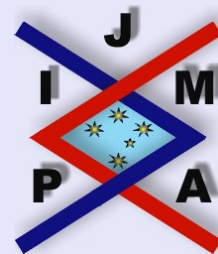
($c > 0$, constant, see [9]), so we have:

$$(5.8) \qquad \log\log T(p-1) > \log\log(p-1) + \frac{c\log p}{\log\log p} - \log 2$$

for infinitely many primes p, implying, e.g.

$$(5.9) \qquad \lim_{\substack{p\to\infty \\ p\,prime}} \sup \frac{\log\log T(p-1)}{\log\log p} = +\infty$$

and

$$(5.10) \qquad \lim_{n\to\infty} \inf \frac{(\log\log T(n))(\log\log n)}{\log n} > 0.$$

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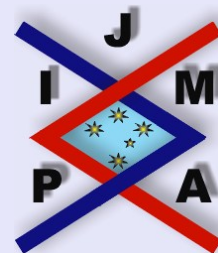
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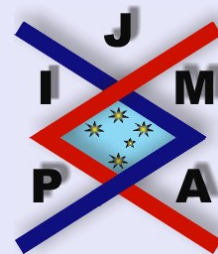
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