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ON ČEBYŠEV-GRÜSS TYPE INEQUALITIES VIA PEČARIĆ'S EXTENSION OF THE MONTGOMERY IDENTITY

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ABSTRACT. In the present note we establish new Čebyšev-Grüss type inequalities by using Pečarič's extension of the Montgomery identity.

Key words and phrases: Čebyšev-Grüss type inequalities, Pečarić's extension, Montgomery identity.

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1. INTRODUCTION

For two absolutely continuous functions $f, g : [a, b] \to \mathbb{R}$ consider the functional

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right),$$

where the involved integrals exist. In 1882, Čebyšev [1] proved that if $f', g' \in L_\infty[a,b]$, then

$$|T(f,g)| \le \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty . \tag{1.1}$$

In 1935, Grüss [2] showed that

$$|T(f,g)| \le \frac{1}{4}(M-m)(N-n), \tag{1.2}$$

provided m, M, n, N are real numbers satisfying the condition $-\infty < m \le M < \infty$, $-\infty < n \le N < \infty$ for $x \in [a,b]$.

Many researchers have given considerable attention to the inequalities (1.1), (1.2) and various generalizations, extensions and variants of these inequalities have appeared in the literature, to mention a few, see [4, 5] and the references cited therein. The aim of this note is to establish two

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new inequalities similar to those of Čebyšev and Grüss inequalities by using Pečarič's extension of the Montgomery identity given in [6].

2. Statement of Results

Let $f : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$ and $f' : [a,b] \to \mathbb{R}$ is integrable on $[a,b]$. Then the Montgomery identity holds [3]:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \int_a^b P(x,t)\,f'(t)\,dt, \tag{2.1}$$

where $P(x,t)$ is the Peano kernel defined by

$$P(x,t) = \begin{cases} \frac{t-a}{b-a}, & a \le t \le x, \\ \frac{t-b}{b-a}, & x < t \le b. \end{cases} \tag{2.2}$$

Let $w : [a,b] \to [0,\infty)$ be some probability density function, that is, an integrable function satisfying $\int_a^b w(t)\,dt = 1$, and $W(t) = \int_a^t w(x)\,dx$ for $t \in [a,b]$, $W(t) = 0$ for $t < a$, and $W(t) = 1$ for $t > b$. In [6] Pečarić has given the following weighted extension of the Montgomery identity:

$$f(x) = \int_a^b w(t)\,f(t)\,dt + \int_a^b P_w(x,t)\,f'(t)\,dt, \tag{2.3}$$

where $P_w(x,t)$ is the weighted Peano kernel defined by

$$P_w(x,t) = \begin{cases} W(t), & a \le t \le x, \\ W(t) - 1, & x < t \le b. \end{cases} \tag{2.4}$$

We use the following notation to simplify the details of presentation. For some suitable functons $w, f, g : [a,b] \to \mathbb{R}$,we set

$$T(w,f,g) = \int_a^b w(x)\,f(x)\,g(x)\,dx - \left(\int_a^b w(x)\,f(x)\,dx\right)\left(\int_a^b w(x)\,g(x)\,dx\right),$$

and define $\|\cdot\|_\infty$ as the usual Lebesgue norm on $L_\infty[a,b]$ that is, $\|h\|_\infty := ess \sup_{t\in[a,b]} |h(t)|$ for $h \in L_\infty[a,b]$.

Our main results are given in the following theorems.

Theorem 2.1. *Let $f, g : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$ and $f', g' : [a,b] \to \mathbb{R}$ are integrable on $[a,b]$. Let $w : [a,b] \to [0,\infty)$ be an integrable function satisfying $\int_a^b w(t)\,dt = 1$. Then*

$$|T(w,f,g)| \le \|f'\|_\infty \|g'\|_\infty \int_a^b w(x)\,H^2(x)\,dx, \tag{2.5}$$

where

$$H(x) = \int_a^b |P_w(x,t)|\,dt \tag{2.6}$$

for $x \in [a,b]$ and $P_w(x,t)$ is the weighted Peano kernel given by (2.4).

Theorem 2.2. *Let f, g, f', g', w be as in Theorem 2.1. Then*

$$|T(w,f,g)| \le \frac{1}{2}\int_a^b w(x)\left[|g(x)|\,\|f'\|_\infty + |f(x)|\,\|g'\|_\infty\right]H(x)\,dx, \tag{2.7}$$

where $H(x)$ is defined by (2.6).

3. Proofs of Theorems 2.1 and 2.2

Proof of Theorem 2.1. From the hypotheses the following identities hold [6]:

$$f(x) = \int_a^b w(t) f(t)\,dt + \int_a^b P_w(x,t) f'(t)\,dt, \tag{3.1}$$

$$g(x) = \int_a^b w(t) g(t)\,dt + \int_a^b P_w(x,t) g'(t)\,dt, \tag{3.2}$$

From (3.1) and (3.2) we observe that

$$\left[f(x) - \int_a^b w(t) f(t)\,dt\right]\left[g(x) - \int_a^b w(t) g(t)\,dt\right] = \left[\int_a^b P_w(x,t) f'(t)\,dt\right]\left[\int_a^b P_w(x,t) g'(t)\,dt\right],$$

i.e.,

$$\begin{aligned} f(x) g(x) - f(x)\int_a^b w(t) g(t)\,dt - g(x)\int_a^b w(t) f(t)\,dt \\ + \left(\int_a^b w(t) f(t)\,dt\right)\left(\int_a^b w(t) g(t)\,dt\right) \\ = \left[\int_a^b P_w(x,t) f'(t)\,dt\right]\left[\int_a^b P_w(x,t) g'(t)\,dt\right]. \end{aligned} \tag{3.3}$$

Multiplying both sides of (3.3) by $w(x)$ and then integrating both sides of the resulting identity with respect to x from a to b and using the fact that $\int_a^b w(t)\,dt = 1$, we have

$$T(w,f,g) = \int_a^b w(x)\left[\int_a^b P_w(x,t) f'(t)\,dt\right]\left[\int_a^b P_w(x,t) g'(t)\,dt\right]dx. \tag{3.4}$$

From (3.4) and using the properties of modulus we observe that

$$\begin{aligned} |T(w,f,g)| &\le \int_a^b w(x)\left[\int_a^b |P_w(x,t)|\,|f'(t)|\,dt\right]\left[\int_a^b |P_w(x,t)|\,|g'(t)|\,dt\right]dx \\ &\le \|f'\|_\infty \|g'\|_\infty \int_a^b w(x) H^2(x)\,dx. \end{aligned}$$

This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. Multiplying both sides of (3.1) and (3.2) by $w(x)g(x)$ and $w(x)f(x)$, adding the resulting identities and rewriting we have

$$\begin{aligned} & w(x) f(x) g(x) \\ &= \frac{1}{2}\left[w(x) g(x)\int_a^b w(t) f(t)\,dt + w(x) f(x)\int_a^b w(t) g(t)\,dt\right] \\ &\quad + \frac{1}{2}\left[w(x) g(x)\int_a^b P_w(x,t) f'(t)\,dt + w(x) f(x)\int_a^b P_w(x,t) g'(t)\,dt\right]. \end{aligned} \tag{3.5}$$

Integrating both sides of (3.5) with respect to x from a to b and rewriting we have

$$T(w,f,g)=\frac{1}{2}\int_a^b\left[w(x)\,g(x)\int_a^b P_w(x,t)\,f'(t)\,dt + w(x)\,f(x)\int_a^b P_w(x,t)\,g'(t)\,dt\right]dx. \tag{3.6}$$

From (3.6) and using the properties of modulus we observe that

$$\begin{aligned}|T(w,f,g)| &\leq \frac{1}{2}\int_a^b w(x)\left[|g(x)|\int_a^b |P_w(x,t)|\,|f'(t)|\,dt + |f(x)|\int_a^b |P_w(x,t)|\,|g'(t)|\,dt\right]dx\\ &\leq \frac{1}{2}\int_a^b w(x)\left[|g(x)|\,\|f'(t)\|_\infty + |f(x)|\,\|g'(t)\|_\infty\right]H(x)\,dx.\end{aligned}$$

The proof of Theorem 2.2 is complete. □

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