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ON THE INEQUALITY OF P. TURÁN FOR LEGENDRE POLYNOMIALS

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Abstract

Our aim is to prove the inequalities

$$\frac{1-x^2}{n(n+1)}h_n \le \begin{vmatrix} P_n(x) & P_{n+1}(x) \\ P_{n-1}(x) & P_n(x) \end{vmatrix} \le \frac{1-x^2}{2}, \quad \forall x \in [-1,1],\ n = 1,2,\dots,$$

where $h_n := \sum_{k=1}^{n} \frac{1}{k}$ and $(P_n)_{n=0}^{\infty}$ are the Legendre polynomials . At the same time, it is shown that the sequence having as general term

$$n(n+1)\begin{vmatrix} P_n(x) & P_{n+1}(x) \\ P_{n-1}(x) & P_n(x) \end{vmatrix}$$

is non-decreasing for $x \in [-1,1]$.

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Contents

1. Introduction

Let $(P_n)_{n=0}^{\infty}$ be the sequence of Legendre polynomials, that is

$$P_n(x) = \frac{1}{n!2^n}\left((x^2-1)^n\right)^{(n)} = {}_2F_1\left(-n, n+1; 1; \frac{1-x}{2}\right),$$

where

$$ {}_2F_1(a, b; c; z) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \cdot \frac{z^k}{k!},$$
$$(a)_k := a(a+1)\cdots(a+k-1), \quad (a)_0 = 1.$$

Denote

$$\Delta_n(x) := \begin{vmatrix} P_n(x) & P_{n+1}(x) \\ P_{n-1}(x) & P_n(x) \end{vmatrix} = [P_n(x)]^2 - P_{n-1}(x)P_{n+1}(x).$$

Note that $P_n(1) = 1$, $P_n(-x) = (-1)^n P_n(-x)$, i.e. $\Delta_n(1) = \Delta_n(-1) = 0$. For instance

$$\Delta_1(x) = \frac{1-x^2}{2}, \quad \Delta_2(x) = \frac{1-x^4}{4}.$$

Paul Turán [3] has proved the following interesting inequality

$$\Delta_n(x) > 0, \quad \forall x \in (-1, 1), \quad n \in \{1, 2, \dots\}. \tag{1.1}$$

In [1] – [2] are given the following remarkable representations of $\Delta_n(x)$.

Lemma 1.1 (A. Lupaş). *Suppose* $\varphi(x,t) := x^2 + t(1-x^2)$ *and* $P_n(x_k) = 0$. *Then*

$$\Delta_n(x) = \frac{1}{\pi n(n+1)} \int_{-1}^{1} \frac{1 - P_n(\varphi(x,t))}{1-t} \cdot \frac{dt}{\sqrt{1-t^2}} \tag{1.2}$$

and

$$\Delta_n(x) = \frac{1-x^2}{n(n+1)} \sum_{k=1}^{n} \left(\frac{P_n(x)}{x - x_k} \right)^2 (1 - x x_k) \ . \tag{1.3}$$

2. Main Results

In this article our aim is to improve the Turán inequality (1.1).

Theorem 2.1. *If* $x \in [-1,1]$, $n \in \mathbb{N}$, $h_n := \sum_{k=1}^{n} \frac{1}{k}$, *then*

$$\frac{1-x^2}{n(n+1)} h_n \leq \Delta_n(x) \leq \frac{1-x^2}{2}. \tag{2.1}$$

Proof. Let us denote $T_k(t) = \cos(k \cdot \arccos t)$, $\gamma_0 = \frac{1}{\pi}$, $\gamma_k = \frac{2}{\pi}$ for $k \geq 1$, and $\varphi(x,t) = x^2 + t(1-x^2)$. According to addition formula for Legendre polynomials, we have

$$P_n(\varphi(x,t)) = \pi \sum_{k=0}^{n} \frac{(n-k)!}{(n+k)!} (1-x^2)^k \left[P_n^{(k)}(x)\right]^2 \gamma_k T_k(t).$$

If $t = 1$ we find

$$1 = \pi \sum_{k=0}^{n} \frac{(n-k)!}{(n+k)!} (1-x^2)^k \left[P_n^{(k)}(x)\right]^2 \gamma_k.$$

Therefore

$$\begin{aligned} \frac{1-P_n(\varphi(x,t))}{1-t} &= 2 \sum_{k=1}^{n} \frac{(n-k)!}{(n+k)!} (1-x^2)^k \left[P_n^{(k)}(x)\right]^2 \frac{1-T_k(t)}{1-t} \\ &= 2\pi \sum_{k=1}^{n} \frac{(n-k)!}{(n+k)!} (1-x^2)^k \left[P_n^{(k)}(x)\right]^2 \sum_{\nu=0}^{k} (k-\nu)\gamma_\nu T_\nu(t). \end{aligned}$$

This shows us that

$$\max_{t\in[-1,1]}\left\{\frac{1-P_n\left(\varphi(x,t)\right)}{1-t}\right\} = \left.\frac{1-P_n\left(\varphi(x,t)\right)}{1-t}\right|_{t=1}$$
$$= (1-x^2)P_n'(1)$$
$$= \frac{n(n+1)}{2}(1-x^2).$$

Using the Lupaş identity (1.2) we obtain

$$\Delta_n(x) \le \frac{1-x^2}{2}, \quad (n\ge 1, \quad x\in[-1,1]).$$

Taking into account the following well-known equalities

$$P_n(x) = \frac{2n-1}{n}xP_{n-1}(x) - \frac{n-1}{n}P_{n-2}(x), \quad P_0(x)=1,\ P_1(x)=x,$$
$$(1-x^2)P_n'(x) = n\left(P_{n-1}(x)-xP_n(x)\right) = (n+1)\left(xP_n(x)-P_{n+1}(x)\right),$$

we obtain

$$k(k+1)\Delta_k(x)-(k-1)k\Delta_{k-1}(x) = (1-x^2)\left[P_k'(x)P_{k-1}(x)-P_k(x)P_{k-1}'(x)\right].$$

The Christofell-Darboux formula for Legendre polynomials enables us to write

$$k(k+1)\Delta_k(x)-(k-1)k\Delta_{k-1}(x) = \frac{1-x^2}{k}\sum_{j=0}^{k-1}(2j+1)\left[P_j(x)\right]^2, \quad k\ge 2.$$

By summing for $k \in \{2,3,\dots,n\}$ we give

$$n(n+1)\Delta_n(x) = (1-x^2)h_n + (1-x^2)\sum_{k=1}^{n-1}\frac{1}{k+1}\sum_{j=1}^{k}(2j+1)\left[P_j(x)\right]^2,$$

which implies $\Delta_n(x) \geq \frac{(1-x^2)h_n}{n(n+1)}$ for $x \in [-1,1]$. □

Another remark regarding $\Delta_n(x)$ is the following :

Theorem 2.2. *The sequence* $(n(n+1)\Delta_n(x))_{n=1}^{\infty}$, $x \in [-1,1]$, *is non-decreasing, i.e.*

$$\Delta_n(x) \geq \frac{n-1}{n+1}\Delta_{n-1}(x), \quad x \in [-1,1],\ n \geq 2.$$

Proof. Let Π_m be the linear space of all polynomials, of degree $\leq m$, having real coefficients. Using a Lagrange-Hermite interpolation formula, every polynomial f from Π_{2n+1} with $f(-1) = f(1) = 0$ may be written as

$$f(x) = (1-x^2)\sum_{k=1}^{n}\left(\frac{P_n(x)}{P_n'(x_k)(x-x_k)}\right)^2 A_k(f;x), \tag{2.2}$$

where

$$A_k(f;x) = \frac{f(x_k) + (x-x_k)f'(x_k)}{1-x_k^2}.$$

Let us observe that

$$(2.3)\qquad \begin{aligned} P_{n-1}(x_k) &= \frac{1-x_k^2}{n}P_n'(x_k), \quad P_{n+1}(x_k) = -\frac{1-x_k^2}{n+1}P_n'(x_k), \\ P_{n-2}(x_k) &= \frac{2n-1}{n(n-1)}x_k(1-x_k^2)P_n'(x_k), \\ P_{n-1}'(x_k) &= P_{n+1}'(x_k) = x_k P_n'(x_k). \end{aligned}$$

In (2.2) let us consider $f \in \Pi_{2n}$, where

$$f(x) = n(n+1)\Delta_n(x) - n(n-1)\Delta_{n-1}(x).$$

From (2.3) we find

$$f(x_k) = \frac{(1-x_k^2)^2}{n}\left[P_n'(x_k)\right]^2, \quad f'(x_k) = 0.$$

Because $A_k(f;x) = \frac{1-x_k^2}{n}\left[P_n'(x_k)\right]^2$, using (2.2) we give

$$f(x) = \frac{1-x^2}{n}\sum_{k=1}^{n}\left(\frac{P_n(x)}{x-x_k}\right)^2(1-x_k^2) \ge 0, \quad x \in [-1,1].$$

Therefore

$$(n+1)\Delta_n(x) - (n-1)\Delta_{n-1}(x) \ge 0 \quad \text{for} \quad x \in [-1,1].$$

$\square$

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