

Journal of Inequalities in Pure and Applied Mathematics

A NEW PROOF OF THE MONOTONICITY OF POWER MEANS

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©2000 Victoria University
ISSN (electronic): 1443-5756
029-04



volume 5, issue 1, article 6,
2004.

*Received 14 February, 2004;
accepted 14 February, 2004.*

Communicated by: P. Bullen

Abstract

Contents



Home Page

Go Back

Close

Quit

Abstract

The author uses certain property of convex functions to prove Bernoulli's inequality and to obtain a simple proof of monotonicity of power means.

2000 Mathematics Subject Classification: 26D15, 26D10.

Key words: Power means, Convex functions.

For positive numbers $a_1, \dots, a_n, p_1, \dots, p_n$, with $p_1 + \dots + p_n = 1$, the weighted power mean of order r , $r \in \mathbb{R}$, is defined by

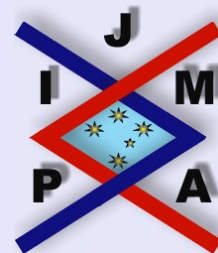
$$(1) \quad M(r) = \begin{cases} \left(\frac{p_1 a_1^r + \dots + p_n a_n^r}{n} \right)^{\frac{1}{r}} & \text{for } r \neq 0, \\ \exp(p_1 \log a_1 + \dots + p_n \log a_n) & \text{for } r = 0. \end{cases}$$

Replacing summation in (1) with integration we obtain integral power means.

It is well known that M is strictly increasing if not all a_i 's are equal. All proofs known to the author use the Cauchy-Schwarz, the Hölder or the Bernoulli inequality (see [1, 2, 3, 4]) to prove this fact.

The aim of this note is to show how to deduce monotonicity of M from convexity of the exponential function. In addition, this method gives a simple proof of Bernoulli's inequality.

The main tool we use is the following well-known property of convex functions, [1, p.26]:



A New Proof of the Monotonicity of Power Means

Alfred Witkowski

Title Page

Contents



Go Back

Close

Quit

Page 2 of 5

Property 1. If f is a (strictly) convex function then the function

$$(2) \quad g(r, s) = \frac{f(s) - f(r)}{s - r}, \quad s \neq r$$

is (strictly) increasing in both variables r and s .

Lemma 1. For $x > 0$ and real r let

$$w_r(x) = \begin{cases} \frac{x^r - 1}{r} & \text{for } r \neq 0, \\ \log x & \text{for } r = 0. \end{cases}$$

Then for $r < s$ we have $w_r(x) \leq w_s(x)$ with equality for $x = 1$ only.

Proof. Applying the Property 1 to the convex function $f(t) = x^t$ we obtain that $g(0, s) = w_s(x)$ is monotone in s for $s \neq 0$. Observation that $\lim_{s \rightarrow 0} w_s(x) = w_0(x)$ completes the proof. Alternatively we may notice that $w_r(x) = \int_1^x t^{r-1} dt$, which is easily seen to be increasing as a function of r . \square

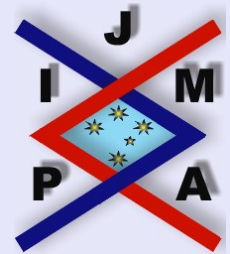
As an immediate consequence we obtain

Corollary 2 (The Bernoulli inequality). For $t > -1$ and $s > 1$ or $s < 0$

$$(1 + t)^s \geq 1 + st,$$

for $0 < s < 1$

$$(1 + t)^s \leq 1 + st.$$



A New Proof of the
Monotonicity of Power Means

Alfred Witkowski

Title Page

Contents



Go Back

Close

Quit

Page 3 of 5

Proof. Substitute $x = 1 + t$ in the inequality between w_s and w_1 . □

Now it is time to formulate the main result.

Let I be a linear functional defined on the subspace of all real-valued functions on X satisfying $I(1) = 1$ and $I(f) \geq 0$ for $f \geq 0$.

For real r and positive f we define the power mean of order r as

$$M(r, f) = \begin{cases} I(f^r)^{1/r} & \text{for } r \neq 0, \\ \exp(I(\log f)) & \text{for } r = 0. \end{cases}$$

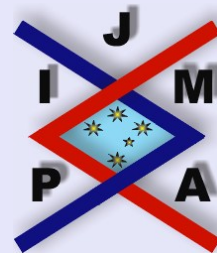
Of course, M may be undefined for some r , but if M is well defined then the following holds:

Theorem 3. *If $r < s$ then $M(r, f) \leq M(s, f)$.*

Proof. If $M(r, f) = 0$ then the conclusion is evident, so we may assume that $M(r, f) > 0$. Substituting $x = f/M(r, f)$ in Lemma 1 we obtain

$$(3) \quad 0 = I\left(w_r\left(\frac{f}{M(r, f)}\right)\right) \leq I\left(w_s\left(\frac{f}{M(r, f)}\right)\right) = \begin{cases} \frac{\left(\frac{M(s, f)}{M(r, f)}\right)^s - 1}{s} & \text{for } s \neq 0, \\ \log \frac{M(0, f)}{M(r, f)} & \text{for } s = 0, \end{cases}$$

which is equivalent to $M(r, f) \leq M(s, f)$. □



A New Proof of the Monotonicity of Power Means

Alfred Witkowski

Title Page

Contents



Go Back

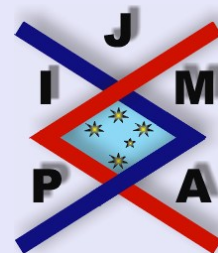
Close

Quit

Page 4 of 5

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A New Proof of the Monotonicity of Power Means

Alfred Witkowski

Title Page

Contents



Go Back

Close

Quit

Page 5 of 5