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A NEW PROOF OF THE MONOTONICITY OF POWER MEANS

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Abstract

The author uses certain property of convex functions to prove Bernoulli's inequality and to obtain a simple proof of monotonicity of power means.

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For positive numbers $a_1, \dots, a_n, p_1, \dots, p_n$, with $p_1 + \cdots + p_n = 1$, the weighted power mean of order r, $r \in \mathbb{R}$, is defined by

$$M(r) = \begin{cases} \left(\dfrac{p_1 a_1^r + \cdots + p_n a_n^r}{n} \right)^{\frac{1}{r}} & \text{for } r \neq 0, \\ \exp(p_1 \log a_1 + \cdots + p_n \log a_n) & \text{for } r = 0. \end{cases} \tag{1}$$

Replacing summation in (1) with integration we obtain integral power means.

It is well known that M is strictly increasing if not all a_i's are equal. All proofs known to the author use the Cauchy-Schwarz, the Hölder or the Bernoulli inequality (see [1, 2, 3, 4]) to prove this fact.

The aim of this note is to show how to deduce monotonicity of M from convexity of the exponential function. In addition, this method gives a simple proof of Bernoulli's inequality.

The main tool we use is the following well-known property of convex functions, [1, p.26]:

Property 1. *If f is a (strictly) convex function then the function*

(2) $$g(r,s) = \frac{f(s)-f(r)}{s-r}, \quad s \neq r$$

is (strictly) increasing in both variables r and s.

Lemma 1. *For $x > 0$ and real r let*

$$w_r(x) = \begin{cases} \dfrac{x^r - 1}{r} & \text{for } r \neq 0, \\ \log x & \text{for } r = 0. \end{cases}$$

Then for $r < s$ we have $w_r(x) \leq w_s(x)$ with equality for $x = 1$ only.

Proof. Applying the Property 1 to the convex function $f(t) = x^t$ we obtain that $g(0,s) = w_s(x)$ is monotone in s for $s \neq 0$. Observation that $\lim_{s\to 0} w_s(x) = w_0(x)$ completes the proof. Alternatively we may notice that $w_r(x) = \int_1^x t^{r-1}dt$, which is easily seen to be increasing as a function of r. □

As an immediate consequence we obtain

Corollary 2 (The Bernoulli inequality). *For $t > -1$ and $s > 1$ or $s < 0$*

$$(1+t)^s \geq 1 + st,$$

for $0 < s < 1$

$$(1+t)^s \leq 1 + st.$$

Proof. Substitute $x = 1 + t$ in the inequality between w_s and w_1. □

Now it is time to formulate the main result.

Let I be a linear functional defined on the subspace of all real-valued functions on X satisfying $I(1) = 1$ and $I(f) \geq 0$ for $f \geq 0$.

For real r and positive f we define the power mean of order r as

$$M(r,f) = \begin{cases} I(f^r)^{1/r} & \text{for} \quad r \neq 0, \\ \exp(I(\log f)) & \text{for} \quad r = 0. \end{cases}$$

Of course, M may be undefined for some r, but if M is well defined then the following holds:

Theorem 3. *If $r < s$ then $M(r,f) \leq M(s,f)$.*

Proof. If $M(r,f) = 0$ then the conclusion is evident, so we may assume that $M(r,f) > 0$. Substituting $x = f/M(r,f)$ in Lemma 1 we obtain

$$0 = I\left(w_r\left(\frac{f}{M(r,f)}\right)\right)$$

$$\leq I\left(w_s\left(\frac{f}{M(r,f)}\right)\right) = \begin{cases} \dfrac{\left(\frac{M(s,f)}{M(r,f)}\right)^s - 1}{s} & \text{for} \quad s \neq 0, \\ \log \dfrac{M(0,f)}{M(r,f)} & \text{for} \quad s = 0, \end{cases} \tag{3}$$

which is equivalent to $M(r,f) \leq M(s,f)$. □

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