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COEFFICIENT INEQUALITIES FOR CERTAIN CLASSES OF RUSCHEWEYH TYPE ANALYTIC FUNCTIONS

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ABSTRACT. A class of univalent functions which provides an interesting transition from starlike functions to convex functions is defined by making use of the Ruscheweyh derivative. Some coefficient inequalities for functions in these classes are discussed which generalize the coefficient inequalities considered by Owa, Polatoğlu and Yavuz.

Key words and phrases: Convolution, Ruscheweyh derivative, Uniformly starlike and Uniformly convex.

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1. INTRODUCTION

Let $\mathcal{N}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.

We designate $\mathcal{V}(\beta, b, \delta)$ as the subclass of $\mathcal{N}$ consisting of functions f obeying the condition

$$\Re\left\{1 - \frac{2}{b} + \frac{2}{b}\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)}\right\} > \beta \tag{1.2}$$

where, $b \neq 0,\ \delta > -1,\ 0 \leq \beta < 1$ and $D^{\delta}f$ is the Rushceweyh derivative of f [5] given by,

$$D^{\delta}f(z) = \frac{z}{(1-z)^{1+\delta}} * f(z) = z + \sum_{n=2}^{\infty} a_n B_n(\delta) z^n, \tag{1.3}$$

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where $*$ stands for the convolution or Hadamard product of two power series and

$$B_n(\delta) = \frac{(\delta+1)(\delta+2)\cdots(\delta+n-1)}{(n-1)!}.$$

This class is obtained by putting $k=2$ and $\lambda=0$ in the class $\mathcal{V}_k^{\lambda}(\beta,b,\delta)$ introduced by Latha and Nanjunda Rao [2]. The class $\mathcal{V}_k^{\lambda}(\beta,b,\delta)$ is of special interest for it contains many well known as well as new classes of analytic univalent functions studied in literature. It provides a transition from starlike functions to convex functions. More specifically, $\mathcal{V}_2^0(\beta,2,0)$ is the family of starlike functions of order β and $\mathcal{V}_2^0(\beta,1,1)$ is the class of convex functions of order β. Shams, Kulkarni and Jahangiri [6] introduced the subclass $\mathcal{SD}(\alpha,\beta)$ of $\mathcal{N}$ consisting of functions f satisfying

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha\left|\frac{zf'(z)}{f(z)}-1\right|+\beta \tag{1.4}$$

for some $\alpha \ge 0,\ 0\le\beta<1$ and $z\in\mathcal{U}$.

The class $\mathcal{KD}(\alpha,\beta),$ another subclass of $\mathcal{N},$ is defined as the set of all functions f obeying

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \alpha\left|\frac{zf''(z)}{f'(z)}-1\right|+\beta \tag{1.5}$$

for some $\alpha \ge 0,\ 0\le\beta<1$ and $z\in\mathcal{U}$.

We introduce the class $\mathcal{VD}(\alpha,\beta,b,\delta)$ as the subclass of $\mathcal{N}$ consisting of functions f which satisfy

$$\Re\left\{1-\frac{2}{b}+\frac{2}{b}\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)}\right\} > \alpha\left|\frac{2}{b}\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)}-1\right|+\beta$$

where, $b\neq 0,\ \alpha\ge 0,$ and $0\le\beta<1$.
For the parametric values $b=2,\ \delta=0$ and $b=\delta=1$ we obtain the classes $\mathcal{SD}(\alpha,\beta)$ and $\mathcal{KD}(\alpha,\beta)$ respectively.

2. MAIN RESULTS

We prove some coefficient inequalities for functions in the class $\mathcal{VD}(\alpha,\beta,b,\delta)$.

Theorem 2.1. *If* $f(z)\in\mathcal{VD}(\alpha,\beta,b,\delta)$ *with* $0\le\alpha\le\beta$, *or,* $\alpha>\frac{1+\beta}{2}$, *then* $f(z)\in\mathcal{V}\left(\frac{\beta-\alpha}{1-\alpha},b,\delta\right)$.

Proof. Since $\Re\{\omega\}\le|\omega|$ for any complex number $\omega,$ $f(z)\in\mathcal{VD}(\alpha,\beta,b,\delta)$ implies that

$$\Re\left\{1-\frac{2}{b}+\frac{2}{b}\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)}\right\} > \alpha\left|\frac{2}{b}\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)}-\frac{2}{b}\right|+\beta. \tag{2.1}$$

Equivalently,

$$\Re\left\{1-\frac{2}{b}+\frac{2}{b}\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)}\right\} > \frac{\beta-\alpha}{1-\alpha}, \quad (z\in\mathcal{U}). \tag{2.2}$$

If $0\le\alpha\le\beta,$ we have, $0\le\frac{\beta-\alpha}{1-\alpha}<1,$ and if $\alpha>\frac{1+\beta}{2},$ then we have $-1<\frac{\alpha-\beta}{\alpha-1}\le 0.$ ☐

Corollary 2.2. *For the parametric values* $b=2$ *and* $\delta=0,$ *we get Theorem* 2.1 *in* [3] *which reads as:*
If $f(z)\in\mathcal{SD}(\alpha,\beta)$ *with* $0\le\alpha\le\beta,$ *or,* $\alpha>\frac{1+\beta}{2},$ *then* $f(z)\in\mathcal{S}^*\left(\frac{\beta-\alpha}{1-\alpha}\right)$.

Corollary 2.3. *The parametric values* $b=\delta=1,$ *yield the Corollary* 2.2 *in* [3] *stated as:*
If $f(z)\in\mathcal{KD}(\alpha,\beta)$ *with* $0\le\alpha\le\beta,$ *or,* $\alpha>\frac{1+\beta}{2},$ *then* $f(z)\in\mathcal{K}\left(\frac{\beta-\alpha}{1-\alpha}\right)$.

Theorem 2.4. *If* $f(z) \in \mathcal{VD}(\alpha, \beta, b, \delta)$ *then,*

$$|a_2| \leq \frac{b(1-\beta)}{|1-\alpha|} \tag{2.3}$$

and

$$|a_n| \leq \frac{b(1-\beta)(\delta+1)}{(n-1)|1-\alpha|B_n(\delta)} \prod_{j=1}^{n-2} \left(1 + \frac{b(\delta+1)(1-\beta)}{j|1-\alpha|}\right), \quad (n \geq 3). \tag{2.4}$$

Proof. We note that for $f(z) \in \mathcal{VD}(\alpha, \beta, b, \delta)$,

$$\Re\left\{1 - \frac{2}{b} + \frac{2}{b}\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)}\right\} > \frac{\beta-\alpha}{1-\alpha}, \quad (z \in \mathcal{U}).$$

We define the function $p(z)$ by

$$p(z) = \frac{(1-\alpha)\left[1 - \frac{2}{b} + \frac{2}{b}\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)}\right] - (\beta-\alpha)}{(1-\beta)}, \quad (z \in \mathcal{U}). \tag{2.5}$$

Then, $p(z)$ is analytic in $\mathcal{U}$ with $p(0) = 1$ and $\Re\{p(z)\} > 0$ and $z \in \mathcal{U}$.
Let $p(z) = 1 + p_1 z + p_1 z^2 + \cdots$. We have

$$1 - \frac{2}{b} + \frac{2}{b}\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)} = 1 + \left(\frac{1-\beta}{1-\alpha}\right)\sum_{n=1}^{\infty} p_n z^n. \tag{2.6}$$

That is,

$$2(D^{\delta+1}f(z) - D^{\delta}f(z)) = bD^{\delta}f(z)\left(\frac{1-\beta}{1-\alpha}\sum_{n=1}^{\infty} p_n z^n\right).$$

which implies that

$$\frac{2B_n(\delta)(n-1)a_n}{(\delta+1)} = \frac{b(1-\beta)}{(1-\alpha)}\left[p_{n-1} + B_2(\delta) + a_2 p_{n-2} + B_3(\delta)a_3 p_{n-3} + \cdots + B_{n-1}(\delta)a_{n-1}p_1\right].$$

Applying the coefficient estimates $|p_n| \leq 2$ for Carathéodory functions [1], we obtain,

$$|a_n| \leq \frac{b(1-\beta)(\delta+1)}{|1-\alpha|(n-1)B_n(\delta)}\left[1 + B_2(\delta)|a_2| + B_3(\delta)|a_3| + \cdots + B_{n-1}(\delta)|a_{n-1}|\right]. \tag{2.7}$$

For $n = 2$, $|a_2| \leq \frac{b(1-\beta)}{|1-\alpha|}$, which proves (2.3).
For $n = 3$,

$$|a_3| \leq \frac{b(1-\beta)(\delta+1)}{2|1-\alpha|B_3(\delta)}\left[1 + \frac{b(1-\beta)(\delta+1)}{|1-\alpha|}\right].$$

Therefore (2.4) holds for $n = 3$.
Assume that (2.4) is true for $n = k$.

Consider,

$$\begin{aligned}|a_{k+1}| &\leq \frac{b(1-\beta)(\delta+1)}{kB_{k+1}(\delta)}\left\{\left(1+\frac{b(1-\beta)(\delta+1)}{|1-\alpha|}\right)\right.\\ &\qquad + \frac{b(1-\beta)(\delta+1)}{|1-\alpha|B_2(\delta)}\left(1+\frac{b(1-\beta)(\delta+1)}{|1-\alpha|}\right)\\ &\qquad \left. + \cdots + \frac{b(1-\beta)(\delta+1)}{(k-1)!|1-\alpha|B_k(\delta)}\prod_{j=1}^{k-2}\left(1+\frac{b(1-\beta)(\delta+1)}{j(|1-\alpha|)}\right)\right\}\\ &= \frac{b(1-\beta)(\delta+1)}{kB_{k+1}(\delta)}\prod_{j=1}^{k-1}\left[1+\frac{b(1-\beta)(\delta+1)}{j(|1-\alpha|)}\right].\end{aligned}$$

Therefore, the result is true for $n = k+1$. Using mathematical induction, (2.4) holds true for all $n \geq 3$. □

Corollary 2.5. *The parametric values* $b = 2$ *and* $\delta = 0$ *yield Theorem* 2.3 *in* [3] *which states that:*
If $f(z) \in \mathcal{SD}(\alpha, \beta)$, *then*

$$|a_2| \leq \frac{2(1-\beta)}{|1-\alpha|} \tag{2.8}$$

and

$$|a_n| \leq \frac{2(1-\beta)}{(n-1)|1-\alpha|}\prod_{j=1}^{n-2}\left(1+\frac{2(1-\beta)}{j|1-\alpha|}\right), \quad (n \geq 3). \tag{2.9}$$

Corollary 2.6. *Putting* $\alpha = 0$ *in Corollary* 2.5, *we get*

$$|a_n| \leq \frac{\prod_{j=1}^{n}(j-2\beta)}{(n-1)!}, \quad (n \geq 2), \tag{2.10}$$

a result by Robertson [4].

Corollary 2.7. *For the parametric values* $b = \delta = 1$ *we obtain Corollary* 2.5 *in* [3] *given by:*
If $f(z) \in \mathcal{KD}(\alpha, \beta)$ *then,*

$$|a_2| \leq \frac{(1-\beta)}{|1-\alpha|} \tag{2.11}$$

and

$$|a_n| \leq \frac{2(1-\beta)}{n(n-1)|1-\alpha|}\prod_{j=1}^{n-2}\left(1+\frac{2(1-\beta)}{j|1-\alpha|}\right), \quad (n \geq 3). \tag{2.12}$$

Corollary 2.8. *Letting* $\alpha = 0$ *in Corollary* 2.7, *we get the inequality by Robertson* [4] *given by:*

$$|a_n| \leq \frac{\prod_{j=1}^{n}(j-2\beta)}{n!}, \quad (n \geq 2). \tag{2.13}$$

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