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COMPLETE SYSTEMS OF INEQUALITIES

MARÍA A. HERNÁNDEZ CIFRE, GUILLERMO SALINAS, AND SALVADOR SEGURA GOMIS

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE MURCIA, CAMPUS DE ESPINARDO, 30071 MURCIA, SPAIN
mhcifre@um.es

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE MURCIA, CAMPUS DE ESPINARDO, 30100 MURCIA, SPAIN
guisamar@navegalia.com

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO Y MATEMÁTICA APLICADA, UNIVERSIDAD DE ALICANTE, CAMPUS DE SAN VICENTE DEL RASPEIG, APDO. DE CORREOS 99, E-03080 ALICANTE, SPAIN
Salvador.Segura@ua.es

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ABSTRACT. In this paper we summarize the known results and the main tools concerning complete systems of inequalities for families of convex sets. We discuss also the possibility of using these systems to determine particular subfamilies of planar convex sets with specific geometric significance. We also analyze complete systems of inequalities for 3-rotationally symmetric planar convex sets concerning the area, the perimeter, the circumradius, the inradius, the diameter and the minimal width; we give a list of new inequalities concerning these parameters and we point out which are the cases that are still open.

Key words and phrases: Inequality, complete system, planar convex set, area, perimeter, diameter, width, inradius, circumradius.

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1. INTRODUCTION

For many years mathematicians have been interested in inequalities involving geometric functionals of convex figures ([11]). These inequalities connect several geometric quantities and in many cases determine the extremal sets which satisfy the equality conditions.

Each new inequality obtained is interesting on its own, but it is also possible to ask if a finite collection of inequalities concerning several geometric magnitudes is large enough to determine the existence of the figure. Such a collection is called a *complete system of inequalities*: a system of inequalities relating all the geometric characteristics such that for any set of numbers satisfying those conditions, a convex set with these values of the characteristics exists.

Historically the first mathematician who studied this type of problems was Blaschke ([1]). He considered a compact convex set K in the Euclidean 3-space $\mathbb{E}^3$, with volume $V = V(K)$, surface area $F = F(K)$, and integral of the mean curvature $M = M(K)$. He asked for a

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characterization of the set of all points in $\mathbb{E}^3$ of the form $(V(K), F(K), M(K))$ as K ranges on the family of all compact convex sets in $\mathbb{E}^3$. Some recent progress has been made by Sangwine-Yager ([9]) and Martínez-Maure ([8]), but the problem still remains open.

A related family of problems was proposed by Santaló ([10]): for a compact convex set K in $\mathbb{E}^2$, let $A = A(K)$, $p = p(K)$, $d = d(K)$, $\omega = \omega(K)$, $R = R(K)$ and $r = r(K)$ denote the area, perimeter, diameter, minimum width, circumradius and inradius of K, respectively. The problem is to find a complete system of inequalities for any triple of $\{A, p, d, \omega, R, r\}$.

Santaló provided the solution for (A, p, ω), (A, p, r), (A, p, R), (A, d, ω), (p, d, ω) and (d, r, R). Recently ([4], [5], [6]) solutions have been found for the cases (ω, R, r), (d, ω, R), (d, ω, r), (A, d, R), (p, d, R). There are still nine open cases: (A, p, d), (A, d, r), (A, ω, R), (A, ω, r), (A, R, r), (p, d, r), (p, ω, R), (p, ω, r), (p, R, r).

Let (a_1, a_2, a_3) be any triple of the measures that we are considering. The problem of finding a complete system of inequalities for (a_1, a_2, a_3) can be expressed by mapping each compact convex set K to a point $(x, y) \in [0, 1] \times [0, 1]$. In this diagram x and y represent particular functionals of two of the measures a_1, a_2 and a_3 which are invariant under dilatations.

Blaschke convergence theorem states that an infinite uniformly bounded family of compact convex sets converges in the Hausdorff metric to a convex set. So by Blaschke theorem, the range of this map $D(K)$ is a closed subset of the square $[0, 1] \times [0, 1]$. It is also easy to prove that $D(K)$ is arcwise connected. Each of the optimal inequalities relating a_1, a_2, a_3 determines part of the boundary of $D(K)$ if and only if these inequalities form a complete system; if one inequality is missing, some part of the boundary of $D(K)$ remains unknown.

In order to know $D(K)$ at least four inequalities are needed, two of them to determine the coordinate functions x and y; the third one to determine the “upper" part of the boundary and the fourth one to determine the “lower" part of the boundary. Sometimes, either the “upper" part, or the “lower" part or both of them require more than one inequality. So the number of four inequalities is always necessary but may be sometimes not sufficient to determine a complete system.

Very often the easiest inequalities involving three geometric functionals a_1, a_2, a_3 are the inequalities concerning pairs of these functionals of the type:

$$a_i^{m(i,j)} \leq \alpha_{ij}\, a_j,$$

the exponent $m(i, j)$ guarantees that the image of the sets is preserved under dilatations.

So, although there is not a unique choice of the coordinate functions x and y, there are at most six canonical choices to express these coordinates as quotients of the type $a_i^{m(i,j)}/\alpha_{ij} a_j$ which guarantee that $D(K) \subset [0, 1] \times [0, 1]$. The difference among these six choices (when the six cases are possible) does not have any relevant geometric significance.

If instead of considering *triples* of measures we consider *pairs* of measures, then the Blaschke-Santaló diagram would be a segment of a straight line. On the other hand, if we consider groups of *four* magnitudes (or more) the Blaschke-Santaló diagram would be part of the unit cube (or hypercube).

2. AN EXAMPLE: THE COMPLETE SYSTEM OF INEQUALITIES FOR (A, d, ω)

For the area, the diameter, and the width of a compact convex set K, the relationships between pairs of these geometric measures are:

(2.1)	$4A \leq \pi d^2$	Equality for the circle
(2.2)	$\omega^2 \leq \sqrt{3} A$	Equality for the equilateral triangle
(2.3)	$\omega \leq d$	Equality for sets of constant width

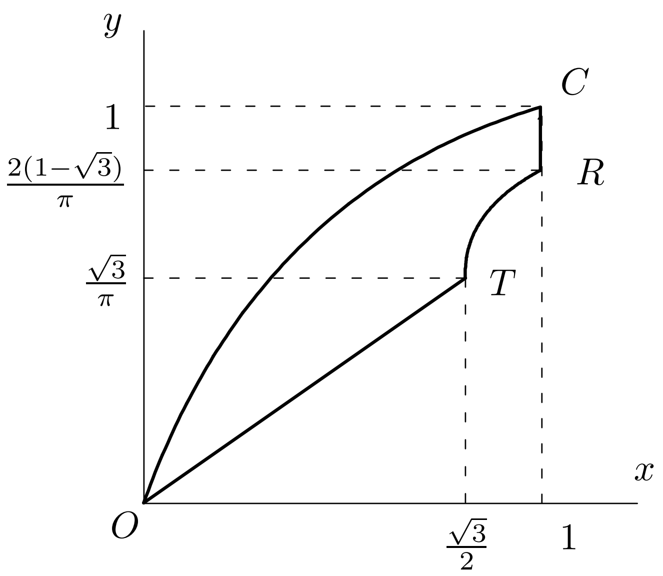

Figure 2.1: Blaschke-Santaló Diagram for the case (A, d, ω).

And the relationships between three of those measures are:

$$2A \leq \omega\sqrt{d^2-\omega^2} + d^2 \arcsin\left(\frac{\omega}{d}\right), \tag{2.4}$$

equality for the intersection of a disk and a symmetrically placed strip,

$$d\omega \leq 2A, \quad \text{if } 2\omega \leq \sqrt{3}d \tag{2.5}$$

equality for the triangles,

$$A \geq 3\omega\left[\sqrt{d^2-\omega^2} + \omega\left(\arcsin\left(\frac{\omega}{d}\right) - \frac{\pi}{3}\right)\right] - \frac{\sqrt{3}}{2}d^2, \quad \text{if } 2\omega > \sqrt{3}d \tag{2.6}$$

equality for the Yamanouti sets.

Let

$$x = \frac{\omega}{d} \quad \text{and} \quad y = \frac{4A}{\pi d^2}.$$

Clearly, from (2.3) and (2.1), $0 \leq x \leq 1$ and $0 \leq y \leq 1$. From inequality (2.4) we obtain

$$y \leq \frac{2}{\pi}(x\sqrt{1-x^2} + \arcsin x) \quad \text{for all } 0 \leq x \leq 1.$$

The curve

$$y = \frac{2}{\pi}(x\sqrt{1-x^2} + \arcsin x)$$

determines the upper part of the boundary of $D(K)$. This curve connects point $O = (0,0)$ (corresponding to line segments) with point $C = (1,1)$ (corresponding to the circle), and the intersections of a disk and a symmetrically placed strip are mapped to the points of this curve. The lower part of the boundary is determined by two curves obtained from inequalities (2.5) and (2.6). The first one is the line segment

$$y = \frac{2}{\pi}x \quad \text{where} \quad 0 \leq x \leq \frac{\sqrt{3}}{2},$$

which joins the points O and $T = (\sqrt{3}/2, \sqrt{3}/\pi)$ (equilateral triangle), and its points represent the triangles. The second curve is

$$y = \frac{12}{\pi}x\left[\sqrt{1-x^2} + x\left(\arcsin x - \frac{\pi}{3}\right)\right] - 2\frac{\sqrt{3}}{\pi} \quad \text{where } \frac{\sqrt{3}}{2} \leq x \leq 1.$$

This curve completes the lower part of the boundary, from the point T to $R = (1, 2(1 - \sqrt{3}/\pi))$ (Reuleaux triangle). The Yamanouti sets are mapped to the points of this curve.

The boundary of $D(K)$ is completed with the line segment $\overline{RC}$ which represents the sets of constant width, from the Reuleaux triangle (minimum area) to the circle (maximum area).

Finally we have to see that the domain $D(K)$ is simply connected, i.e., there are convex sets which are mapped to any of its interior points.

Let us consider the following two assertions:

(1) Let K be a compact convex set in the plane and $K^c = \frac{1}{2}(K - K)$ (the centrally symmetral set of K). If we consider

$$K_\lambda = \lambda K + (1 - \lambda) K^c$$

then, for all $0 \le \lambda \le 1$ the convex set K_λ has the same width and diameter as K (see [5]).

(2) Let K be a centrally symmetric convex set. Then K is contained in the intersection of a disk and a symmetrically placed strip, S, with the same width and diameter as K. Let

$$K_\lambda = \lambda K + (1 - \lambda) S.$$

Then for all $0 \le \lambda \le 1$, the convex set K_λ has the same width and diameter as K.

Then it is easy to find examples of convex sets which are mapped into any of the interior points of $D(K)$.

So, the inequalities (2.1), (2.3), (2.4), (2.5), (2.6) determine a complete system of inequalities for the case (A, d, ω).

3. GOOD FAMILIES FOR COMPLETE SYSTEMS OF INEQUALITIES

Although the concept of complete system of inequalities was developed for general convex sets, it is also interesting to characterize other families of convex sets. Burago and Zalgaller ([2]) state the problem as: “Fixing *any class* of figures and any finite set of numerical characteristics of those figures... finding a complete system of inequalities between them".

So it is interesting to ask if all the classes of figures can be characterized by complete systems of inequalities. In general the answer to this question turns out to be negative. For instance, if we consider the family of all convex regular polygons and any triple of the classical geometric magnitudes $\{A, p, d, \omega, R, r\}$, the image of this family under Blaschke-Santaló map is a sequence of points inside the unit square, which certainly cannot be determined by a finite number of curves (inequalities). On the other hand, if we consider the family of all convex polygons, by the polygonal approximation theorem, any convex set can be approximated by a sequence of polygons. Then the image of this family under the Blaschke-Santaló map is not very much different from the image of the family of general convex sets (in many cases the difference is only part of the boundary of the diagram); so it does not involve the number of inequalities considered, but only if these inequalities are strict or not. The question makes sense if we consider general (not necessarily convex) sets, but in this case there are some technical difficulties:

i) The classical functionals have nice monotonicity properties for convex sets, but not in the general cases.
ii) Geometric symmetrizations behave well for convex sets; these tools are important to obtain in some cases the inequalities.

So, which kind of families can be characterized in an interesting way by complete systems of inequalities? Many well-known families are included here. For instance, it is possible to obtain good results for families with special kinds of symmetry (centrally symmetric convex sets, 3-rotationally symmetric convex sets, convex sets which are symmetric with respect to a straight line...).

It seems also interesting (although no result has been yet obtained) to consider families of sets which satisfy some lattice constraints.

For some of the families that we have already mentioned we are going to make the following remarks:

1) Centrally symmetric planar convex sets:
   If we consider the six classic geometric magnitudes in this family, then

$$\omega = 2r \quad \text{and} \quad d = 2R,$$

   So, instead of having 6 free parameters we just have 4, and there are only 4 possible cases of complete systems of inequalities. These cases have been solved in [7].
2) 3-rotationally symmetric planar convex sets:
   This family turns out to be very interesting because there is no reduction in the number of free parameters, and so there are 20 cases. The knowledge of the Blaschke-Santaló diagram for these cases helps us understand the problem in the cases that are still open for general planar convex sets.

Because of this reason we are going to summarize in the next section the known results for this last case.

4. COMPLETE SYSTEMS OF INEQUALITIES FOR 3-ROTATIONALLY SYMMETRIC PLANAR CONVEX SETS

Besides being a good family to be characterized for complete systems of inequalities, 3-rotationally symmetric convex sets are also interesting in their own right.

i) They provide extremal sets for many optimization problems for general convex sets.
ii) 3-rotational symmetry is preserved by many interesting geometric transformations, like Minkowski addition and others.
iii) They provide interesting solutions for lattice problems or for packing and covering problems.

If we continue considering pairs of the 6 classical geometric magnitudes $\{A, p, \omega, d, R, r\}$, then the 15 cases of complete systems of inequalities are completely solved. In the table 4.1 we provide the inequalities that determine these cases and the extremal sets for these inequalities.

Let us remark that for the 3-rotationally symmetric case we have two (finite) inequalities for each pair of magnitudes (which is completely different from the general case in which this happens only in four cases) ([10]); this is because 3-rotational symmetry determines more control on the convex sets in the sense that it does not allow “elongated" sets. The proofs of these inequalities can be found in section 5.

If we now consider triples of the magnitudes, the situation becomes more interesting.

Table 4.2 lists all the known inequalities and the corresponding extremal sets. Proofs of these inequalities can be found in section 5.

	Parameters	**Inequalities**	**Extremal Sets**
1)	A, ω	$\frac{\omega^2}{\sqrt{3}} \le A \le \frac{\sqrt{3}}{2}\omega^2$	$T \mid \mathcal{H}$
2)	A, r	$\pi r^2 \le A \le 3\sqrt{3}r^2$	$C \mid T$
3)	A, R	$\frac{3}{4}\sqrt{3}R^2 \le A \le \pi R^2$	$T \mid C$
4)	A, p	$4\pi A \le p^2 \le 12\sqrt{3}A$	$C \mid T$
5)	A, d	$\frac{4}{\pi}A \le d^2 \le \frac{4}{\sqrt{3}}A$	$C \mid T$
6)	p, r	$2\pi r \le p \le 6\sqrt{3}r$	$C \mid T$
7)	p, R	$3\sqrt{3}R \le p \le 2\pi R$	$T \mid C$
8)	p, ω	$\pi\omega \le p \le 2\sqrt{3}\omega$	$W \mid H, T$
9)	p, d	$3d \le p \le \pi d$	$H, T \mid W$
10)	d, r	$2r \le d \le 2\sqrt{3}r$	$C \mid T$
11)	d, R	$\sqrt{3}R \le d \le 2R$	$Y^* \mid {R_6}^*$
12)	d, ω	$\omega \le d \le \frac{2}{\sqrt{3}}\omega$	$W \mid H, T$
13)	ω, r	$2r \le \omega \le 3r$	${R_6}^* \mid T$
14)	ω, R	$\frac{3}{2}R \le \omega \le 2R$	$T \mid C$
15)	R, r	$r \le R \le 2r$	$C \mid T$

Table 4.1: Inequalities for 3-rotationally symmetric convex sets relating 2 parameters.

* There are more extremal sets

Note on Extremal Sets: The sets which are at the left of the vertical bar are extremal sets for the left inequality; the sets which are at the right of the vertical bar are extremal sets for the right inequality. The sets are described after Table 4.2.

	Param.	**Condition**	**Inequality**	**Ext. Sets**
16)	A, d, p		$8A \le 3\sqrt{3}d^2 + \frac{p}{3\alpha}(p - 3d\cos\alpha)$ **(1)**	$\mathcal{H}_C$
17)	A, d, r	$2r \le d \le \frac{4}{\sqrt{3}}r$	$A \ge 3r[\sqrt{d^2 - 4r^2} + r(\frac{\pi}{3} - 2\arccos(\frac{2r}{d}))]$	C_B^6
18)	A, d, R		$A \ge \frac{3\sqrt{3}}{4}(d^2 - 2R^2)$ $A \le \frac{3\sqrt{3}}{4}R^2 + 3\int_{R/2}^{x_0}\sqrt{d^2 - 4(x+a)^2}\,dx + +6\int_{x_0}^{d-R}\sqrt{d^2 - (x+R)^2}\,dx$ **(2)**	H, T W
19)	A, d, ω		$A \ge 3\omega[\sqrt{d^2 - \omega^2} + \omega(\arcsin(\frac{\omega}{d}) - \frac{\pi}{3})] - \frac{\sqrt{3}}{2}d^2$ $A \le 3[\frac{\omega}{2}\sqrt{d^2 - \omega^2} + \frac{d^2}{4}(\frac{\pi}{3} - 2\arccos(\frac{\omega}{d}))]$	Y $\mathcal{H} \cap C$
20)	A, p, r		$pr \le 2A$ $4(3\sqrt{3} - \pi)A \le 12\sqrt{3}r(p - \pi r) - p^2$	${C_B}^*$ T_R
21)	A, p, R		$A \le \frac{3\sqrt{3}}{4}R^2 + \frac{p}{12\phi}(p - 3\sqrt{3}R\cos\phi)$ **(3)**	T_C
22)	A, p, ω		$2A \ge \omega p - \sqrt{3}\omega^2\sec^2\theta$ **(4)** $4(2\sqrt{3} - \pi)A \le 2\sqrt{3}\omega(2p - \pi\omega) - p^2$	Y $\mathcal{H}_R$
23)	A, R, r		$A \ge 3[r\sqrt{R^2 - r^2} + r^2(\frac{\pi}{3} - \arcsin(\frac{\sqrt{R^2-r^2}}{R}))]$ $A \le R^2(3\arcsin(\frac{r}{R}) + 3\frac{r}{R^2}\sqrt{R^2 - r^2} - \frac{\pi}{2})$	C_B^3 $T \cap C$
24)	A, R, ω	$\frac{3}{2}R \le \omega \le \sqrt{3}R$ $\sqrt{3}R \le \omega \le 2R$	$A \le \sqrt{3}\omega^2 - \frac{3\sqrt{3}}{2}R^2$ $A \le 3[\frac{\omega}{2}\sqrt{4R^2 - \omega^2} + R^2(\frac{\pi}{3} - 2\arccos(\frac{\omega}{2R}))]$	H, T $\mathcal{H} \cap C$
25)	p, d, r		$p \ge 6\sqrt{d^2 - 4r^2} + 2r(\pi - 6\arccos(\frac{2r}{d}))$	C_B^6

	Param.	Condition	Inequality	Ext. Sets
26)	p, d, ω		$p \le 6[\sqrt{d^2-\omega^2} + d(\frac{\pi}{6} - \arccos(\frac{\omega}{d}))]$	$\mathcal{H} \cap C$
			$p \ge 6\sqrt{d^2-\omega^2} + \omega(\pi - 6\arccos(\frac{\omega}{d}))$	C_B^6
27)	d, R, r	$\frac{R}{2} \le r \le (\sqrt{3}-1)R$	$d \ge \sqrt{3}R$	Y^*
		$(\sqrt{3}-1)R \le r \le R$	$d \ge R + r$	W^*
		$\frac{R}{2} \le r \le \frac{\sqrt{3}}{2}R$	$d \le \sqrt{3}r + \sqrt{R^2-r^2}$	$T \cap C$, H^*
		$\frac{\sqrt{3}}{2}R \le r \le R$	$d \le 2R$	R_6^*
28)	d, r, ω		$\omega - r \le \frac{\sqrt{3}}{3}d$	Y
29)	p, R, r		$p \ge 6[\sqrt{R^2-r^2} + r(\frac{\pi}{3} - \arcsin(\frac{\sqrt{R^2-r^2}}{R}))]$	C_B^3
			$p \le 6[\sqrt{R^2-r^2} + R(\frac{\pi}{3} - \arcsin(\frac{\sqrt{R^2-r^2}}{R}))]$	$T \cap C$
30)	p, R, ω	$\frac{3}{2}R \le \omega \le \sqrt{3}R$	$p \le 2\sqrt{3}\omega$	H, T
		$\sqrt{3}R \le \omega \le 2R$	$p \le 6[\sqrt{4R^2-\omega^2} + R(\frac{\pi}{3} - 2\arccos(\frac{\omega}{2R}))]$	$\mathcal{H} \cap C$
		$\frac{3}{2}R \le \omega \le \sqrt{3}R$	$p \ge 6[\sqrt{3R^2-\omega^2} + \omega(\frac{\pi}{6} - \arccos(\frac{\omega}{\sqrt{3}R}))]$	Y
31)	p, r, ω	$\frac{3+\sqrt{3}}{2}r \le \omega \le 3r$	$p \ge 6[\sqrt{3(\omega-r)^2-\omega^2} + \omega(\frac{\pi}{6} - \arccos(\frac{\omega}{\sqrt{3}(\omega-r)}))]$	Y
32)	ω, R, r	$r \le R \le \frac{2}{\sqrt{3}}r$	$\omega \ge 2r$	R_6^*
		$\frac{2}{\sqrt{3}}r \le R \le 2r$	$\omega \ge \frac{\sqrt{3}}{2}(\sqrt{3}r + \sqrt{R^2-r^2})$	C_B^3, H^*
			$\omega \le R + r$	W^*

Table 4.2: Inequalities for 3-rotationally symmetric convex sets relating 3 parameters.

* There are more extremal sets

(1) $p\sin\alpha = 3\alpha d$ (2) $a = \frac{\sqrt{d^2-3R^2}-R}{2}$, $x_0 = \frac{2d^2-3R^2-R\sqrt{d^2-3R^2}}{2\left(3R-\sqrt{d^2-3R^2}\right)}$

(3) $p\sin\phi = 3\sqrt{3}R\phi$ (4) $6\omega(\tan\theta - \theta) = p - \pi\omega$

Extremal Sets:

C	Disk	T	Equilateral triangle
$\mathcal{H}$	Regular hexagon	W	Constant width sets
C_B	3-Rotationally symmetric cap bodies (convex hull of the circle and a finite number of points)	H	3-Rotationally symmetric hexagon with parallel opposite sides
C_B^3	Cap bodies with three vertices	C_B^6	Cap bodies with six vertices
$T \cap C$	Intersection of T with a disk with the same center	$\mathcal{H} \cap C$	Intersection of $\mathcal{H}$ with a disk with the same center
T_C	Convex sets obtained from T replacing the edges by three equal circular arcs	$\mathcal{H}_C$	Convex sets obtained from $\mathcal{H}$ replacing the edges by six equal circular arcs
T_R	Convex set obtained from T replacing the vertices by three equal circular arcs tangent to the edges	H_R	Convex set obtained from $\mathcal{H}$ replacing the vertices by six equal circular arcs tangent to the edges
R_6	6-Rotationally symmetric convex sets	Y	Yamanouti sets

5. PROOFS OF THE INEQUALITIES

In this section we are going to give a sketch of the proofs of the inequalities collected in tables 4.1 and 4.2.

For the sake of brevity we will label the inequalities of tables 4.1 and 4.2 in the following way:

In table 4.1, for each numbered case we will label with L the inequality corresponding to the left-hand side and with R the inequality corresponding to the right-hand side.

In table 4.2, for each numbered case we will enumerate in order the corresponding inequalities (for instance, (27.1) corresponds to the inequality $d \geq \sqrt{3}R$, (27.2) corresponds to the inequality $d \geq R + r$, and so on).

First, we are going to list a number of properties that verify the 3-rotationally symmetric convex sets. They will be useful to prove these inequalities.

Let $K \subset \mathbb{R}^2$ be a 3-rotationally symmetric convex set. Then K has the following properties:

1) The incircle and circumcircle of K are concentric.
2) If R is the circumradius of K then K contains an equilateral triangle with the same circumradius as K.
3) If r is the inradius of K then it is contained in an equilateral triangle with inradius r.
4) If ω is the minimal width of K, then it is contained in a 3-rotationally symmetric hexagon with parallel opposite sides and minimal width ω (which can degenerate to an equilateral triangle).
5) If d is the diameter of K, then it contains a 3-rotationally symmetric hexagon with diameter d (that can degenerate to an equilateral triangle).

The centrally symmetral set of K, K^c, is a 6-rotationally symmetric convex set, and the following properties hold:

6) $\omega(K^c) = \omega(K)$
7) $p(K^c) = p(K)$
8) $d(K^c) = d(K)$
9) $A(K^c) \geq A(K)$
10) $r(K^c) \geq r(K)$
11) $R(K^c) \leq R(K)$

Let $K \subset \mathbb{R}^2$ be a 6-rotationally symmetric convex set with minimal width ω and diameter d. Then:

12) K is contained in a regular hexagon with minimal width ω.
13) K contains a regular hexagon with diameter d.

INEQUALITIES OF TABLE 4.1

The inequalities 1L, 2L, 3R, 4L, 5L, 6L, 7R, 8L, 9R, 10L, 11L, 11R, 12L, 13L, 13R, 14R and 15L are true for arbitrary planar convex sets (see [10]).

The inequalities 2R, 6R and 10R are obtained from 3). Inequalities 3L, 7L, 14L and 15R follow from 2). 8R and 12R are obtained from 4).

From 8R and 1L we can deduce 4R and from 12R and 1L we can obtain 5R. 1L is a consequence of 9) and 12).

9L is obtained from 5). An analytical calculation shows that for 3-rotationally symmetric hexagons with diameter d and perimeter p, $p \geq 3d$, and the equality is attained for the hexagon with parallel opposite sides.

INEQUALITIES OF TABLE 4.2

The inequalities (18.2), (19.1), (20.1), (22.1), (27.1), (27.2), (27.4), (28.1), (32.1) and (32.3) are true for arbitrary planar convex sets (see [5], [6] and [10]).

Inequalities (23.1), (29.1), (32.2): Let C be the incircle of K and x_1, x_2, $x_3 \in K$ be the vertices of the equilateral triangle with circumradius R that is contained in K. Then

$$C_B^3 = \operatorname{conv}\{x_1, x_2, x_3, C\} \subset K.$$

Inequalities (23.2), (27.3), (29.2): Let C be the circumcircle of K and T be the equilateral triangle with inradius r that contains K. Then $K \subset T \cap C$.

Inequalities (20.2), (21.1): They are obtained from 3) and 2) respectively, and the isoperimetric properties of the arcs of circle.

Inequalities (24.1), (24.2): K is contained in the intersection of a circle with radius R and a 3-rotationally symmetric hexagon with parallel opposite sides. An analytical calculation of optimization completes the proof.

Inequalities (30.1), (30.2): The proofs are similar to the ones of (24.1) and (24.2).

Inequality (31.1): It is obtained from (30.3) and (32.3).

From 6), 7), 8), 9) and 10) it is sufficient to check that the inequalities (16.1), (19.2), (22.2), (25.1), (26.1) and (26.2) are true for 6-rotationally symmetric convex sets.

So, from now on, K will be a 6-rotationally symmetric planar convex set.

Inequality (16.1): Let C be the circumcircle of K and H be the regular hexagon with diameter d contained in K. Then $H \subset K \subset C$ and because of the isoperimetric properties of the arcs of circle, the set with maximum area is $\mathcal{H}_C$ ($H \subset \mathcal{H}_C \subset C$).

Inequality (19.2): Let H be the regular hexagon with minimal width ω such that $K \subset H$, and let C be the circle with radius $d/2$ that contains K. Then $K \subset H \cap C$.

Inequality (22.2): This inequality is obtained from 12) and the isoperimetric properties of the arcs of circle.

Inequality (25.1): Let C be the incircle of K and H be the regular hexagon with diameter d contained in K. Then

$$C_B^6 = \operatorname{conv}(H \cup C) \subset K.$$

Inequality (26.1): K lies in a regular hexagon with minimal width ω and in a circle with radius $d/2$.

Inequality (26.2): It is obtained from the inequality (25.1) and the equality $\omega = 2r$.

Inequality (17.1): It is obtained from (20.1) and (25.1).

Inequality (18.1): We prove the following theorem.

Theorem 5.1. *Let $K \subset \mathbb{R}^2$ be a 3-rotationally symmetric convex set. Then*

$$A \geq \frac{3\sqrt{3}}{4}\left(d^2 - 2R^2\right),$$

with equality when and only when K is a 3-rotationally symmetric hexagon with parallel opposite sides.

Proof. We can suppose that the center of symmetry of K is the origin of coordinates O. Let C_R be the circle whose center is the origin and with radius R. Then there exist x_1, x_2, $x_3 \in \operatorname{bd}(C_R) \cap K$ such that $\operatorname{conv}(x_1, x_2, x_3)$ is an equilateral triangle (see figure 5.1).

Let $P, Q \in \operatorname{bd}(K)$ such that the diameter of K is given by the distance between P and Q, $d(K) = d(P,Q)$.

Now, let P' and P'' (Q' and Q'' respectively) be the rotations of P (rotations of Q) with angles $2\pi/3$ and $4\pi/3$ respectively.

If $K_1 = \operatorname{conv}\{x_1, x_2, x_3, P, P', P'', Q, Q', Q''\}$, then $K_1 \subseteq K$, and hence $A(K) \geq A(K_1)$.

Without lost of generality we can suppose that $d(P,O) \geq d(Q,O)$. Let P_1 be intersection point, closest to P, of the straight line that passes through P and Q'' with $\operatorname{bd}(C_R)$. Let

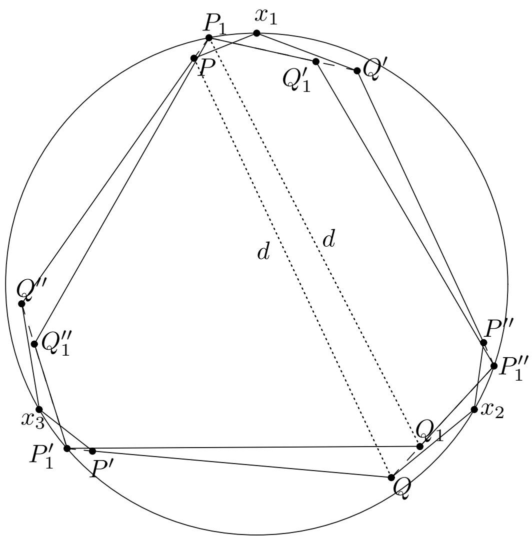

Figure 5.1: Reduction to 3-rotationally symmetric hexagons

P_1' and P_1'' be the rotations of P_1 with angles $2\pi/3$ and $4\pi/3$ respectively, and let $K_2 = \operatorname{conv}\{P_1, P_1', P_1'', Q, Q', Q''\}$. The triangles $\operatorname{conv}\{P, Q', x_1\}$ and $\operatorname{conv}\{P, Q', P_1\}$ have the same basis but different heights, so, $A(\operatorname{conv}\{P, Q', x_1\}) \geq A(\operatorname{conv}\{P, Q', P_1\})$. Therefore, $A(K_1) \geq A(K_2)$.

Also, we have that $d(K_2) = d(P_1, Q) \geq d(K) \geq d(P_1, P_1'') = \sqrt{3}R$. Then there exists a point Q_1 lying in the straight line segment $\overline{QP_1''}$ such that $d(P_1, Q_1) = d(K)$.

Now, let Q_1' and Q_1'' be the rotations of Q_1 with angles $2\pi/3$ and $4\pi/3$ respectively and let $K_3 = \operatorname{conv}\{P_1, P_1', P_1'', Q_1, Q_1', Q_1''\}$. Then, K_3 is a 3-rotationally symmetric hexagon with diameter d and circumradius R that lies into K_2. So, $A(K_3) \leq A(K_2)$.

Therefore it is sufficient to check that the inequality is true for the family of the 3-rotationally symmetric hexagons.

To this end, let $K = \operatorname{conv}\{P, P', P'', Q, Q', Q''\}$ be a 3-rotationally symmetric hexagon (with respect to O) with diameter d and circumradius R. We can suppose that $d(P, O) = R$ and $d(Q, O) = a \leq R$ (see figure 5.2).

Then it is easy to check that

$$A(K) = \frac{3\sqrt{3}}{4}(d^2 - R^2 - a^2).$$

Since $0 \leq a \leq R$, we obtain that

$$A \geq \frac{3\sqrt{3}}{4}(d^2 - 2R^2),$$

where the equality is attained when K is a hexagon with parallel opposite sides. □

Inequality (30.3): We prove the following theorem.

Theorem 5.2. *Let C be a circle with radius R and T an equilateral triangle inscribed in C. Let K be a planar convex set (not necessarily 3-rotationally symmetric) and Y a Yamanouti set both of them with minimal width ω and such that $T \subset K \subset C$ and $T \subset Y \subset C$. Let $\overline{\Omega}$ and Ω be the breadth functions of K and Y respectively. Then*

$$\Omega(\theta) \leq \overline{\Omega}(\theta) \quad \forall\, \theta \in [0, 2\pi].$$

Proof. Let x_1, x_2 and x_3 be the vertices of T.

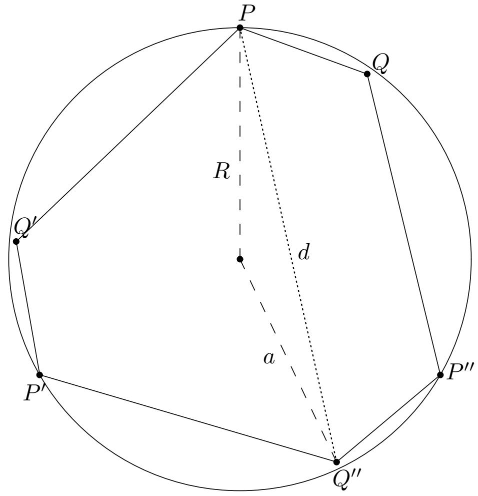

Figure 5.2: Obtaining the extremal sets

$\omega(K) = \omega(Y) = \omega$, hence $\Omega(\theta) \geq \omega$ and $\overline{\Omega}(\theta) \geq \omega$. Therefore it is sufficient to check that $\Omega(\theta) \leq \overline{\Omega}(\theta)$ when θ is an angle such that $\Omega(\theta) > \omega$.

$\Omega(\theta)$ is given by the distance between two parallel support lines of Y in the direction θ (r_Y^θ and s_Y^θ). Then, since $\Omega(\theta) > \omega$, there exist $i, j \in \{1, 2, 3\}$ such that $x_i \in r_Y^\theta$ and $x_j \in s_Y^\theta$.

Let r_K^θ and s_K^θ be the two parallel support lines of K in the direction θ. Since x_i, $x_j \in K$, then x_i, x_j lie in the strip determined by r_K^θ and s_K^θ. Therefore

$$d(r_Y^\theta, s_Y^\theta) \leq d(r_K^\theta, s_K^\theta)$$

Then, $\Omega(\theta) \leq \overline{\Omega}(\theta)$. With this result and the equality

$$p(K) = \frac{1}{2}\int_0^{2\pi} \overline{\Omega} d\theta,$$

the inequality (30.3) is obtained. □

6. THE COMPLETE SYSTEMS OF INEQUALITIES FOR THE 3-ROTATIONALLY SYMMETRIC CONVEX SETS

We have obtained complete systems of inequalities for fourteen cases: (A, R, r), (d, R, r), (p, R, r), (ω, R, r), (A, p, r), (d, ω, R), (d, ω, r), (p, d, R), (A, p, ω), (A, d, ω), (p, d, ω), (A, d, R), (p, ω, R) and (p, ω, r).

The six cases (A, p, R), (A, ω, r), (A, ω, R), (p, d, r), (A, p, d), (A, d, r) are still open.

The inequalities listed above determine complete systems for each of the cases. Blaschke diagram shows that for that choice of coordinates, the curves representing these inequalities bound a region. It is easy to see that this region is simply connected: with a suitable choice of extremal sets K and K', the linear family $\lambda K + (1-\lambda)K'$ fills the interior of the diagram.

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Cases	**Complete Systems of Inequalities**	**Coordinates**
A, d, R	3R, 11L, 11R, 18.1, 18.2	$x = \frac{A}{\pi R^2}, y = \frac{d}{2R}$
A, d, ω	5L, 12L, 12R, 19.1, 19.2	$x = \frac{\omega}{d}, y = \frac{4A}{\pi d^2}$
A, p, r	4L, 6L, 20.1, 20.2	$x = \frac{2\pi r}{p}, y = \frac{4\pi A}{p^2}$
A, p, ω	1L, 1R, 8L, 8R, 22.1, 22.2	$x = \frac{A}{\pi R^2}, y = \frac{d}{2R}$
A, R, r	3R, 15L, 23.1, 23.2	$x = \frac{A}{\pi R^2}, y = \frac{r}{R}$
p, d, ω	9R, 12L, 26.1, 26.2	$x = \frac{\omega}{d}, y = \frac{p}{\pi d}$
d, R, r	11L, 11R, 15L, 27.2, 27.3	$x = \frac{d}{2R}, y = \frac{r}{R}$
d, ω, r	12L, 12R, 13L, 28.1	$x = \frac{\sqrt{3}d}{2\omega}, y = \frac{2r}{\omega}$
p, R, r	7R, 15L, 29.1, 29.2	$x = \frac{p}{2\pi R}, y = \frac{r}{R}$
p, ω, R	7L, 8R, 30.2, 30.3	$x = \frac{3\sqrt{3}R}{p}, y = \frac{\pi\omega}{p}$
p, ω, r	8L, 8R, 13L, 31.1	$x = \frac{p}{2\sqrt{3}\omega}, y = \frac{2r}{\omega}$
ω, R, r	13L, 14L, 32.2, 32.3	$x = \frac{2r}{\omega}, y = \frac{3R}{2\omega}$
p, d, R	9L, 9R, 11L, 11R	$x = \frac{p}{\pi d}, y = \frac{\sqrt{3}R}{d}$
d, ω, R	11L, 11R, 12L, 12R, 14R	$x = \frac{\omega}{2R}, y = \frac{d}{2R}$

Table 6.1: The complete systems of inequalities for 3-rotationally symmetric convex sets

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