Journal of Inequalities in Pure and Applied Mathematics

SOME GRÜSS TYPE INEQUALITIES IN INNER PRODUCT SPACES

S.S. DRAGOMIR

School of Computer Science & Mathematics
Victoria University,
PO Box 14428, MCMC
Melbourne, Victoria 8001,
Australia.
E-Mail: sever.dragomir@vu.edu.au
URL: http://rgmia.vu.edu.au/SSDragomirWeb.html

ISSN (electronic): 1443-5756
032-03

volume 4, issue 2, article 42, 2003.

Received 12 March, 2003; accepted 12 April, 2003.

Communicated by: B. Mond

Abstract

Some new Grüss type inequalities in inner product spaces and applications for integrals are given.

2000 Mathematics Subject Classification: Primary 26D15, 46D05 Secondary 46C99
Key words: Grüss' Inequality, Inner products, Integral inequalities, Discrete Inequalities

Contents

1. Introduction

In [1], the author has proved the following Grüss type inequality in real or complex inner product spaces.

Theorem 1.1. *Let* $(H, \langle \cdot, \cdot \rangle)$ *be an inner product space over* $\mathbb{K}\,(\mathbb{K} = \mathbb{R},\mathbb{C})$ *and* $e \in H,\ \|e\| = 1$. *If* $\varphi, \gamma, \Phi, \Gamma$ *are real or complex numbers and* x, y *are vectors in* H *such that the conditions*

$$\operatorname{Re}\langle \Phi e - x, x - \varphi e\rangle \geq 0 \text{ and } \operatorname{Re}\langle \Gamma e - y, y - \gamma e\rangle \geq 0 \tag{1.1}$$

hold, then we have the inequality

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma| . \tag{1.2}$$

The constant $\frac{1}{4}$ *is best possible in the sense that it cannot be replaced by a smaller constant.*

Some particular cases of interest for integrable functions with real or complex values and the corresponding discrete versions are listed below.

Corollary 1.2. *Let* $f, g : [a, b] \to \mathbb{K}\,(\mathbb{K} = \mathbb{R},\mathbb{C})$ *be Lebesgue integrable and such that*

$$\operatorname{Re}\left[(\Phi - f(x))\left(\overline{f(x)} - \overline{\varphi}\right)\right] \geq 0,\ \operatorname{Re}\left[(\Gamma - g(x))\left(\overline{g(x)} - \overline{\gamma}\right)\right] \geq 0 \tag{1.3}$$

for a.e. $x \in [a, b]$, *where* $\varphi, \gamma, \Phi, \Gamma$ *are real or complex numbers and* $\bar{z}$ *denotes* *the complex conjugate of z. Then we have the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(x)\overline{g(x)}dx - \frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b \overline{g(x)}dx\right| \leq \frac{1}{4}\left|\Phi-\varphi\right|\cdot\left|\Gamma-\gamma\right|. \tag{1.4}$$

The constant $\frac{1}{4}$ is best possible.

The discrete case is embodied in

Corollary 1.3. *Let $\mathbf{x},\mathbf{y}\in\mathbb{K}^n$ and $\varphi,\gamma,\Phi,\Gamma$ are real or complex numbers such that*

$$\operatorname{Re}\left[(\Phi-x_i)(\overline{x_i}-\overline{\varphi})\right]\geq 0,\qquad \operatorname{Re}\left[(\Gamma-y_i)(\overline{y_i}-\overline{\gamma})\right]\geq 0 \tag{1.5}$$

for each $i\in\{1,\dots,n\}$. Then we have the inequality

$$\left|\frac{1}{n}\sum_{i=1}^n x_i\overline{y_i} - \frac{1}{n}\sum_{i=1}^n x_i\cdot\frac{1}{n}\sum_{i=1}^n \overline{y_i}\right| \leq \frac{1}{4}\left|\Phi-\varphi\right|\cdot\left|\Gamma-\gamma\right|. \tag{1.6}$$

The constant $\frac{1}{4}$ is best possible.

For other applications of Theorem 1.1, see the recent paper [2].

In the present paper we show that the condition (1.1) may be replaced by an equivalent but simpler assumption and a new proof of Theorem 1.1 is produced. A refinement of the Grüss type inequality (1.2), some companions and applications for integrals are pointed out as well.

2. An Equivalent Assumption

The following lemma holds.

Lemma 2.1. *Let a, x, A be vectors in the inner product space $(H, \langle\cdot,\cdot\rangle)$ over $\mathbb{K}$ $(\mathbb{K} = \mathbb{R},\mathbb{C})$ with $a \neq A$. Then*

$$\operatorname{Re}\langle A - x, x - a\rangle \geq 0$$

if and only if

$$\left\| x - \frac{a+A}{2} \right\| \leq \frac{1}{2}\|A - a\|.$$

Proof. Define

$$I_1 := \operatorname{Re}\langle A - x, x - a\rangle, \quad I_2 := \frac{1}{4}\|A - a\|^2 - \left\| x - \frac{a+A}{2} \right\|^2.$$

A simple calculation shows that

$$I_1 = I_2 = \operatorname{Re}[\langle x, a\rangle + \langle A, x\rangle] - \operatorname{Re}\langle A, a\rangle - \|x\|^2$$

and thus, obviously, $I_1 \geq 0$ iff $I_2 \geq 0$, showing the required equivalence. □

The following corollary is obvious

Corollary 2.2. *Let $x, e \in H$ with $\|e\| = 1$ and $\delta, \Delta \in \mathbb{K}$ with $\delta \neq \Delta$. Then*

$$\operatorname{Re}\langle \Delta e - x, x - \delta e\rangle \geq 0$$

if and only if

$$\left\| x - \frac{\delta + \Delta}{2} \cdot e \right\| \leq \frac{1}{2}|\Delta - \delta|.$$

Remark 2.1. *If* $H = \mathbb{C}$, *then*

$$\operatorname{Re}\left[(A - x)\left(\bar{x} - \bar{a}\right)\right] \geq 0$$

if and only if

$$\left| x - \frac{a + A}{2} \right| \leq \frac{1}{2} \left| A - a \right|,$$

where $a, x, A \in \mathbb{C}$. *If* $H = \mathbb{R}$, *and* $A > a$ *then* $a \leq x \leq A$ *if and only if* $\left| x - \frac{a+A}{2} \right| \leq \frac{1}{2} \left| A - a \right|$.

The following lemma also holds.

Lemma 2.3. *Let* $x, e \in H$ *with* $\|e\| = 1$. *Then one has the following representation*

$$0 \leq \|x\|^2 - |\langle x, e\rangle|^2 = \inf_{\lambda \in \mathbb{K}} \|x - \lambda e\|^2. \tag{2.1}$$

Proof. Observe, for any $\lambda \in \mathbb{K}$, that

$$\begin{aligned} \langle x - \lambda e, x - \langle x, e\rangle e\rangle &= \|x\|^2 - |\langle x, e\rangle|^2 - \lambda \left[\langle e, x\rangle - \langle e, x\rangle \|e\|^2\right] \\ &= \|x\|^2 - |\langle x, e\rangle|^2. \end{aligned}$$

Using Schwarz's inequality, we have

$$\begin{aligned} \left[\|x\|^2 - |\langle x, e\rangle|^2\right]^2 &= |\langle x - \lambda e, x - \langle x, e\rangle e\rangle|^2 \\ &\leq \|x - \lambda e\|^2 \|x - \langle x, e\rangle e\|^2 \\ &= \|x - \lambda e\|^2 \left[\|x\|^2 - |\langle x, e\rangle|^2\right] \end{aligned}$$

giving the bound

$$\|x\|^2 - |\langle x, e\rangle|^2 \leq \|x - \lambda e\|^2, \lambda \in \mathbb{K}. \tag{2.2}$$

Taking the infimum in (2.2) over $\lambda \in \mathbb{K}$, we deduce

$$\|x\|^2 - |\langle x, e\rangle|^2 \leq \inf_{\lambda \in \mathbb{K}} \|x - \lambda e\|^2 .$$

Since, for $\lambda_0 = \langle x, e\rangle$, we get $\|x - \lambda_0 e\|^2 = \|x\|^2 - |\langle x, e\rangle|^2$, then the representation (2.1) is proved. □

We are able now to provide a different proof for the Grüss type inequality in inner product spaces mentioned in the Introduction, than the one from paper [1].

Theorem 2.4. *Let* $(H, \langle\cdot,\cdot\rangle)$ *be an inner product space over* $\mathbb{K}$ $(\mathbb{K} = \mathbb{R}, \mathbb{C})$ *and* $e \in H$, $\|e\| = 1$. *If* $\varphi, \gamma, \Phi, \Gamma$ *are real or complex numbers and* x, y *are vectors in* H *such that the conditions* (1.1) *hold, or, equivalently, the following assumptions*

$$\left\| x - \frac{\varphi + \Phi}{2} \cdot e \right\| \leq \frac{1}{2} |\Phi - \varphi|, \qquad \left\| y - \frac{\gamma + \Gamma}{2} \cdot e \right\| \leq \frac{1}{2} |\Gamma - \gamma| \tag{2.3}$$

are valid, then one has the inequality

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma| . \tag{2.4}$$

The constant $\frac{1}{4}$ *is best possible.*

Proof. It can be easily shown (see for example the proof of Theorem 1 from [1]) that

$$\left|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle\right| \le \left[\|x\|^2 - |\langle x,e\rangle|^2\right]^{\frac{1}{2}} \left[\|y\|^2 - |\langle y,e\rangle|^2\right]^{\frac{1}{2}}, \tag{2.5}$$

for any $x, y \in H$ and $e \in H$, $\|e\| = 1$. Using Lemma 2.3 and the conditions (2.3) we obviously have that

$$\left[\|x\|^2 - |\langle x,e\rangle|^2\right]^{\frac{1}{2}} = \inf_{\lambda\in\mathbb{K}} \|x - \lambda e\| \le \left\| x - \frac{\varphi+\Phi}{2}\cdot e \right\| \le \frac{1}{2}\left|\Phi - \varphi\right|$$

and

$$\left[\|y\|^2 - |\langle y,e\rangle|^2\right]^{\frac{1}{2}} = \inf_{\lambda\in\mathbb{K}} \|y - \lambda e\| \le \left\| y - \frac{\gamma+\Gamma}{2}\cdot e \right\| \le \frac{1}{2}\left|\Gamma - \gamma\right|$$

and by (2.5) the desired inequality (2.4) is obtained.

The fact that $\frac{1}{4}$ is the best possible constant, has been shown in [1] and we omit the details. □

3. A Refinement of the Grüss Inequality

The following result improving (1.1) holds

Theorem 3.1. *Let* $(H, \langle\cdot,\cdot\rangle)$ *be an inner product space over* $\mathbb{K}\,(\mathbb{K}=\mathbb{R},\mathbb{C})$ *and* $e \in H,\ \|e\| = 1$. *If* $\varphi, \gamma, \Phi, \Gamma$ *are real or complex numbers and* x, y *are vectors in* H *such that the conditions* (1.1), *or, equivalently,* (2.3) *hold, then we have the inequality*

$$|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle| \le \frac{1}{4}|\Phi-\varphi|\cdot|\Gamma-\gamma| - [\operatorname{Re}\langle \Phi e - x, x - \varphi e\rangle]^{\frac{1}{2}} [\operatorname{Re}\langle \Gamma e - y, y - \gamma e\rangle]^{\frac{1}{2}}\,. \tag{3.1}$$

Proof. As in [1], we have

$$|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle|^2 \le \left[\|x\|^2 - |\langle x,e\rangle|^2\right]\left[\|y\|^2 - |\langle y,e\rangle|^2\right], \tag{3.2}$$

$$\|x\|^2 - |\langle x,e\rangle|^2 = \operatorname{Re}\left[(\Phi - \langle x,e\rangle)\left(\overline{\langle x,e\rangle} - \overline{\varphi}\right)\right] - \operatorname{Re}\langle \Phi e - x, x - \varphi e\rangle \tag{3.3}$$

and

$$\|y\|^2 - |\langle y,e\rangle|^2 = \operatorname{Re}\left[(\Gamma - \langle y,e\rangle)\left(\overline{\langle y,e\rangle} - \overline{\gamma}\right)\right] - \operatorname{Re}\langle \Gamma e - x, x - \gamma e\rangle\,. \tag{3.4}$$

Using the elementary inequality

$$4\,\mathrm{Re}\left(a\bar{b}\right) \leq |a+b|^2;\ \ a,b \in \mathbb{K}\,(\mathbb{K}=\mathbb{R},\mathbb{C})$$

we may state that

$$\mathrm{Re}\left[\left(\Phi - \langle x,e\rangle\right)\left(\overline{\langle x,e\rangle} - \overline{\varphi}\right)\right] \leq \frac{1}{4}\left|\Phi-\varphi\right|^2 \tag{3.5}$$

and

$$\mathrm{Re}\left[\left(\Gamma - \langle y,e\rangle\right)\left(\overline{\langle y,e\rangle} - \overline{\gamma}\right)\right] \leq \frac{1}{4}\left|\Gamma-\gamma\right|^2. \tag{3.6}$$

Consequently, by (3.2) – (3.6) we may state that

$$\begin{aligned} (3.7)\quad & \left|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle\right|^2 \\ & \leq \left[\frac{1}{4}\left|\Phi-\varphi\right|^2 - \left(\left[\mathrm{Re}\langle \Phi e - x, x - \varphi e\rangle\right]^{\frac{1}{2}}\right)^2\right] \\ & \qquad \times \left[\frac{1}{4}\left|\Gamma-\gamma\right|^2 - \left(\left[\mathrm{Re}\langle \Gamma e - y, y - \gamma e\rangle\right]^{\frac{1}{2}}\right)^2\right]. \end{aligned}$$

Finally, using the elementary inequality for positive real numbers

$$\left(m^2-n^2\right)\left(p^2-q^2\right) \leq (mp-nq)^2$$

we have

$$\left[\frac{1}{4}\left|\Phi-\varphi\right|^2-\left(\left[\operatorname{Re}\left\langle \Phi e-x,x-\varphi e\right\rangle\right]^{\frac{1}{2}}\right)^2\right] \times\left[\frac{1}{4}\left|\Gamma-\gamma\right|^2-\left(\left[\operatorname{Re}\left\langle \Gamma e-y,y-\gamma e\right\rangle\right]^{\frac{1}{2}}\right)^2\right]$$
$$\leq\left(\frac{1}{4}\left|\Phi-\varphi\right|\cdot\left|\Gamma-\gamma\right|-\left[\operatorname{Re}\left\langle \Phi e-x,x-\varphi e\right\rangle\right]^{\frac{1}{2}}\left[\operatorname{Re}\left\langle \Gamma e-y,y-\gamma e\right\rangle\right]^{\frac{1}{2}}\right)^2,$$

giving the desired inequality (3.1) . □

4. Some Companion Inequalities

The following companion of the Grüss inequality in inner product spaces holds.

Theorem 4.1. *Let* $(H, \langle \cdot, \cdot \rangle)$ *be an inner product space over* $\mathbb{K}$ $(\mathbb{K} = \mathbb{R}, \mathbb{C})$ *and* $e \in H$, $\|e\| = 1$. *If* $\gamma, \Gamma \in \mathbb{K}$ *and* $x, y \in H$ *are such that*

$$\operatorname{Re} \left\langle \Gamma e - \frac{x+y}{2}, \frac{x+y}{2} - \gamma e \right\rangle \geq 0 \tag{4.1}$$

or, equivalently,

$$\left\| \frac{x+y}{2} - \frac{\gamma + \Gamma}{2} \cdot e \right\| \leq \frac{1}{2} \left| \Gamma - \gamma \right|, \tag{4.2}$$

then we have the inequality

$$\operatorname{Re} \left[\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle \right] \leq \frac{1}{4} \left| \Gamma - \gamma \right|^2. \tag{4.3}$$

The constant $\frac{1}{4}$ *is best possible in the sense that it cannot be replaced by a smaller constant.*

Proof. Start with the well known inequality

$$\operatorname{Re} \langle z, u \rangle \leq \frac{1}{4} \|z + u\|^2; \quad z, u \in H. \tag{4.4}$$

Since

$$\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle = \langle x - \langle x, e \rangle e, y - \langle y, e \rangle e \rangle$$

then using (4.4) we may write

$$\begin{aligned}\operatorname{Re}\left[\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle\right] &= \operatorname{Re}\left[\langle x-\langle x,e\rangle e, y-\langle y,e\rangle e\rangle\right] \\ &\le \frac{1}{4}\left\|x-\langle x,e\rangle e+y-\langle y,e\rangle e\right\|^2 \\ &= \left\|\frac{x+y}{2}-\left\langle\frac{x+y}{2},e\right\rangle\cdot e\right\|^2 \\ &= \left\|\frac{x+y}{2}\right\|^2-\left|\left\langle\frac{x+y}{2},e\right\rangle\right|^2. \end{aligned} \tag{4.5}$$

If we apply Grüss' inequality in inner product spaces for, say, $a=b=\frac{x+y}{2}$, we get

$$\left\|\frac{x+y}{2}\right\|^2-\left|\left\langle\frac{x+y}{2},e\right\rangle\right|^2 \le \frac{1}{4}\left|\Gamma-\gamma\right|^2. \tag{4.6}$$

Making use of (4.5) and (4.6) we deduce (4.3).

The fact that $\frac{1}{4}$ is the best possible constant in (4.3) follows by the fact that if in (4.1) we choose $x=y$, then it becomes $\operatorname{Re}\langle\Gamma e-x, x-\gamma e\rangle\ge 0$, implying $0\le\|x\|^2-|\langle x,e\rangle|^2\le\frac{1}{4}|\Gamma-\gamma|^2$, for which, by Grüss' inequality in inner product spaces, we know that the constant $\frac{1}{4}$ is best possible. □

The following corollary might be of interest if one wanted to evaluate the absolute value of

$$\operatorname{Re}\left[\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle\right].$$

Corollary 4.2. *Let* $(H, \langle\cdot,\cdot\rangle)$ *be an inner product space over* $\mathbb{K}$ $(\mathbb{K} = \mathbb{R}, \mathbb{C})$ *and* $e \in H$, $\|e\| = 1$. *If* $\gamma, \Gamma \in \mathbb{K}$ *and* $x, y \in H$ *are such that*

$$\operatorname{Re}\left\langle \Gamma e - \frac{x \pm y}{2}, \frac{x \pm y}{2} - \gamma e \right\rangle \geq 0 \tag{4.7}$$

or, equivalently,

$$\left\| \frac{x \pm y}{2} - \frac{\gamma + \Gamma}{2} \cdot e \right\| \leq \frac{1}{2} |\Gamma - \gamma|, \tag{4.8}$$

then we have the inequality

$$|\operatorname{Re}[\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle]| \leq \frac{1}{4} |\Gamma - \gamma|^2. \tag{4.9}$$

If the inner product space H *is real, then* (*for* $m, M \in \mathbb{R}$, $M > m$)

$$\left\langle Me - \frac{x \pm y}{2}, \frac{x \pm y}{2} - me \right\rangle \geq 0 \tag{4.10}$$

or, equivalently,

$$\left\| \frac{x \pm y}{2} - \frac{m + M}{2} \cdot e \right\| \leq \frac{1}{2} (M - m), \tag{4.11}$$

implies

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \leq \frac{1}{4} (M - m)^2. \tag{4.12}$$

In both inequalities (4.9) *and* (4.12), *the constant* $\frac{1}{4}$ *is best possible.*

Proof. We only remark that, if

$$\operatorname{Re}\left\langle \Gamma e-\frac{x-y}{2},\frac{x-y}{2}-\gamma e\right\rangle \geq 0$$

holds, then by Theorem 4.1, we get

$$\operatorname{Re}\left[-\langle x,y\rangle+\langle x,e\rangle\langle e,y\rangle\right]\leq\frac{1}{4}\left|\Gamma-\gamma\right|^{2},$$

showing that

$$\operatorname{Re}\left[\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle\right]\geq-\frac{1}{4}\left|\Gamma-\gamma\right|^{2}. \tag{4.13}$$

Making use of (4.3) and (4.13) we deduce the desired result (4.9) . □

Finally, we may state and prove the following dual result as well

Proposition 4.3. *Let* $(H,\langle\cdot,\cdot\rangle)$ *be an inner product space over* $\mathbb{K}\,(\mathbb{K}=\mathbb{R},\mathbb{C})$ *and* $e\in H$, $\|e\|=1$. *If* $\varphi,\Phi\in\mathbb{K}$ *and* $x,y\in H$ *are such that*

$$\operatorname{Re}\left[(\Phi-\langle x,e\rangle)\left(\overline{\langle x,e\rangle}-\overline{\varphi}\right)\right]\leq 0, \tag{4.14}$$

then we have the inequalities

$$\begin{aligned}\|x-\langle x,e\rangle e\| &\leq \left[\operatorname{Re}\langle x-\Phi e,x-\varphi e\rangle\right]^{\frac{1}{2}}\\ &\leq\frac{\sqrt{2}}{2}\left[\|x-\Phi e\|^{2}+\|x-\varphi e\|^{2}\right]^{\frac{1}{2}}.\end{aligned} \tag{4.15}$$

Proof. We know that the following identity holds true (see (3.3))

$$\|x\|^2 - |\langle x, e\rangle|^2 = \operatorname{Re}\left[(\Phi - \langle x, e\rangle)\left(\overline{\langle x, e\rangle} - \overline{\varphi}\right)\right] + \operatorname{Re}\langle x - \Phi e, x - \varphi e\rangle . \tag{4.16}$$

Using the assumption (4.14) and the fact that

$$\|x\|^2 - |\langle x, e\rangle|^2 = \|x - \langle x, e\rangle e\|^2 ,$$

by (4.16) we deduce the first inequality in (4.15) .

The second inequality in (4.15) follows by the fact that for any $v, w \in H$ one has

$$\operatorname{Re}\langle w, v\rangle \leq \frac{1}{2}\left(\|w\|^2 + \|v\|^2\right) .$$

The proposition is thus proved. □

5. Integral Inequalities

Let (Ω, Σ, μ) be a measure space consisting of a set Ω, a $\sigma-$algebra of parts Σ and a countably additive and positive measure μ on Σ with values in $\mathbb{R} \cup \{\infty\}$. Denote by $L^2(\Omega, \mathbb{K})$ the Hilbert space of all real or complex valued functions f defined on Ω and $2-$integrable on Ω, i.e.,

$$\int_\Omega |f(s)|^2 \, d\mu(s) < \infty.$$

The following proposition holds

Proposition 5.1. *If* $f, g, h \in L^2(\Omega, \mathbb{K})$ *and* $\varphi, \Phi, \gamma, \Gamma \in \mathbb{K}$, *are such that* $\int_\Omega |h(s)|^2 \, d\mu(s) = 1$ *and*

$$\int_\Omega \operatorname{Re}\left[(\Phi h(s) - f(s))\left(\overline{f(s)} - \overline{\varphi}\overline{h(s)}\right)\right] d\mu(s) \geq 0, \tag{5.1}$$
$$\int_\Omega \operatorname{Re}\left[(\Gamma h(s) - g(s))\left(\overline{g(s)} - \overline{\gamma}\overline{h(s)}\right)\right] d\mu(s) \geq 0$$

or, equivalently

$$\left(\int_\Omega \left| f(s) - \frac{\Phi + \varphi}{2} h(s) \right|^2 d\mu(s)\right)^{\frac{1}{2}} \leq \frac{1}{2} |\Phi - \varphi|, \tag{5.2}$$
$$\left(\int_\Omega \left| g(s) - \frac{\Gamma + \gamma}{2} h(s) \right|^2 d\mu(s)\right)^{\frac{1}{2}} \leq \frac{1}{2} |\Gamma - \gamma|,$$

then we have the following refinement of the Grüss integral inequality

$$\left|\int_{\Omega} f(s)\overline{g(s)}d\mu(s) - \int_{\Omega} f(s)\overline{h(s)}d\mu(s)\int_{\Omega} h(s)\overline{g(s)}d\mu(s)\right| \tag{5.3}$$
$$\leq \frac{1}{4}|\Phi-\varphi|\cdot|\Gamma-\gamma|$$
$$-\left[\int_{\Omega}\operatorname{Re}\left[(\Phi h(s)-f(s))\left(\overline{f(s)}-\overline{\varphi}\overline{h(s)}\right)\right]d\mu(s)\right.$$
$$\left.\times\int_{\Omega}\operatorname{Re}\left[(\Gamma h(s)-g(s))\left(\overline{g(s)}-\overline{\gamma}\overline{h(s)}\right)\right]d\mu(s)\right]^{\frac{1}{2}}.$$

The constant $\frac{1}{4}$ is best possible.

The proof follows by Theorem 3.1 on choosing $H = L^2(\Omega,\mathbb{K})$ with the inner product

$$\langle f,g\rangle := \int_{\Omega} f(s)\overline{g(s)}d\mu(s).$$

We omit the details.

Remark 5.1. *It is obvious that a sufficient condition for (5.1) to hold is*

$$\operatorname{Re}\left[(\Phi h(s)-f(s))\left(\overline{f(s)}-\overline{\varphi}\overline{h(s)}\right)\right]\geq 0,$$

and

$$\operatorname{Re}\left[(\Gamma h(s)-g(s))\left(\overline{g(s)}-\overline{\gamma}\overline{h(s)}\right)\right]\geq 0,$$

for $\mu-$a.e. $s\in\Omega$, or equivalently,

$$\left|f(s)-\frac{\Phi+\varphi}{2}h(s)\right|\leq\frac{1}{2}|\Phi-\varphi||h(s)| \quad \textit{and}$$
$$\left|g(s)-\frac{\Gamma+\gamma}{2}h(s)\right|\leq\frac{1}{2}|\Gamma-\gamma||h(s)|,$$

for $\mu-$a.e. $s\in\Omega$.

The following result may be stated as well.

Corollary 5.2. *If $z,Z,t,T\in\mathbb{K}$, $\mu(\Omega)<\infty$ and $f,g\in L^2(\Omega,\mathbb{K})$ are such that:*

$$\operatorname{Re}\left[(Z-f(s))\left(\overline{f(s)}-\bar{z}\right)\right]\geq 0,\tag{5.4}$$
$$\operatorname{Re}\left[(T-g(s))\left(\overline{g(s)}-\bar{t}\right)\right]\geq 0 \quad \textit{for a.e. } s\in\Omega$$

or, equivalently

$$\left|f(s)-\frac{z+Z}{2}\right|\leq\frac{1}{2}|Z-z|,\tag{5.5}$$
$$\left|g(s)-\frac{t+T}{2}\right|\leq\frac{1}{2}|T-t| \quad \textit{for a.e. } s\in\Omega$$

then we have the inequality

$$\left|\frac{1}{\mu(\Omega)}\int_\Omega f(s)\overline{g(s)}d\mu(s) - \frac{1}{\mu(\Omega)}\int_\Omega f(s)\,d\mu(s)\cdot\frac{1}{\mu(\Omega)}\int_\Omega\overline{g(s)}d\mu(s)\right|\tag{5.6}$$

$$\leq \frac{1}{4}|Z-z||T-t| - \frac{1}{\mu(\Omega)}\left[\int_\Omega \operatorname{Re}\left[(Z-f(s))\left(\overline{f(s)}-\bar{z}\right)\right] d\mu(s)\right.$$
$$\left.\times \int_\Omega \operatorname{Re}\left[(T-g(s))\left(\overline{g(s)}-\bar{t}\right)\right] d\mu(s)\right]^{\frac{1}{2}}.$$

Using Theorem 4.1 we may state the following result as well.

Proposition 5.3. *If* $f, g, h \in L^2(\Omega, \mathbb{K})$ *and* $\gamma, \Gamma \in \mathbb{K}$ *are such that* $\int_\Omega |h(s)|^2 d\mu(s) = 1$ *and*

$$\int_\Omega \operatorname{Re}\left\{\left[\Gamma h(s) - \frac{f(s)+g(s)}{2}\right] \times \left[\frac{\overline{f(s)}+\overline{g(s)}}{2} - \bar{\gamma}\bar{h}(s)\right]\right\} d\mu(s) \geq 0 \tag{5.7}$$

or, equivalently,

$$\left(\int_\Omega \left|\frac{f(s)+g(s)}{2} - \frac{\gamma+\Gamma}{2}h(s)\right|^2 d\mu(s)\right)^{\frac{1}{2}} \leq \frac{1}{2}|\Gamma-\gamma|, \tag{5.8}$$

then we have the inequality

$$I := \int_\Omega \operatorname{Re}\left[f(s)\overline{g(s)}\right] d\mu(s) - \operatorname{Re}\left[\int_\Omega f(s)\overline{h(s)} d\mu(s) \cdot \int_\Omega h(s)\overline{g(s)} d\mu(s)\right] \leq \frac{1}{4}|\Gamma-\gamma|^2. \tag{5.9}$$

If (5.7) and (5.8) hold with " $\pm$ " instead of " $+$ ", then

$$|I| \leq \frac{1}{4} |\Gamma - \gamma|^2 . \tag{5.10}$$

Remark 5.2. *It is obvious that a sufficient condition for (5.7) to hold is*

$$\operatorname{Re}\left\{ \left[\Gamma h(s) - \frac{f(s)+g(s)}{2} \right] \cdot \left[\frac{\overline{f(s)} + \overline{g(s)}}{2} - \bar{\gamma}\bar{h}(s) \right] \right\} \geq 0 \tag{5.11}$$

for a.e. $s \in \Omega$, or equivalently

$$\left| \frac{f(s)+g(s)}{2} - \frac{\gamma+\Gamma}{2} h(s) \right| \leq \frac{1}{2} |\Gamma - \gamma| \, |h(s)| \quad \text{for a.e. } s \in \Omega. \tag{5.12}$$

Finally, the following corollary holds.

Corollary 5.4. *If $Z, z \in \mathbb{K}$, $\mu(\Omega) < \infty$ and $f, g \in L^2(\Omega, \mathbb{K})$ are such that*

$$\operatorname{Re}\left[\left(Z - \frac{f(s)+g(s)}{2} \right) \left(\frac{\overline{f(s)} + \overline{g(s)}}{2} - \bar{z} \right) \right] \geq 0 \quad \text{for a.e. } s \in \Omega \tag{5.13}$$

or, equivalently

$$\left| \frac{f(s)+g(s)}{2} - \frac{z+Z}{2} \right| \leq \frac{1}{2} |Z - z| \quad \text{for a.e. } s \in \Omega, \tag{5.14}$$

then we have the inequality

$$J := \frac{1}{\mu(\Omega)} \int_\Omega \operatorname{Re}\left[f(s)\overline{g(s)}\right] d\mu(s) - \operatorname{Re}\left[\frac{1}{\mu(\Omega)} \int_\Omega f(s)\, d\mu(s) \cdot \frac{1}{\mu(\Omega)} \int_\Omega \overline{g(s)} d\mu(s)\right] \le \frac{1}{4}|Z - z|^2 .$$

If (5.13) and (5.14) hold with " $\pm$ " instead of " $+$ ", then

$$|J| \le \frac{1}{4}|Z - z|^2 . \tag{5.15}$$

Remark 5.3. *It is obvious that if one chooses the discrete measure above, then all the inequalities in this section may be written for sequences of real or complex numbers. We omit the details.*

References

[1] S.S. DRAGOMIR, A generalization of Grüss' inequality in inner product spaces and applications, *J. Math. Anal. Appl.*, **237** (1999), 74–82.

[2] S.S. DRAGOMIR AND I. GOMM, Some integral and discrete versions of the Grüss inequality for real and complex functions and sequences, *RGMIA Res. Rep. Coll.*, **5**(3) (2003), Article [http://rgmia.vu.edu.au/v5n3.html]