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SOME GRÜSS TYPE INEQUALITIES IN INNER PRODUCT SPACES

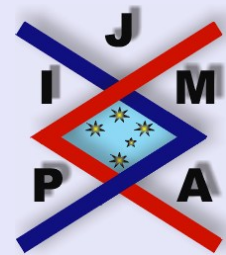
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Abstract

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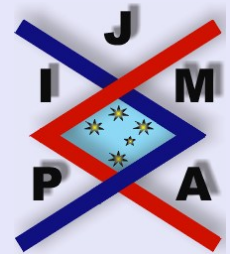
Abstract

Some new Grüss type inequalities in inner product spaces and applications for integrals are given.

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1. Introduction

In [1], the author has proved the following Grüss type inequality in real or complex inner product spaces.

Theorem 1.1. *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over \mathbb{K} ($\mathbb{K} = \mathbb{R}, \mathbb{C}$) and $e \in H$, $\|e\| = 1$. If $\varphi, \gamma, \Phi, \Gamma$ are real or complex numbers and x, y are vectors in H such that the conditions*

$$(1.1) \quad \operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle \geq 0 \text{ and } \operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle \geq 0$$

hold, then we have the inequality

$$(1.2) \quad |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma|.$$

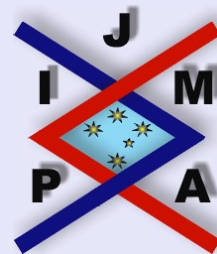
The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

Some particular cases of interest for integrable functions with real or complex values and the corresponding discrete versions are listed below.

Corollary 1.2. *Let $f, g : [a, b] \rightarrow \mathbb{K}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$) be Lebesgue integrable and such that*

$$(1.3) \quad \operatorname{Re} \left[(\Phi - f(x)) \left(\overline{f(x)} - \overline{\varphi} \right) \right] \geq 0, \operatorname{Re} \left[(\Gamma - g(x)) \left(\overline{g(x)} - \overline{\gamma} \right) \right] \geq 0$$

for a.e. $x \in [a, b]$, where $\varphi, \gamma, \Phi, \Gamma$ are real or complex numbers and \bar{z} denotes



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the complex conjugate of z . Then we have the inequality

$$(1.4) \quad \left| \frac{1}{b-a} \int_a^b f(x) \overline{g(x)} dx - \frac{1}{b-a} \int_a^b f(x) dx \cdot \frac{1}{b-a} \int_a^b \overline{g(x)} dx \right| \leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma|.$$

The constant $\frac{1}{4}$ is best possible.

The discrete case is embodied in

Corollary 1.3. Let $\mathbf{x}, \mathbf{y} \in \mathbb{K}^n$ and $\varphi, \gamma, \Phi, \Gamma$ are real or complex numbers such that

$$(1.5) \quad \operatorname{Re} [(\Phi - x_i) (\overline{x_i} - \overline{\varphi})] \geq 0, \quad \operatorname{Re} [(\Gamma - y_i) (\overline{y_i} - \overline{\gamma})] \geq 0$$

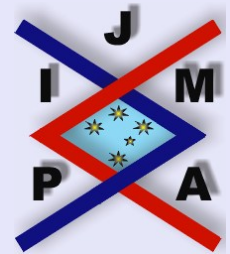
for each $i \in \{1, \dots, n\}$. Then we have the inequality

$$(1.6) \quad \left| \frac{1}{n} \sum_{i=1}^n x_i \overline{y_i} - \frac{1}{n} \sum_{i=1}^n x_i \cdot \frac{1}{n} \sum_{i=1}^n \overline{y_i} \right| \leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma|.$$

The constant $\frac{1}{4}$ is best possible.

For other applications of Theorem 1.1, see the recent paper [2].

In the present paper we show that the condition (1.1) may be replaced by an equivalent but simpler assumption and a new proof of Theorem 1.1 is produced. A refinement of the Grüss type inequality (1.2), some companions and applications for integrals are pointed out as well.



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2. An Equivalent Assumption

The following lemma holds.

Lemma 2.1. *Let a, x, A be vectors in the inner product space $(H, \langle \cdot, \cdot \rangle)$ over \mathbb{K} ($\mathbb{K} = \mathbb{R}, \mathbb{C}$) with $a \neq A$. Then*

$$\operatorname{Re} \langle A - x, x - a \rangle \geq 0$$

if and only if

$$\left\| x - \frac{a + A}{2} \right\| \leq \frac{1}{2} \|A - a\|.$$

Proof. Define

$$I_1 := \operatorname{Re} \langle A - x, x - a \rangle, \quad I_2 := \frac{1}{4} \|A - a\|^2 - \left\| x - \frac{a + A}{2} \right\|^2.$$

A simple calculation shows that

$$I_1 = I_2 = \operatorname{Re} [\langle x, a \rangle + \langle A, x \rangle] - \operatorname{Re} \langle A, a \rangle - \|x\|^2$$

and thus, obviously, $I_1 \geq 0$ iff $I_2 \geq 0$, showing the required equivalence. \square

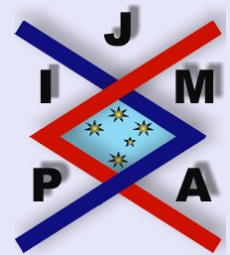
The following corollary is obvious

Corollary 2.2. *Let $x, e \in H$ with $\|e\| = 1$ and $\delta, \Delta \in \mathbb{K}$ with $\delta \neq \Delta$. Then*

$$\operatorname{Re} \langle \Delta e - x, x - \delta e \rangle \geq 0$$

if and only if

$$\left\| x - \frac{\delta + \Delta}{2} \cdot e \right\| \leq \frac{1}{2} |\Delta - \delta|.$$



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Remark 2.1. If $H = \mathbb{C}$, then

$$\operatorname{Re}[(A - x)(\bar{x} - \bar{a})] \geq 0$$

if and only if

$$\left| x - \frac{a + A}{2} \right| \leq \frac{1}{2} |A - a|,$$

where $a, x, A \in \mathbb{C}$. If $H = \mathbb{R}$, and $A > a$ then $a \leq x \leq A$ if and only if $\left| x - \frac{a+A}{2} \right| \leq \frac{1}{2} |A - a|$.

The following lemma also holds.

Lemma 2.3. Let $x, e \in H$ with $\|e\| = 1$. Then one has the following representation

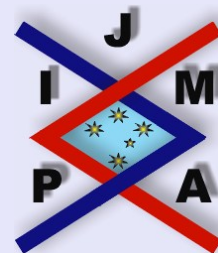
$$(2.1) \quad 0 \leq \|x\|^2 - |\langle x, e \rangle|^2 = \inf_{\lambda \in \mathbb{K}} \|x - \lambda e\|^2.$$

Proof. Observe, for any $\lambda \in \mathbb{K}$, that

$$\begin{aligned} \langle x - \lambda e, x - \langle x, e \rangle e \rangle &= \|x\|^2 - |\langle x, e \rangle|^2 - \lambda [\langle e, x \rangle - \langle e, x \rangle \|e\|^2] \\ &= \|x\|^2 - |\langle x, e \rangle|^2. \end{aligned}$$

Using Schwarz's inequality, we have

$$\begin{aligned} [\|x\|^2 - |\langle x, e \rangle|^2]^2 &= |\langle x - \lambda e, x - \langle x, e \rangle e \rangle|^2 \\ &\leq \|x - \lambda e\|^2 \|x - \langle x, e \rangle e\|^2 \\ &= \|x - \lambda e\|^2 [\|x\|^2 - |\langle x, e \rangle|^2] \end{aligned}$$



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giving the bound

$$(2.2) \quad \|x\|^2 - |\langle x, e \rangle|^2 \leq \|x - \lambda e\|^2, \lambda \in \mathbb{K}.$$

Taking the infimum in (2.2) over $\lambda \in \mathbb{K}$, we deduce

$$\|x\|^2 - |\langle x, e \rangle|^2 \leq \inf_{\lambda \in \mathbb{K}} \|x - \lambda e\|^2.$$

Since, for $\lambda_0 = \langle x, e \rangle$, we get $\|x - \lambda_0 e\|^2 = \|x\|^2 - |\langle x, e \rangle|^2$, then the representation (2.1) is proved. \square

We are able now to provide a different proof for the Grüss type inequality in inner product spaces mentioned in the Introduction, than the one from paper [1].

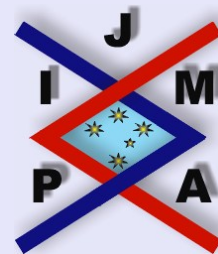
Theorem 2.4. *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over \mathbb{K} ($\mathbb{K} = \mathbb{R}, \mathbb{C}$) and $e \in H$, $\|e\| = 1$. If $\varphi, \gamma, \Phi, \Gamma$ are real or complex numbers and x, y are vectors in H such that the conditions (1.1) hold, or, equivalently, the following assumptions*

$$(2.3) \quad \left\| x - \frac{\varphi + \Phi}{2} \cdot e \right\| \leq \frac{1}{2} |\Phi - \varphi|, \quad \left\| y - \frac{\gamma + \Gamma}{2} \cdot e \right\| \leq \frac{1}{2} |\Gamma - \gamma|$$

are valid, then one has the inequality

$$(2.4) \quad |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma|.$$

The constant $\frac{1}{4}$ is best possible.



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Proof. It can be easily shown (see for example the proof of Theorem 1 from [1]) that

$$(2.5) \quad |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq [\|x\|^2 - |\langle x, e \rangle|^2]^{\frac{1}{2}} [\|y\|^2 - |\langle y, e \rangle|^2]^{\frac{1}{2}},$$

for any $x, y \in H$ and $e \in H$, $\|e\| = 1$. Using Lemma 2.3 and the conditions (2.3) we obviously have that

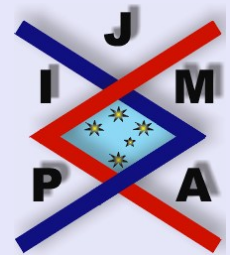
$$[\|x\|^2 - |\langle x, e \rangle|^2]^{\frac{1}{2}} = \inf_{\lambda \in \mathbb{K}} \|x - \lambda e\| \leq \left\| x - \frac{\varphi + \Phi}{2} \cdot e \right\| \leq \frac{1}{2} |\Phi - \varphi|$$

and

$$[\|y\|^2 - |\langle y, e \rangle|^2]^{\frac{1}{2}} = \inf_{\lambda \in \mathbb{K}} \|y - \lambda e\| \leq \left\| y - \frac{\gamma + \Gamma}{2} \cdot e \right\| \leq \frac{1}{2} |\Gamma - \gamma|$$

and by (2.5) the desired inequality (2.4) is obtained.

The fact that $\frac{1}{4}$ is the best possible constant, has been shown in [1] and we omit the details. \square



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3. A Refinement of the Grüss Inequality

The following result improving (1.1) holds

Theorem 3.1. *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over \mathbb{K} ($\mathbb{K} = \mathbb{R}, \mathbb{C}$) and $e \in H$, $\|e\| = 1$. If $\varphi, \gamma, \Phi, \Gamma$ are real or complex numbers and x, y are vectors in H such that the conditions (1.1), or, equivalently, (2.3) hold, then we have the inequality*

$$(3.1) \quad |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma| \\ - [\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle]^{\frac{1}{2}} [\operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle]^{\frac{1}{2}}.$$

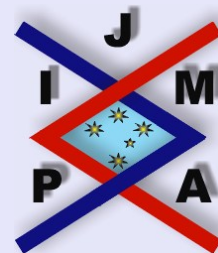
Proof. As in [1], we have

$$(3.2) \quad |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle|^2 \leq [\|x\|^2 - |\langle x, e \rangle|^2] [\|y\|^2 - |\langle y, e \rangle|^2],$$

$$(3.3) \quad \|x\|^2 - |\langle x, e \rangle|^2 \\ = \operatorname{Re} \left[(\Phi - \langle x, e \rangle) \left(\overline{\langle x, e \rangle} - \bar{\varphi} \right) \right] - \operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle$$

and

$$(3.4) \quad \|y\|^2 - |\langle y, e \rangle|^2 \\ = \operatorname{Re} \left[(\Gamma - \langle y, e \rangle) \left(\overline{\langle y, e \rangle} - \bar{\gamma} \right) \right] - \operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle.$$



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Using the elementary inequality

$$4 \operatorname{Re}(a\bar{b}) \leq |a + b|^2; \quad a, b \in \mathbb{K} (\mathbb{K} = \mathbb{R}, \mathbb{C})$$

we may state that

$$(3.5) \quad \operatorname{Re} \left[(\Phi - \langle x, e \rangle) \left(\overline{\langle x, e \rangle} - \bar{\varphi} \right) \right] \leq \frac{1}{4} |\Phi - \varphi|^2$$

and

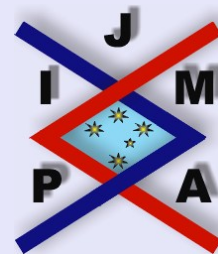
$$(3.6) \quad \operatorname{Re} \left[(\Gamma - \langle y, e \rangle) \left(\overline{\langle y, e \rangle} - \bar{\gamma} \right) \right] \leq \frac{1}{4} |\Gamma - \gamma|^2.$$

Consequently, by (3.2) – (3.6) we may state that

$$(3.7) \quad |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle|^2 \leq \left[\frac{1}{4} |\Phi - \varphi|^2 - \left([\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle]^{\frac{1}{2}} \right)^2 \right] \times \left[\frac{1}{4} |\Gamma - \gamma|^2 - \left([\operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle]^{\frac{1}{2}} \right)^2 \right].$$

Finally, using the elementary inequality for positive real numbers

$$(m^2 - n^2) (p^2 - q^2) \leq (mp - nq)^2$$



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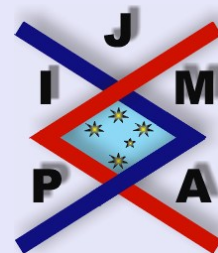
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we have

$$\begin{aligned} & \left[\frac{1}{4} |\Phi - \varphi|^2 - \left([\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle]^{\frac{1}{2}} \right)^2 \right] \\ & \quad \times \left[\frac{1}{4} |\Gamma - \gamma|^2 - \left([\operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle]^{\frac{1}{2}} \right)^2 \right] \\ & \leq \left(\frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma| - [\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle]^{\frac{1}{2}} [\operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle]^{\frac{1}{2}} \right)^2, \end{aligned}$$

giving the desired inequality (3.1). \square



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4. Some Companion Inequalities

The following companion of the Grüss inequality in inner product spaces holds.

Theorem 4.1. *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over \mathbb{K} ($\mathbb{K} = \mathbb{R}, \mathbb{C}$) and $e \in H$, $\|e\| = 1$. If $\gamma, \Gamma \in \mathbb{K}$ and $x, y \in H$ are such that*

$$(4.1) \quad \operatorname{Re} \left\langle \Gamma e - \frac{x+y}{2}, \frac{x+y}{2} - \gamma e \right\rangle \geq 0$$

or, equivalently,

$$(4.2) \quad \left\| \frac{x+y}{2} - \frac{\gamma + \Gamma}{2} \cdot e \right\| \leq \frac{1}{2} |\Gamma - \gamma|,$$

then we have the inequality

$$(4.3) \quad \operatorname{Re} [\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle] \leq \frac{1}{4} |\Gamma - \gamma|^2.$$

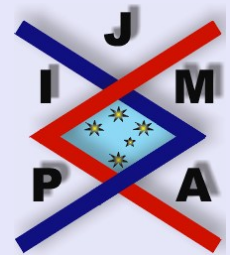
The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. Start with the well known inequality

$$(4.4) \quad \operatorname{Re} \langle z, u \rangle \leq \frac{1}{4} \|z + u\|^2; \quad z, u \in H.$$

Since

$$\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle = \langle x - \langle x, e \rangle e, y - \langle y, e \rangle e \rangle$$



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then using (4.4) we may write

$$\begin{aligned}
 (4.5) \quad \operatorname{Re}[\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle] &= \operatorname{Re}[(x - \langle x, e \rangle e, y - \langle y, e \rangle e)] \\
 &\leq \frac{1}{4} \|x - \langle x, e \rangle e + y - \langle y, e \rangle e\|^2 \\
 &= \left\| \frac{x+y}{2} - \left\langle \frac{x+y}{2}, e \right\rangle \cdot e \right\|^2 \\
 &= \left\| \frac{x+y}{2} \right\|^2 - \left| \left\langle \frac{x+y}{2}, e \right\rangle \right|^2.
 \end{aligned}$$

If we apply Grüss' inequality in inner product spaces for, say, $a = b = \frac{x+y}{2}$, we get

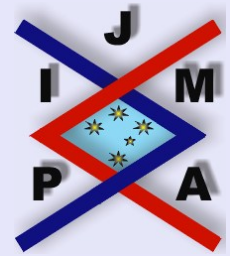
$$(4.6) \quad \left\| \frac{x+y}{2} \right\|^2 - \left| \left\langle \frac{x+y}{2}, e \right\rangle \right|^2 \leq \frac{1}{4} |\Gamma - \gamma|^2.$$

Making use of (4.5) and (4.6) we deduce (4.3).

The fact that $\frac{1}{4}$ is the best possible constant in (4.3) follows by the fact that if in (4.1) we choose $x = y$, then it becomes $\operatorname{Re} \langle \Gamma e - x, x - \gamma e \rangle \geq 0$, implying $0 \leq \|x\|^2 - |\langle x, e \rangle|^2 \leq \frac{1}{4} |\Gamma - \gamma|^2$, for which, by Grüss' inequality in inner product spaces, we know that the constant $\frac{1}{4}$ is best possible. \square

The following corollary might be of interest if one wanted to evaluate the absolute value of

$$\operatorname{Re}[\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle].$$



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Corollary 4.2. Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over \mathbb{K} ($\mathbb{K} = \mathbb{R}, \mathbb{C}$) and $e \in H$, $\|e\| = 1$. If $\gamma, \Gamma \in \mathbb{K}$ and $x, y \in H$ are such that

$$(4.7) \quad \operatorname{Re} \left\langle \Gamma e - \frac{x \pm y}{2}, \frac{x \pm y}{2} - \gamma e \right\rangle \geq 0$$

or, equivalently,

$$(4.8) \quad \left\| \frac{x \pm y}{2} - \frac{\gamma + \Gamma}{2} \cdot e \right\| \leq \frac{1}{2} |\Gamma - \gamma|,$$

then we have the inequality

$$(4.9) \quad |\operatorname{Re} [\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle]| \leq \frac{1}{4} |\Gamma - \gamma|^2.$$

If the inner product space H is real, then (for $m, M \in \mathbb{R}$, $M > m$)

$$(4.10) \quad \left\langle Me - \frac{x \pm y}{2}, \frac{x \pm y}{2} - me \right\rangle \geq 0$$

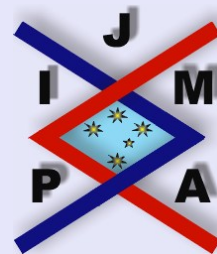
or, equivalently,

$$(4.11) \quad \left\| \frac{x \pm y}{2} - \frac{m + M}{2} \cdot e \right\| \leq \frac{1}{2} (M - m),$$

implies

$$(4.12) \quad |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} (M - m)^2.$$

In both inequalities (4.9) and (4.12), the constant $\frac{1}{4}$ is best possible.



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Proof. We only remark that, if

$$\operatorname{Re} \left\langle \Gamma e - \frac{x-y}{2}, \frac{x-y}{2} - \gamma e \right\rangle \geq 0$$

holds, then by Theorem 4.1, we get

$$\operatorname{Re} [-\langle x, y \rangle + \langle x, e \rangle \langle e, y \rangle] \leq \frac{1}{4} |\Gamma - \gamma|^2,$$

showing that

$$(4.13) \quad \operatorname{Re} [\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle] \geq -\frac{1}{4} |\Gamma - \gamma|^2.$$

Making use of (4.3) and (4.13) we deduce the desired result (4.9). \square

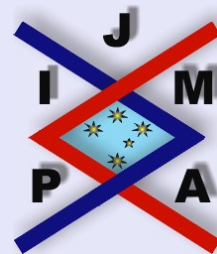
Finally, we may state and prove the following dual result as well

Proposition 4.3. *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over \mathbb{K} ($\mathbb{K} = \mathbb{R}, \mathbb{C}$) and $e \in H$, $\|e\| = 1$. If $\varphi, \Phi \in \mathbb{K}$ and $x, y \in H$ are such that*

$$(4.14) \quad \operatorname{Re} \left[(\Phi - \langle x, e \rangle) \left(\overline{\langle x, e \rangle} - \overline{\varphi} \right) \right] \leq 0,$$

then we have the inequalities

$$(4.15) \quad \begin{aligned} \|x - \langle x, e \rangle e\| &\leq [\operatorname{Re} \langle x - \Phi e, x - \varphi e \rangle]^{\frac{1}{2}} \\ &\leq \frac{\sqrt{2}}{2} [\|x - \Phi e\|^2 + \|x - \varphi e\|^2]^{\frac{1}{2}}. \end{aligned}$$



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Proof. We know that the following identity holds true (see (3.3))

$$(4.16) \quad \|x\|^2 - |\langle x, e \rangle|^2 = \operatorname{Re} \left[(\Phi - \langle x, e \rangle) \left(\overline{\langle x, e \rangle} - \overline{\varphi} \right) \right] + \operatorname{Re} \langle x - \Phi e, x - \varphi e \rangle.$$

Using the assumption (4.14) and the fact that

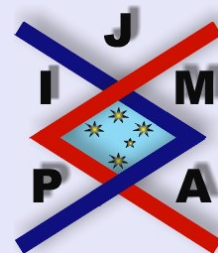
$$\|x\|^2 - |\langle x, e \rangle|^2 = \|x - \langle x, e \rangle e\|^2,$$

by (4.16) we deduce the first inequality in (4.15).

The second inequality in (4.15) follows by the fact that for any $v, w \in H$ one has

$$\operatorname{Re} \langle w, v \rangle \leq \frac{1}{2} (\|w\|^2 + \|v\|^2).$$

The proposition is thus proved. \square



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5. Integral Inequalities

Let (Ω, Σ, μ) be a measure space consisting of a set Ω , a σ -algebra of parts Σ and a countably additive and positive measure μ on Σ with values in $\mathbb{R} \cup \{\infty\}$. Denote by $L^2(\Omega, \mathbb{K})$ the Hilbert space of all real or complex valued functions f defined on Ω and 2-integrable on Ω , i.e.,

$$\int_{\Omega} |f(s)|^2 d\mu(s) < \infty.$$

The following proposition holds

Proposition 5.1. *If $f, g, h \in L^2(\Omega, \mathbb{K})$ and $\varphi, \Phi, \gamma, \Gamma \in \mathbb{K}$, are such that $\int_{\Omega} |h(s)|^2 d\mu(s) = 1$ and*

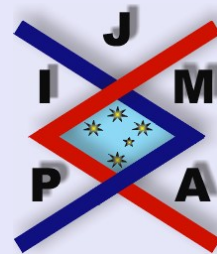
$$(5.1) \quad \int_{\Omega} \operatorname{Re} \left[(\Phi h(s) - f(s)) \left(\overline{f(s)} - \overline{\varphi h(s)} \right) \right] d\mu(s) \geq 0,$$

$$\int_{\Omega} \operatorname{Re} \left[(\Gamma h(s) - g(s)) \left(\overline{g(s)} - \overline{\gamma h(s)} \right) \right] d\mu(s) \geq 0$$

or, equivalently

$$(5.2) \quad \left(\int_{\Omega} \left| f(s) - \frac{\Phi + \varphi}{2} h(s) \right|^2 d\mu(s) \right)^{\frac{1}{2}} \leq \frac{1}{2} |\Phi - \varphi|,$$

$$\left(\int_{\Omega} \left| g(s) - \frac{\Gamma + \gamma}{2} h(s) \right|^2 d\mu(s) \right)^{\frac{1}{2}} \leq \frac{1}{2} |\Gamma - \gamma|,$$



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then we have the following refinement of the Grüss integral inequality

$$(5.3) \quad \left| \int_{\Omega} f(s) \overline{g(s)} d\mu(s) - \int_{\Omega} f(s) \overline{h(s)} d\mu(s) \int_{\Omega} h(s) \overline{g(s)} d\mu(s) \right| \\ \leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma| \\ - \left[\int_{\Omega} \operatorname{Re} \left[(\Phi h(s) - f(s)) \left(\overline{f(s)} - \overline{\varphi h(s)} \right) \right] d\mu(s) \right. \\ \left. \times \int_{\Omega} \operatorname{Re} \left[(\Gamma h(s) - g(s)) \left(\overline{g(s)} - \overline{\gamma h(s)} \right) \right] d\mu(s) \right]^{\frac{1}{2}}.$$

The constant $\frac{1}{4}$ is best possible.

The proof follows by Theorem 3.1 on choosing $H = L^2(\Omega, \mathbb{K})$ with the inner product

$$\langle f, g \rangle := \int_{\Omega} f(s) \overline{g(s)} d\mu(s).$$

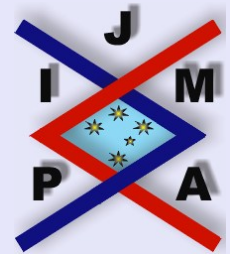
We omit the details.

Remark 5.1. It is obvious that a sufficient condition for (5.1) to hold is

$$\operatorname{Re} \left[(\Phi h(s) - f(s)) \left(\overline{f(s)} - \overline{\varphi h(s)} \right) \right] \geq 0,$$

and

$$\operatorname{Re} \left[(\Gamma h(s) - g(s)) \left(\overline{g(s)} - \overline{\gamma h(s)} \right) \right] \geq 0,$$



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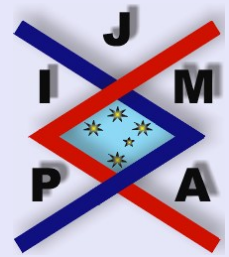


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for μ -a.e. $s \in \Omega$, or equivalently,

$$\left| f(s) - \frac{\Phi + \varphi}{2} h(s) \right| \leq \frac{1}{2} |\Phi - \varphi| |h(s)| \quad \text{and}$$

$$\left| g(s) - \frac{\Gamma + \gamma}{2} h(s) \right| \leq \frac{1}{2} |\Gamma - \gamma| |h(s)|,$$

for μ -a.e. $s \in \Omega$.

The following result may be stated as well.

Corollary 5.2. *If $z, Z, t, T \in \mathbb{K}$, $\mu(\Omega) < \infty$ and $f, g \in L^2(\Omega, \mathbb{K})$ are such that:*

$$(5.4) \quad \operatorname{Re} \left[(Z - f(s)) \left(\overline{f(s)} - \bar{z} \right) \right] \geq 0,$$

$$\operatorname{Re} \left[(T - g(s)) \left(\overline{g(s)} - \bar{t} \right) \right] \geq 0 \quad \text{for a.e. } s \in \Omega$$

or, equivalently

$$(5.5) \quad \left| f(s) - \frac{z + Z}{2} \right| \leq \frac{1}{2} |Z - z|,$$

$$\left| g(s) - \frac{t + T}{2} \right| \leq \frac{1}{2} |T - t| \quad \text{for a.e. } s \in \Omega$$

then we have the inequality

$$(5.6) \quad \left| \frac{1}{\mu(\Omega)} \int_{\Omega} f(s) \overline{g(s)} d\mu(s) - \frac{1}{\mu(\Omega)} \int_{\Omega} f(s) d\mu(s) \cdot \frac{1}{\mu(\Omega)} \int_{\Omega} \overline{g(s)} d\mu(s) \right|$$

$$\leq \frac{1}{4} |Z - z| |T - t| - \frac{1}{\mu(\Omega)} \left[\int_{\Omega} \operatorname{Re} \left[(Z - f(s)) (\overline{f(s)} - \bar{z}) \right] d\mu(s) \right. \\ \left. \times \int_{\Omega} \operatorname{Re} \left[(T - g(s)) (\overline{g(s)} - \bar{t}) \right] d\mu(s) \right]^{\frac{1}{2}}.$$

Using Theorem 4.1 we may state the following result as well.

Proposition 5.3. *If $f, g, h \in L^2(\Omega, \mathbb{K})$ and $\gamma, \Gamma \in \mathbb{K}$ are such that $\int_{\Omega} |h(s)|^2 d\mu(s) = 1$ and*

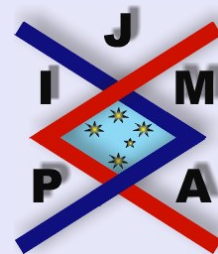
$$(5.7) \quad \int_{\Omega} \operatorname{Re} \left\{ \left[\Gamma h(s) - \frac{f(s) + g(s)}{2} \right] \right. \\ \left. \times \left[\frac{\overline{f(s)} + \overline{g(s)}}{2} - \bar{\gamma} \bar{h}(s) \right] \right\} d\mu(s) \geq 0$$

or, equivalently,

$$(5.8) \quad \left(\int_{\Omega} \left| \frac{f(s) + g(s)}{2} - \frac{\gamma + \Gamma}{2} h(s) \right|^2 d\mu(s) \right)^{\frac{1}{2}} \leq \frac{1}{2} |\Gamma - \gamma|,$$

then we have the inequality

$$(5.9) \quad I := \int_{\Omega} \operatorname{Re} [f(s) \overline{g(s)}] d\mu(s) \\ - \operatorname{Re} \left[\int_{\Omega} f(s) \overline{h(s)} d\mu(s) \cdot \int_{\Omega} h(s) \overline{g(s)} d\mu(s) \right] \\ \leq \frac{1}{4} |\Gamma - \gamma|^2.$$



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If (5.7) and (5.8) hold with “ \pm ” instead of “ $+$ ”, then

$$(5.10) \quad |I| \leq \frac{1}{4} |\Gamma - \gamma|^2.$$

Remark 5.2. It is obvious that a sufficient condition for (5.7) to hold is

$$(5.11) \quad \operatorname{Re} \left\{ \left[\Gamma h(s) - \frac{f(s) + g(s)}{2} \right] \cdot \left[\frac{\overline{f(s)} + \overline{g(s)}}{2} - \bar{\gamma} \bar{h}(s) \right] \right\} \geq 0$$

for a.e. $s \in \Omega$, or equivalently

$$(5.12) \quad \left| \frac{f(s) + g(s)}{2} - \frac{\gamma + \Gamma}{2} h(s) \right| \leq \frac{1}{2} |\Gamma - \gamma| |h(s)| \quad \text{for a.e. } s \in \Omega.$$

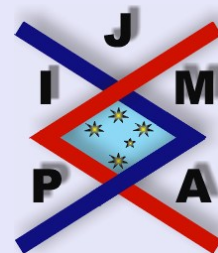
Finally, the following corollary holds.

Corollary 5.4. If $Z, z \in \mathbb{K}$, $\mu(\Omega) < \infty$ and $f, g \in L^2(\Omega, \mathbb{K})$ are such that

$$(5.13) \quad \operatorname{Re} \left[\left(Z - \frac{f(s) + g(s)}{2} \right) \left(\frac{\overline{f(s)} + \overline{g(s)}}{2} - \bar{z} \right) \right] \geq 0 \quad \text{for a.e. } s \in \Omega$$

or, equivalently

$$(5.14) \quad \left| \frac{f(s) + g(s)}{2} - \frac{z + Z}{2} \right| \leq \frac{1}{2} |Z - z| \quad \text{for a.e. } s \in \Omega,$$



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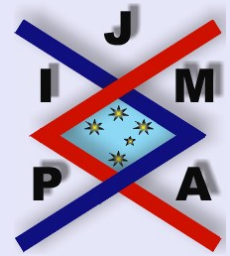
then we have the inequality

$$\begin{aligned}
 J &:= \frac{1}{\mu(\Omega)} \int_{\Omega} \operatorname{Re} \left[f(s) \overline{g(s)} \right] d\mu(s) \\
 &\quad - \operatorname{Re} \left[\frac{1}{\mu(\Omega)} \int_{\Omega} f(s) d\mu(s) \cdot \frac{1}{\mu(\Omega)} \int_{\Omega} \overline{g(s)} d\mu(s) \right] \\
 &\leq \frac{1}{4} |Z - z|^2.
 \end{aligned}$$

If (5.13) and (5.14) hold with “ \pm ” instead of “ $+$ ”, then

$$(5.15) \quad |J| \leq \frac{1}{4} |Z - z|^2.$$

Remark 5.3. *It is obvious that if one chooses the discrete measure above, then all the inequalities in this section may be written for sequences of real or complex numbers. We omit the details.*



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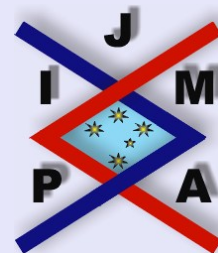
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