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SUB- SUPERSOLUTIONS AND THE EXISTENCE OF EXTREMAL SOLUTIONS IN NONCOERCIVE VARIATIONAL INEQUALITIES

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ABSTRACT. The paper is concerned with the solvability of variational inequalities that contain second-order quasilinear elliptic operators and convex functionals. Appropriate concepts of sub- and supersolutions (for inequalities) are introduced and existence of solutions and extremal solutions are discussed.

Key words and phrases: subsolution, supersolution, extremal solution, variational inequality.

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1. INTRODUCTION

We are concerned in this paper with the existence of solutions and extremal solutions of noncoercive variational inequalities of the form:

$$\begin{cases} \langle L(u), v-u\rangle - \langle G(u), v-u\rangle + j(v) - j(u) \geq 0, \ \forall v \in W_0^{1,p}(\Omega) \\ u \in W_0^{1,p}(\Omega). \end{cases} \tag{1.1}$$

Here Ω is a bounded region in $\mathbb{R}^N$ with smooth boundary. L is (the weak form of) the second order quasi-linear elliptic operator

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}[A_i(x,u,\nabla u)] + A_0(x,u,\nabla u) \tag{1.2}$$

and G is the lower-order term (cf. (2.1) and (2.6)). j is a convex functional, representing obstacles or unilateral conditions imposed on the solutions. Depending on the choice of j, the variational inequality (1.1) is the weak form of an equation or a complementarity problem that contains the operator (1.2) with various types of free boundaries or constraints (cf. e.g. [14, 3, 12]).

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Since G may have superlinear growth, the operator $L-G$ is noncoercive in general. The solvability of problem (1.1) can be studied by several approaches, for example, bifurcation methods (cf. [16, 27, 29, 19], etc.), recession arguments ([4, 2, 28, 1, 21, 22, 20], etc.), variational approaches ([24, 32, 21], etc.), or topological/fixed point methods ([30, 31], etc.).

We are concerned here with another way to study the solvability of (1.1), based on sub- and supersolutions. Recession methods have been quite popular recently in studying noncoercive problems. There are essential differences between these two approaches. Following recession approaches, the solvability of the problem is usually established by assuming conditions on asymptotic behaviors (i.e., behaviors when the involved variables are large) of the lower order terms (e.g., G in problem (1.1)). Problems of type (1.1) have been investigated in detail by recession arguments in [21, 22]. Improvements on the existence results based on recession approaches in [2, 28, 1, 21, 22] were presented recently in [20].

Compared to other methods, the sub-supersolution approach when applicable (i.e, when sub- and supersolutions exist) usually permits more flexible requirements on the growth rate of the perturbing term G (normally, one only needs to know the behaviors of G on bounded intervals). Moreover, based on the lattice structure of the space $W^{1,p}(\Omega)$, the sub- and supersolution method could also give insight into the ordering properties of the solution set between the sub- and supersolutions, and especially, the existence of maximal and minimal solutions. We refer the reader to [26] or [18] for more discussions on the difficulties arising when the sub-supersolution method is extended from equations (with natural symmetric structure) to variational inequalities (without symmetric settings), together with advantages of the method. More remarks on our approach here for (1.1), compared with the recession approach, are given in Remark 4.3.

This paper is the next step of our study plan proposed in [18] on sub- supersolution methods applied to variational inequalities. In that paper, we consider inequalities on closed convex sets, that is the particular case where j is the indicator function of a closed convex set K:

$$j(u) = \begin{cases} 0 & \text{if} \quad u \in K \\ \infty & \text{if} \quad u \notin K. \end{cases}$$

However, many interesting problems in mechanics and applied mathematics lead to other types of convex functionals, for example,

$$j(u) = \int_\Omega \psi(x, u(x))dx \ \text{ or } \ j(u) = \int_{\partial\Omega} \psi(x, u|_{\partial\Omega}(x))dS,$$

(cf. [12, 11]). Because of the nonsymmetric nature of the problem, sub- supersolution methods for smooth equations (cf. e.g. [13], [10], [9], or [15]) and also the arguments in [18] for inequalities on convex sets are not directly applicable to (1.1). The goal of this paper is to study the variational inequality (1.1) with more generality on the convex functional j by a sub- supersolution approach. The main difficulty we face here is defining sub- and supersolutions for the inequality (1.1) in an appropriate way such that the truncation–penalization machinery used for smooth, symmetric equations and for inequalities on convex sets can be extended to our nonsmooth, nonsymmetric case. Basically, we need to define sub- and supersolutions of (1.1) such that: (i) under reasonable conditions, one can show the existence of solutions and extremal solutions between sub- and supersolutions, (ii) there is some way to find sub- supersolutions or to check whether a given function is a sub- or supersolution, and (iii) sub- and supersolutions in inequalities extend those in equations.

To meet these requirements, in the next section, we need to make non straightforward extensions on the usual sub- and supersolution concepts for equations and also on those presented in

[18] to the more general situation of inequality (1.1) (cf. Definition 2.1). In Section 3, we prove several existence results for the inequality (1.1) based on the sub- supersolution concepts in Section 2. It is shown in Theorems 3.1 and 3.3 that if there exist a subsolution and a supersolution or merely a subsolution (or a supersolution) and a one-sided growth condition, then problem (1.1) is solvable. We also consider the existence of maximal or minimal (extremal) solutions, which are the biggest and smallest solutions of (1.1) (in certain ordering) within the interval between a subsolution and a supersolution (Theorems 3.2, 3.4). In Section 4, we consider some examples where one can actually find sub- and supersolutions. Combining with the results in Section 3, we obtain the existence of nonnegative nontrivial solutions and extremal solutions in eigenvalue problems for variational inequalities. The first problem is about an inequality containing a quasilinear elliptic operator and the convex term is given by an integral. By using constants as sub- and supersolutions, we find conditions such that the inequality has bounded solutions. The second example is an eigenvalue problem for an inequality that contains the p-Laplacian. By using sub- and supersolutions constructed from the principal eigenfunctions of the p-Laplacian, we show the existence of positive solutions of the inequality.

Compared with sub- supersolution methods for equations or for inequalities on convex sets, the development of the method for inequalities with general convex functionals (not necessarily indicators of convex sets) requires some nontrivial adaptation and modifications and new arguments in several places. Note that our presentation here is somewhat related to the results in [6, 8, 7] about sub- supersolution methods for differential inclusions with convex terms given by certain integrals. The concepts of sub- and supersolutions there are for (pointwise) inclusion are defined mostly pointwise, while our concepts here are for inequalities and are based on the dual between $W^{1,p}(\Omega)$ and $[W^{1,p}(\Omega)]^*$. An interesting question is to possibly compare the approach here with that in [6, 8, 7].

2. GENERAL SETTINGS

In this section, we consider the assumptions imposed on the inequality (1.1) and next define sub- and supersolutions for it. We use the notation $X := W^{1,p}(\Omega)$ and $X_0 := W_0^{1,p}(\Omega)$ for the usual first-order Sobolev spaces. In (1.1), L is a mapping from X to X^*, defined by

$$\langle L(u), v\rangle = \int_\Omega \left[\sum_{i=1}^N A_i(x,u,\nabla u)\partial_i v + A_0(x,u,\nabla u)v\right] dx,\ \forall u,v \in X, \tag{2.1}$$

where, for each $i \in \{0,1,\dots,N\}$, A_i is a Carathéodory function from $\Omega \times \mathbb{R}^{N+1}$ to $\mathbb{R}$. For $i \in \{1,\dots,N\}N$,

$$|A_i(x,u,\xi)| \le a_0(x) + b_0(|u|^{p-1} + |\xi|^{p-1}), \tag{2.2}$$

and

$$|A_0(x,u,\xi)| \le a_1(x) + b_1(|u|^{q-1} + |\xi|^{\frac{p}{q'}}), \tag{2.3}$$

for almost all $x \in \Omega$, all $u \in \mathbb{R},\ \xi \in \mathbb{R}^N$ with $b_0, b_1 > 0$, $a_0 \in L^{p'}(\Omega), a_1 \in L^{p'}(\Omega),\ 1 \le q < p^*$. (As usual, p' is the Hölder conjugate of p and p^* is its Sobolev conjugate.) Moreover,

$$\sum_{i=1}^N [A_i(x,u,\xi) - A_i(x,u',\xi')](\xi_i - \xi_i') + [A_0(x,u,\xi) - A_0(x,u',\xi')](u-u') > 0, \tag{2.4}$$

if $(u,\xi) \ne (u',\xi')$, and

$$\sum_{i=1}^N A_i(x,u,\xi)\xi_i + A_0(x,u,\xi)u \ge \alpha(|\xi|^p + |u|^p) - \beta(x), \tag{2.5}$$

for a.e. $x \in \Omega$, all $u \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, where $\alpha > 0$ and $\beta \in L^1(\Omega)$. The lower-order operator G is defined by

$$\langle G(u), v\rangle = \int_\Omega F(x, u, \nabla u) v dx, \tag{2.6}$$

where $F : \Omega \times \mathbb{R}^{N+1} \to \mathbb{R}$ is a Carathéodory function with certain growth conditions to be specified later. We also assume that j is a mapping from X to $\mathbb{R} \cup \{\infty\}$ such that the restriction $j|_{W_0^{1,p}(\Omega)}$ is convex and lower semicontinuous on $W_0^{1,p}(\Omega)$ with non empty effective domain. Before stating our theorem about existence of solutions, we need to define subsolutions and supersolutions for inequalities with convex functionals. These definitions extend those definitions presented in [18] for inequalities on closed convex sets. As seen in the following definitions, they are more complicated. As usual, we use the notation

$$u \vee v = \max\{u, v\}, \; u \wedge v = \min\{u, v\}.$$

and

$$A * B = \{a * b : a \in A, \; b \in B\},$$

where $A, B \subset W^{1,p}(\Omega)$ and $* \in \{\wedge, \vee\}$. As is well known, $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ are closed under the operations $\vee$ and $\wedge$, that is,

$$u, v \in W^{1,p}(\Omega)(\text{resp. } W_0^{1,p}(\Omega)) \Longrightarrow u \vee v, u \wedge v \in W^{1,p}(\Omega)(\text{resp. } W_0^{1,p}(\Omega)).$$

Definition 2.1. A function $\underline{u} \in W^{1,p}(\Omega)$ is called a W-subsolution of (1.1) if there exists a functional J (depending on u):

$$J = J_{\underline{u}} : W^{1,p}(\Omega) \to \mathbb{R} \cup \{\infty\},$$

such that

$$\begin{aligned} &\text{(i)} \quad \underline{u} \le 0 \text{ on } \partial\Omega \\ &\text{(ii)} \quad F(\cdot, \underline{u}, \nabla \underline{u}) \in L^{q'}(\Omega) \\ &\text{(iii)} \quad J(\underline{u}) < \infty, \quad \text{and} \end{aligned} \tag{2.7}$$

$$j(v \vee \underline{u}) + J(v \wedge \underline{u}) \le j(v) + J(\underline{u}), \; \forall v \in W_0^{1,p}(\Omega) \cap D(j), \tag{2.8}$$

and

$$\text{(iv)} \qquad \langle L(\underline{u}), v - \underline{u}\rangle - \langle G(\underline{u}), v - \underline{u}\rangle + J(v) - J(\underline{u}) \ge 0, \; v \in \underline{u} \wedge [W_0^{1,p}(\Omega) \cap D(j)], \tag{2.9}$$

($D(j) = \{v \in X : j(v) < \infty\}$ is the effective domain of j). We have a similar definition for W-supersolution $\overline{u}$: $\overline{u}$ is a W-supersolution of (1.1) if there exists $J = J_{\overline{u}} : W^{1,p}(\Omega) \to \mathbb{R} \cup \{\infty\}$ such that:

$$\begin{aligned} &\text{(i)} \quad \overline{u} \ge 0 \text{ on } \partial\Omega \\ &\text{(ii)} \quad F(\cdot, \overline{u}, \nabla \overline{u}) \in L^{q'}(\Omega) \\ &\text{(iii)} \quad J(\overline{u}) < \infty, \quad \text{and} \end{aligned} \tag{2.10}$$

$$j(v \wedge \overline{u}) + J(v \vee \underline{u}) \le j(v) + J(\overline{u}), \; \forall v \in W_0^{1,p}(\Omega) \cap D(j), \tag{2.11}$$

and

$$\text{(iv)} \qquad \langle L(\overline{u}), v - \overline{u}\rangle - \langle G(\overline{u}), v - \overline{u}\rangle + J(v) - J(\overline{u}) \ge 0, \; v \in \overline{u} \vee [W_0^{1,p}(\Omega) \cap D(j)]. \tag{2.12}$$

A subsolution of (1.1) is a finite maximum of W-subsolutions and a supersolution is a finite minimum of W-supersolutions.

Suppose there exist a subsolution $\underline{u} = \max\{\underline{u}_i : 1 \le i \le k\}$ and a supersolution $\overline{u} = \min\{\overline{u}_l : 1 \le l \le m\}$ of (1.1). We assume that F has the following growth condition:

$$|F(x,u,\xi)| \le a_2(x) + b_2|\xi|^{p/q'} \tag{2.13}$$

for a.e. $x \in \Omega$, all $\xi \in \mathbb{R}^N$, all u such that $\underline{u}_0(x) \le u \le \overline{u}_0(x))$, where $a_2 \in L^{q'}(\Omega)$, $b_2 \ge 0$, $q < p^*$ (p^* is the Sobolev conjugate of p), and

$$\underline{u}_0 = \min\{\underline{u}_i : 1 \le i \le k\},\ \overline{u}_0 = \max\{\overline{u}_l : 1 \le l \le m\}.$$

We conclude this section with some remarks.

Remark 2.1. (i) If u is a solution of (1.1), then u is a subsolution of (1.1), provided j satisfies the following condition:

$$j(v \vee u) + j(v \wedge u) \le j(v) + j(u), \tag{2.14}$$

for all $u, v \in W^{1,p}(\Omega)$. In fact, if u is a solution of (1.1) then it satisfies (i) – (ii). By choosing $J = j$, we see that (2.8) follows from (2.14). If $v = u \wedge w, w \in W_0^{1,p}(\Omega)$, then $v = 0$ on $\partial\Omega$, i.e., $v \in W_0^{1,p}(\Omega)$. Hence, (2.9) is a consequence of (1.1). Similarly, if (2.14) holds, then any solution is a supersolution.

(ii) (2.14) is satisfied for several usual convex functionals j. For example, if j is given by

$$j(u) = \int_E \psi(x,u)dx, \tag{2.15}$$

where E is a subset of Ω or $\partial\Omega$, $\psi : \Omega \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}$, is a Carathéodory function such that

$$\psi(x,u) \ge a_3(x) + b_3|u|^s,\ x \in \Omega,\ u \in \mathbb{R}, \tag{2.16}$$

where $a_3 \in L^1(\Omega)$ and $0 \le s < p^*$. j is well defined from $W^{1,p}(\Omega)$ to $\mathbb{R} \cup \{\infty\}$ and j is convex if $\psi(x,\cdot)$ is convex for a.e. $x \in \Omega$. Also, by Fatou's lemma, j is weakly lower semicontinuous. Let $u, v \in W^{1,p}(\Omega)$ and denote

$$\Omega_1 = \{x \in \Omega : v(x) < u(x)\},\ \Omega_2 = \{x \in \Omega : v(x) \ge u(x)\}.$$

Then,

$$\begin{aligned} j(v \wedge u) + j(v \vee u) &= \left(\int_{\Omega_1} + \int_{\Omega_2}\right)\psi(v \wedge u) + \left(\int_{\Omega_1} + \int_{\Omega_2}\right)\psi(v \vee u) \\ &= \int_{\Omega_1}\psi(x,u) + \int_{\Omega_2}\psi(x,v) + \int_{\Omega_1}\psi(x,v) + \int_{\Omega_2}\psi(x,u) \\ &= \int_{\Omega}\psi(x,u) + \int_{\Omega}\psi(x,v) \\ &= j(u) + j(v). \end{aligned} \tag{2.17}$$

Hence, (2.14) is satisfied. Note that from (2.16), $\psi(x,u)$ is bounded from below by a function in $L^1(\Omega)$. Thus, the integrals in (2.17) are in $\mathbb{R} \cup \{\infty\}$ and we can split and combine them as done.

(iii) If $j = I_K$, K is a closed convex set in $W_0^{1,p}(\Omega)$, then we recover the cases considered in [18]. Moreover, (2.14) holds provided K satisfied the condition

$$u, v \in K \Longrightarrow u \wedge v,\ u \vee v \in K. \tag{2.18}$$

As noted in [18], (2.18) is satisfied whenever K is defined by obstacles or by certain conditions on the gradients. We can also check that by using (2.15).

(iv) If $j = 0$, we have an equation in (1.1). By choosing $J = 0$ also, we see that (2.8) – (2.11) obviously hold and (2.9) – (2.12) reduce to the usual definitions of sub- and supersolutions of equations. If $j = I_K$ as in (iii), then by choosing $J = 0$, we see that the definition of subsolutions in [18] is equivalent to the definition in (i) – (iv) here. Thus, Definition 2.1 is an extension of that in [18].

(v) By choosing $J = 0$ in (2.8) and (2.9), we see that if $\underline{u}$ is a subsolution of the equation

$$\langle L(u), v\rangle - \langle G(u), v\rangle = 0,\ \forall v \in W_0^{1,p}(\Omega)$$

and $j(v \vee \underline{u}) \le j(v),\ \forall v \in W_0^{1,p}(\Omega) \cap D(j)$, then $\underline{u}$ is a subsolution of (1.1). Similar observations hold for supersolutions.

(vi) Compared to the definitions in [13, 10, 9, 15, 18], the new ingredient here is the introduction of the functional J in Definition 2.1, which permits more flexibility in constructing sub- and supersolutions (by choosing different J).

3. Main Existence Results

In this section, we state and prove our existence results for solutions and extremal solutions of (1.1), based on the concepts of sub- and supersolutions in Section 2.

Theorem 3.1. *Assume (1.1) has a subsolution $\underline{u}$ and a supersolution $\overline{u}$ such that $\underline{u} \le \overline{u}$ and that (2.13) holds. Then, (1.1) has a solution u such that $\underline{u} \le u \le \overline{u}$.*

Proof. We follow the usual truncation–penalization technique as in [13, 9, 15] or [18]. Therefore, we just outline the main arguments and present only the different points and modifications needed for our situation here. Let b be defined by (cf. [18])

$$b(x,t) = \begin{cases} [t - \overline{u}(x)]^{q-1} & \text{if } \ t > \overline{u}(x) \\ 0 & \text{if } \ \underline{u}(x) \le t \le \overline{u}(x) \\ -[-t + \underline{u}(x)]^{q-1} & \text{if } \ t < \underline{u}(x). \end{cases} \tag{3.1}$$

We have the following estimates (cf. (49) and (50) in [18]):

$$|b(x,t)| \le a_3(x) + c_3|t|^{q-1},$$

with $a_3 \in L^{q'}(\Omega)$, and

$$\int_\Omega b(\cdot, u)u \ge c_4\|u\|^q_{L^q(\Omega)} - c_5,$$

for all $u \in L^q(\Omega)$, where the c_i's ($i = 3, 4, 5$) are positive constants independent of u. We define $T_{il}\ (1 \le i \le k,\ 1 \le l \le m)$ and T by:

$$T_{il}(u)(x) = \begin{cases} \underline{u}_i(x) & \text{if } \ u(x) < \underline{u}_i(x) \\ u(x) & \text{if } \ \underline{u}_i(x) \le u(x) \le \overline{u}_l(x) \\ \overline{u}_l(x) & \text{if } \ u(x) > \overline{u}_l(x), \end{cases}$$

and

$$T(u)(x) = \begin{cases} \underline{u}(x) & \text{if } \ u(x) < \underline{u}(x) \\ u(x) & \text{if } \ \underline{u}(x) \le u(x) \le \overline{u}(x) \\ \overline{u}(x) & \text{if } \ u(x) > \overline{u}(x). \end{cases}$$

Let us consider the variational inequality

$$\begin{cases} \langle L(u) + \beta B(u) - C(u), v - u\rangle + j(u) - j(u) \ge 0,\ \forall v \in W_0^{1,p}(\Omega) \\ u \in W_0^{1,p}(\Omega), \end{cases} \tag{3.2}$$

with $\beta > 0$ sufficiently large. Consider the (nonlinear) operators B and C given by

$$\langle B(u), \phi\rangle = \int_\Omega b(\cdot, u)\phi,$$

and

$$\begin{aligned} \langle C(u), \phi\rangle &= \int_\Omega [F(\cdot, T(u), \nabla T(u)) \\ &\quad + \sum_{i,l} |F(\cdot, T_{il}(u), \nabla T_{il}(u)) - F(\cdot, T(u), \nabla T(u))|]\phi, \\ &\quad \forall u, \phi \in W_0^{1,p}(\Omega). \end{aligned} \tag{3.3}$$

Let us prove that $H = L + \beta B - C$ is pseudo-monotone on $W^{1,p}(\Omega)$. In fact, assume $w_n \rightharpoonup w$ in $W^{1,p}(\Omega)$ ("$\rightharpoonup$" denotes the weak convergence) and

$$\limsup_{n\to\infty} \langle H(w_n), w_n - w\rangle \leq 0. \tag{3.4}$$

We show that

$$\lim_{n\to\infty} \langle H(w_n), w_n - v\rangle \geq \langle H(w), w - v\rangle,\ \forall v \in W^{1,p}(\Omega). \tag{3.5}$$

Since the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, we have $w_n \to w$ in $L^p(\Omega)$. By passing to a subsequence, if necessary, we can assume that there is a function h in $L^p(\Omega)$ such that

$$\begin{cases} w_n \to w \ \text{ a.e. in } \Omega, \ \text{ and} \\ |w_n| \leq h \ \text{ a.e. in } \ \Omega,\ \forall n. \end{cases} \tag{3.6}$$

Since the sequence $\{w_n\}$ is bounded in $W^{1,p}(\Omega)$, the sequences $\{F(\cdot, T_{il}(w_n), \nabla T_{il}(w_n))\}$ and $\{F(\cdot, T(w_n), \nabla T(w_n))\}$ are uniformly bounded in $L^{q'}(\Omega)$. From (3.3), it follows that the sequence $\{(\beta B - C)(w_n)\}$ is bounded in $L^{q'}(\Omega)$. (3.6) thus implies that

$$\langle(\beta B - C)(w_n), w_n - w\rangle \to 0.$$

Hence, from (3.4),

$$\limsup\langle L(w_n) - L(w), w_n - w\rangle = \limsup\langle L(w_n), w_n - w\rangle \leq 0. \tag{3.7}$$

Since $\{A_i\}$ $(i = 0, 1, \ldots, n)$ satisfy (2.2) – (2.4), it follows from (3.6) and (3.7) that $w_n \to w$ in $W^{1,p}(\Omega)$. Consequently, $H(w_n) \to H(w)$ in $L^{q'}(\Omega)$ and (3.5) follows. This shows that $L + \beta B - C$ is pseudo-monotone. Using arguments similar to those in [18], we can prove that $L + \beta B - C$ is coercive on $W_0^{1,p}(\Omega)$. Moreover, this mapping is obviously continuous and bounded. Classical existence results for variational inequalities (cf. e.g. [23, 14]) give the existence of at least one solution $u \in W_0^{1,p}(\Omega)$ of (3.2). Also, it is clear that $u \in D(j)$. We prove that $\underline{u} \leq u$. Let $\underline{u}_q$ $(1 \leq q \leq k)$ be a W-subsolution. Since $u \in W_0^{1,p}(\Omega) \cap D(j)$, (2.9) with $\underline{u} = \underline{u}_q$ and $v = \underline{u} \wedge u$ gives

$$\langle(\underline{u}_q), \underline{u}_q \wedge u - \underline{u}_q\rangle - \langle G(\underline{u}_q), \underline{u}_q \wedge u - \underline{u}_q\rangle + J(\underline{u}_q \wedge u) - J(\underline{u}_q) \geq 0.$$

Since $\underline{u}_q \wedge u = \underline{u}_q - (\underline{u}_q - u)^+$, the above inequality becomes

$$-\langle(\underline{u}_q), (\underline{u}_q - u)^+\rangle - \langle G(\underline{u}_q), (\underline{u}_q - u)^+\rangle + J(\underline{u}_q \wedge u) - J(\underline{u}_q) \geq 0. \tag{3.8}$$

On the other hand, since $\underline{u}_q \vee u = 0$ on $\partial\Omega$, $v = \underline{u}_q \vee u \in W_0^{1,p}(\Omega)$. Letting v into (3.2) and noting that $\underline{u}_q \vee u = u + (\underline{u}_q - u)^+$, we get

$$\langle L(u) + \beta B(u) - C(u), (\underline{u}_q - u)^+\rangle + j(\underline{u}_q \vee u) - j(u) \geq 0. \tag{3.9}$$

Adding (3.8) and (3.9), one gets

$$\langle L(u)-L(\underline{u}_q),(\underline{u}_q-u)^+\rangle+\langle G(\underline{u}_q),(\underline{u}_q-u)^+\rangle \\ +\langle \beta B(u)-C(u),(\underline{u}_q-u)^+\rangle j(\underline{u}_q\vee u)-j(u)+J(\underline{u}_q\wedge u)-J(\underline{u}_q)\geq 0.$$

From (2.8),

$$j(\underline{u}_q\vee u)-j(u)+J(\underline{u}_q\wedge u)-J(\underline{u}_q)\leq 0.$$

Using the integral formulation of B, C and G, we get
(3.10)

$$\langle L(u)-L(\underline{u}_q),(\underline{u}_q-u)^+\rangle+\int_\Omega F(x,\underline{u}_q,\nabla\underline{u}_q)(\underline{u}_q-u)^+ +\beta\int_\Omega b(x,u)(\underline{u}_q-u)^+ \\ -\int_\Omega\left[F(x,T(u),\nabla T(u))+\sum_{i,l}|F(x,T_{il}(u),\nabla T_{il}(u))-F(x,T(u),\nabla T(u))|\right](\underline{u}_q-u)^+ \\ \geq 0.$$

We have

$$\langle L(u)-L(\underline{u}_q),(\underline{u}_q-u)^+\rangle+\beta\int_\Omega b(\cdot,u)(\underline{u}_q-u)^+ \\ +\int_\Omega\left[F(\cdot,\underline{u}_q)-F(\cdot,T(u))-\sum_{i,l}|F(\cdot,T_{il}(u))-F(\cdot,T(u))|\right](\underline{u}_q-u)^+\geq 0.$$

Using

$$\begin{aligned} F(x,\underline{u}_q(x))-F(x,T(u)(x))-\sum_{i,l}|F(x,T_{il}(u)(x))-F(x,T(u)(x))| \\ &\leq F(x,\underline{u}_q(x))-F(x,T(u)(x))-|F(x,T_{q_1}(u)(x))-F(x,T(u)(x))| \\ &= F(x,\underline{u}_q(x))-F(x,T(\underline{u})(x))-|F(x,\underline{u}_q(x))-F(x,\underline{u}(x))| \\ &\leq 0, \end{aligned}$$

we obtain

$$\begin{aligned} (3.11)\quad \int_\Omega\left[F(\cdot,\underline{u}_q)-F(\cdot,T(u))-\sum_{i,l}|F(\cdot,T_{il}(u))-F(\cdot,T(u))|\right](\underline{u}_q-u)^+ \\ &=\int_{\{\underline{u}_q>u\}}\left[F(\cdot,\underline{u}_q)-F(\cdot,T(u))-\sum_{i,l}|F(\cdot,T_{il}(u))-F(\cdot,T(u))|\right](\underline{u}_q-u)^+ \\ &\leq 0. \end{aligned}$$

Using the fact that

$$\begin{aligned} \langle L(u)-L(\underline{u}_q),(\underline{u}_q-u)^+\rangle \\ &=-\int_{\{\underline{u}_q-u>0\}}\Bigg\{\sum_{i=1}^N[A_i(x,\underline{u}_q,\nabla\underline{u}_q)-A_i(x,u,\nabla u)](\partial_i\underline{u}_q-\partial_i u) \\ &\quad +[A_0(x,\underline{u}_q,\nabla\underline{u}_q)-A_0(x,u,\nabla u)](\underline{u}_q-u)]\Bigg\} \\ &\leq 0, \end{aligned}$$

(by (2.4)), we have from (3.10) and (3.11) the following estimate

$$\begin{aligned} 0 &\le \int_\Omega b(\cdot,u)(\underline{u}_q - u)^+ \\ &= \int_{\{\underline{u}_q > u\}} b(\cdot,u)(\underline{u}_q - u) \\ &= -\int_{\{\underline{u}_q > u\}} (\underline{u} - u)^{q-1}(\underline{u}_q - u) \ \text{(since } a < \underline{u}_q \le \underline{u}) \\ &\le 0. \end{aligned}$$

Thus,

$$0 = \int_\Omega [(\underline{u}_q - u)^+]^q dx,$$

and $(\underline{u}_q - u)^+ = 0$ a.e. in Ω, i.e., $u \ge \underline{u}_q$ a.e. in Ω. Using these arguments for all $q \in \{1, \dots, k\}$, we see that $u \ge \underline{u}$. We can show in the same way that $u \le \overline{u}$. Now, from (3.1), we have $b(x, u(x)) = 0$ for almost all $x \in \Omega$, i.e., $B = 0$. Also, $T_{il}(u) = T(u) = u$, for all i, l and thus

$$\langle C(u), \phi\rangle = \int_\Omega F(\cdot, u, \nabla u)\phi = \langle G(u), \phi\rangle.$$

Hence, since u satisfies (3.2), it also satisfies (1.1), i.e., u is a solution of (1.1) and $\underline{u} \le u \le \overline{u}$. □

We now prove that (1.1) has a maximal and a minimal solution within the interval between $\underline{u}$ and $\overline{u}$.

Theorem 3.2. *Assume (1.1) has a subsolution $\underline{u}$ and a supersolution $\overline{u}$ such that $\underline{u} \le \overline{u}$. Moreover, (2.13) and (2.14) hold. Then, (1.1) has a maximal solution u^* and a minimal solution u_* such that*

$$\underline{u} \le u_* \le u^* \le \overline{u}, \tag{3.12}$$

that is, u_* and u^* are solutions of (1.1) that satisfy (3.12) and if u is a solution of (1.1) such that $\underline{u} \le u \le \overline{u}$ then $u_* \le u \le u^*$ on Ω.

The proof is similar to that of the particular case $j = I_K$, which was already presented in [18]. Therefore, it is omitted.

As in the case of variational inequalities on convex sets, we still have existence of solutions and extremal solutions provided only subsolutions (or supersolutions) exist together with certain one-sided growth conditions. We have in fact the following result.

Theorem 3.3. *Assume (1.1) has a subsolution $\underline{u}$ and F has the growth condition*

$$|F(x,u,\xi)| \le a_3(x) + b_3(|u|^\sigma + |\xi|^\sigma) \tag{3.13}$$

for a.e. $x \in \Omega$, all u such that $\underline{u}_0(x) \le u$, all $\xi \in \mathbb{R}^N$, where $0 \le \sigma < p-1,\ a \in L^{p'}(\Omega)$, and

$$\underline{u}_0 = \min\{\underline{u}_i : 1 \le i \le k\}.$$

Hence, (1.1) has a solution u such that $u \ge \underline{u}$.

The idea of the proof of this result is a combination of Theorem 3.1 stated above and an extension of Theorem 1 in [18]. We omit the proof and refer the reader to [18] for more details.

By looking closely at the set of solutions of (1.1), one can improve Theorem 3.3 and get the following stronger result.

Theorem 3.4. *Under the assumptions of Theorem 3.3, (1.1) has a maximal solution u^* and a minimal solution u_* such that*

$$\underline{u} \leq u_* \leq u^* \leq \overline{u}, \tag{3.14}$$

that is, u_* and u^* are solutions of (1.1) that satisfy (3.14) and if u is a solution of (1.1) such that $\underline{u} \leq u \leq \overline{u}$ then $u_* \leq u \leq u^*$ on Ω.

Proof. The proof follows the same line as that in Theorem 2, [18]. A main ingredient of the proof is the boundedness of the set

$$S = \{u \in W_0^{1,p}(\Omega) : u \geq \underline{u},\ u \text{ is a solution of (1.1)}\}$$

in $W_0^{1,p}(\Omega)$. Proving that S is bounded requires some different arguments from those in [18]. From (1.1) with $v = \phi$ being a fixed element in $D(j)$, we have

$$\langle L(u), \phi - u\rangle - \langle G(u), \phi - u\rangle + j(\phi) - j(u) \geq 0.$$

Therefore,

$$\begin{aligned}
\langle L(u), \phi\rangle &= \int_\Omega \left[\sum_i A_i(x, u, \nabla u)\partial_i u + A_0(x, u, \nabla u)u\right] \\
&\geq \alpha \int_\Omega (|\nabla u|^p + |u|^p) - \int_\Omega \beta dx \ \ \text{(by (2.5))} \\
&\geq \alpha \|u\|^p_{W_0^{1,p}(\Omega)} - c.
\end{aligned}$$

Hence,

$$\begin{aligned}
|\langle L(u), \phi\rangle| &\leq c[\sum_i \|A_i(\cdot, u, \nabla u)\|_{L^{p'}(\Omega)} \|\partial_i \phi\|_{L^p(\Omega)} + \|A_0(\cdot, u, \nabla u)\|_{L^{p'}(\Omega)} \|\phi\|_{L^p(\Omega)}] \\
&\leq c[\sum_i \|a_0 + b_0|u|^{p-1} + b_0|\nabla u|^{p-1}\|_{L^{p'}(\Omega)} \|\partial_i \phi\|_{L^p(\Omega)} \\
&\quad + \|a_1 + b_1|u|^{p-1} + |\nabla u|^{p-1}\|_{L^{p'}(\Omega)} \|\phi\|_{L^p(\Omega)}] \\
&\leq c(1 + \|u\|^{p-1}_{L^p(\Omega)} + \||\nabla u|\|^{p-1}_{L^p(\Omega)}) \\
&\leq c(1 + \|u\|^{p-1}_{W^{1,p}(\Omega)}),
\end{aligned}$$

(c denotes a generic constant). From (3.13), we have

$$|\langle G(u), \phi\rangle| \leq c(1 + \|u\|^\sigma_{W^{1,p}(\Omega)})$$

and

$$|\langle G(u), u\rangle| \leq c(1 + \|u\|^\sigma_{W^{1,p}(\Omega)})\|u\|_{W^{1,p}(\Omega)} \leq c(1 + \|u\|^{\sigma+1}_{W^{1,p}(\Omega)}).$$

Since j is convex and lower semi-continuous, there exist $a_4, b_4 \in \mathbb{R}$ such that

$$j(u) \geq a_4 + b_4 \|u\|_{W_0^{1,p}(\Omega)},\ \forall u \in W_0^{1,p}(\Omega).$$

Hence,

$$\alpha \|u\|^p_{W^{1,p}(\Omega)} - c \leq c(1 + \|u\|^{p-1}_{W^{1,p}(\Omega)} + \|u\|^\sigma_{W^{1,p}(\Omega)} + \|u\|_{W^{1,p}(\Omega)}).$$

Since $\sigma < p$, this shows that $\|u\|_{W^{1,p}(\Omega)} \leq c$ for all solutions u of (1.1) such that $u \geq \underline{u}$. Hence, S is bounded in $W^{1,p}(\Omega)$.

The remainder of the proof is similar to that of Theorem 3 in [18]. □

Remark 3.5. Note that if $A_i = A_i(x, \xi)$ ($i = 1, \ldots, N$) do not depend on u, then we can choose $A_0 = 0$ and all the results stated above still hold.

4. Some Examples

We now apply these general results to establish the existence of solutions and extremal solutions in some particular variational inequalities.

4.1. In this example, we study a quasi-linear elliptic variational inequality that contains a "unilateral" term given by an integral. Assume that for $i = 0, 1, \dots, N$, A_i satisfies

$$A_i(x, u, 0) = 0 \tag{4.1}$$

for a.e. $x \in \Omega$, all $u \in \mathbb{R}$ and consider the variational inequality

$$\begin{cases} \langle L(u), v - u\rangle - \lambda \int_\Omega F(x, u, \nabla u)(v - u) + j(v) - j(u) \geq 0, \ \forall v \in W_0^{1,p}(\Omega) \\ u \in W_0^{1,p}(\Omega). \end{cases} \tag{4.2}$$

Here, L and A are defined as in (2.1), (2.2), and (2.3) of Section 2. λ is a real parameter and

$$j(u) = \int_\Omega \psi(x, u(x))dx, \tag{4.3}$$

where $\psi : \Omega \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is a Carathéodory function such that

$$\psi(x, u) \geq -a(x) - b|u|^p, \tag{4.4}$$

where $a \in L^1(\Omega), b \geq 0$. It follows from this inequality that for $u \in W^{1,p}(\Omega)$, $\psi(x, u(x))$ is measurable and since $-a(x) - b|u(x)|^p \in L^1(\Omega)$, j is well defined and $j(u) \in \mathbb{R} \cup \{\infty\}$. Assume also that for almost all $x \in \Omega$, $\psi(x, \cdot)$ is convex. Hence, j is convex on $W^{1,p}(\Omega)$. It follows from Fatou's lemma that j is lower semicontinuous on that space. The following lemma shows the existence of constant sub- and supersolutions of (4.2).

Lemma 4.1. *(a) Assume $B \in \mathbb{R}, \ B \leq 0$ is such that*

$$\begin{aligned} &\text{(i)} \quad F(x, B, 0) \geq 0 \ \textit{ for a.e. } x \in \Omega \\ &\text{(ii)} \quad F(\cdot, B, 0) \in L^{q'}(\Omega) \end{aligned} \tag{4.5}$$

and

$$\text{(iii)} \qquad \psi(x, B) \leq \psi(x, v), \ \forall v \leq B, \tag{4.6}$$

then B is a subsolution of (4.2).

(b) Similarly, if $A \in \mathbb{R}, \ A \geq 0$ and

$$\begin{aligned} &\text{(i)} \quad F(x, A, 0) \leq 0 \ \textit{ for a.e. } x \in \Omega \\ &\text{(ii)} \quad F(\cdot, A, 0) \in L^{q'}(\Omega) \end{aligned} \tag{4.7}$$

and

$$\text{(iii)} \qquad \psi(x, A) \geq \psi(x, v), \ \forall v \geq A, \tag{4.8}$$

then A is a supersolution of (4.2).

Proof. (a) Choosing $J = 0$, we see that $\underline{u} = B$ satisfies conditions (i) – (iii) of Definition 2.1. Moreover, (2.8) becomes, in this case,

$$j(v \vee B) \leq j(v), \ v \in W_0^{1,p}(\Omega) \cap D(j), \tag{4.9}$$

i.e.,

$$\int_\Omega \psi(x, v(x) \vee B)dx \leq \int_\Omega \psi(x, v(x))dx.$$

In view of (4.3) and (4.4), this is equivalent to

$$(4.10)\quad \int_{\{x\in\Omega:v(x)>B\}}\psi(x,v)dx+\int_{\{x\in\Omega:v(x)\le B\}}\psi(x,B)dx \le \int_{\{x\in\Omega:v(x)>B\}}\psi(x,v)dx+\int_{\{x\in\Omega:v(x)\le B\}}\psi(x,v)dx.$$

Now, from (4.6), we have

$$\psi(x,B)\le\psi(x,v(x)) \text{ on } \{x\in\Omega: v(x)\le B\}$$

and thus

$$\int_{\{x\in\Omega:v(x)\le B\}}\psi(x,B)\le\int_{\{x\in\Omega:v(x)\le B\}}\psi(x,v),$$

which implies (4.10) and thus (4.9).

To check (2.9), we assume that $v=B\wedge w$ with some $w\in W_0^{1,p}(\Omega)\cap D(j)$. From (4.1) and the definition of L, $L(B)=0$. Since $v-B\le 0$, we have from (4.5)(i) that

$$\langle G(B),v-B\rangle=\int_\Omega F(x,B,0)(v-B)dx\le 0.$$

This implies (2.9), completing the proof of (a). The proof of (b) is similar. □

By using Theorems 3.2, 3.4, and Lemma 4.1, we have the following existence result for (4.2).

Theorem 4.2. *(a) Assume $B\in\mathbb{R}$ satisfies (4.5), ψ satisfies (4.6), and that*

$$|F(x,u,\xi)|\le a(x)+b(|u|^\sigma+|\xi|^\sigma)$$

for a.e. $x\in\Omega,\ u\ge B,\ \xi\in\mathbb{R}^N$, with $0\le\sigma<p-1$, $a\in L^{p'}(\Omega)$. Then, (4.2) has a minimal solution u_* and a maximal solution u^* such that $B\le u_*\le u^*$.

(b) Assume $A,B\in\mathbb{R}$ ($A\ge B$) satisfy (4.5) – (4.8) and that F has the growth condition

$$|F(x,u,\xi)|\le a(x)+b(|\xi|^{p/q'})$$

for a.e. $x\in\Omega,\ \xi\in\mathbb{R}^N,\ u\in[A,B]$ with $q<p^*$, $a\in L^{p'}(\Omega)$. Then, (4.2) has a minimal solution u_* and a maximal solution u^* such that $B\le u_*\le u^*\le A$.

Remark 4.3. As shown in Theorem 4.2 (see also Theorem 4.5 below), comparing sub-supersolution with recession method, we note that more flexible conditions are usually required in the first method. In fact, in Theorem 4.2, if there are $A,B\in\mathbb{R}$ satisfying (4.5) – (4.8), then the growth condition for F is limited to only $u\in[A,B]$. On the other hand, when recession arguments are used (cf. e.g. Theorems 3.4, 3.16 in [2], Theorems 2.3, 4.3 in [5], Theorems 2.5, 4.4, Corollary 6.10 in [21], or Theorems 1, 2, 3 in [20], etc.) conditions on behaviors of the functional G containing F at infinity are assumed, which is completely different from our approach here.

Another advantage of the method here is that we obtain, in addition to the solvability of (1.1), ordering properties of the solution sets, especially the existence of maximal and minimal solutions. This cannot be obtained by recession arguments. However, sub-supersolution method works only in function spaces with some lattice structure (such as $W^{1,p}(\Omega)$). That is the reason why the method is normally restricted to problems with second-order operators, such as (1.1). Recession methods, on the other hand, are applicable to higher order problems.

4.2. We consider in this example a variational inequality that contains the p-Laplacian, that is, the inequality (1.1) with

$$\langle L(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v\, dx,$$

In this case, $A_i = |\nabla u|^{p-2}\partial_i u,\ (1 \le i \le N)$ and $A_0 = 0$. The coefficients A_i $(i = 0, 1, \dots, N)$ clearly satisfy (2.2) and (2.3). For each $K > 0$, suppose that the function

$$x \mapsto \sup\{|F(x, u, \xi)| : 0 \le u \le K,\ |\xi| \le K\} \tag{4.11}$$

belongs to $L^{q'}(\Omega)$. We also assume the following behavior of $F(x, u, \xi)$ when u is very small or very large:

$$\liminf_{u\to 0^+, |\xi|\to 0} \frac{F(x, u, \xi)}{u^{p-1}} > \frac{\lambda_0}{\lambda} > \limsup_{u\to\infty, \xi\in\mathbb{R}^N} \frac{F(x, u, \xi)}{u^{p-1}}, \tag{4.12}$$

where λ_0 is the principal eigenvalue of the p-Laplacian,

$$\lambda_0 = \inf\left\{ \left(\int_\Omega |u|^p dx\right)^{-1} \int_\Omega |\nabla u|^p dx : u \in W_0^{1,p}(\Omega) \setminus \{0\} \right\}.$$

Let ϕ_0 be the (unique) eigenfunction corresponding to λ_0 such that $\phi_0(x) > 0$ for all $x \in \Omega$. (It is known, see e.g. [25], that $\phi_0 \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$.) By choosing $J = 0$ and using the arguments in [17] (Lemma 1), we can show that the function $\underline{u} = \epsilon\phi_0$ satisfies (2.9) for all $\epsilon > 0$ sufficiently small. On the other hand, let $\tilde{\Omega}$ be a bounded open region that contains $\overline{\Omega}$ and let $\tilde{\lambda}$ be the principal eigenvalue of the p-Laplacian on $\tilde{\Omega}$ and $\tilde{\phi}$ the corresponding eigenfunction on $\tilde{\Omega}$ such that $\tilde{\phi} > 0$ on $\tilde{\Omega}$. Then, we can prove that $\overline{u} = R\tilde{\phi}|_{\overline{\Omega}}$ satisfies (2.12) (with $J = 0$) for $R > 0$, sufficiently large. The proofs of these statements are somewhat lengthy; we refer the reader to [17] for more details. The following lemma is about the construction of sub- and supersolutions of (1.1) based on the eigenfunctions ϕ_0 and $\tilde{\phi}$ of the p-Laplacian.

Lemma 4.4. *(a) If there exists $C_1 > 0$ such that ψ is nonincreasing on $(-\infty, C_1)$, i.e.,*

$$\psi(x, u) \le \psi(x, v),\ \text{for a.e. } x \in \Omega, \text{ for all } u, v \text{ such that } v \le u < C_1, \tag{4.13}$$

then, for $\epsilon > 0$ sufficiently small, $\underline{u} = \epsilon\phi_0$ is a subsolution of (4.2).

(b) Similarly, if there exists $C_2 > 0$ such that ψ is nondecreasing on (C_2, ∞), i.e.,

$$\psi(x, u) \ge \psi(x, v),\ \text{for a.e. } x \in \Omega, \text{ for all } u, v \text{ such that } u \ge v > C_2, \tag{4.14}$$

then for $R > 0$ sufficiently large, $\overline{u} = R\tilde{\phi}|_\Omega$ is a supersolution of (4.2).

Proof. (a) We need only to check (2.8), i.e.,

$$j(v \vee \epsilon\phi_0) \le j(v),\ \forall v \in W_0^{1,p}(\Omega) \cap D(j).$$

This is equivalent to

$$\int_{\{x\in\Omega: v<\epsilon\phi_0\}} \psi(x, \epsilon\phi_0)dx + \int_{\{x\in\Omega: v\ge\epsilon\phi_0\}} \psi(x, v)dx \le \left(\int_{\{x\in\Omega: v<\epsilon\phi_0\}} + \int_{\{x\in\Omega: v\ge\epsilon\phi_0\}}\right) \psi(x, v)dx,$$

that is,

$$\int_{\{x\in\Omega: v<\epsilon\phi_0\}} \psi(x, \epsilon\phi_0)dx \le \int_{\{x\in\Omega: v<\epsilon\phi_0\}} \psi(x, v)dx. \tag{4.15}$$

Now, since $\phi_0 \in L^\infty(\Omega)$, $\epsilon\phi_0(x) < C_1$, for a.e. $x \in \Omega$ for $\epsilon > 0$ small. Hence, for $v < \epsilon\phi_0 < C_1$, (4.13) implies $\psi(x, v(x)) \geq \psi(x, \epsilon\phi_0(x))$ for a.e. $x \in \Omega$. This implies (4.15). Hence, $\epsilon\phi_0$ is a subsolution of (4.2). The proof of (b) is similar. □

As a consequence of Lemma 4.4 and Theorem 3.2, we have the following result.

Theorem 4.5. *Under the conditions (4.12) and (4.11), there exist a subsolution u_* and a supersolution u^* of (4.2) such that*

$$(0 <)\epsilon\phi_0 \leq u_* \leq u^* \leq R\tilde{\phi}|_\Omega,$$

where $\epsilon > 0$ sufficiently small and $R > 0$ sufficiently large. In particular, if F has the growth condition (4.11) and λ satisfies (4.12), then, (4.2) has a positive solution.

Remark 4.6. (a) (4.2) can be seen as an eigenvalue problem for a variational inequality. We have proved that for λ in certain appropriate interval (given by (4.12)) , then (4.2) has positive eigenfunction.

(b) One can replace (2.4) by a somewhat different condition, concentrating only on the higher-order coefficients A_i $(1 \leq i \leq N)$. Namely, we assume that $A_0 = 0$ and instead of (2.4),

$$\sum_{i=1}^{N} [A_i(x, u, \xi) - A_i(x, u', \xi)](\xi_i - \xi_i') > 0,$$

for a.e. $x \in \Omega$, all $u \in \mathbb{R}$, all $\xi, \xi' \in \mathbb{R}^N$, $\xi \neq \xi'$, and in (2.5),

$$\sum_{i=1}^{N} A_i(x, u, \xi)\xi_i \geq \alpha|\xi|^p - \beta, \text{ a.e. } x \in \Omega, \ \forall u \in \mathbb{R}, \ \forall \xi \in \mathbb{R}^N,$$

$(\alpha > 0)$. Also, we need a Hölder-continuity type of assumption with respect to u:

$$|A_i(x, u, \xi) - A_i(x, u', \xi)| \leq [k(x) + |u|^{p-1} + |u'|^{p-1} + |\xi|^{p-1}]\omega(|u - u'|),$$

for a.e. $x \in \Omega$, all $u, u' \in \mathbb{R}$, all $\xi \in \mathbb{R}^N$, where $\omega : [0, \infty) \to [0, \infty)$ satisfies

$$\int_{0+} \frac{dr}{\omega^{p'}(r)} = \infty$$

(cf. [7]). Assume also that j is given by an integral:

$$j(u) = \int_\Omega \psi(x, u(x))dx,$$

where ψ satisfies the following growth condition (instead of (2.16)):

$$|\psi(x, u)| \leq a_3(x) + b_3|u|^s, \text{ a.e. } x \in \Omega, \forall u \in \mathbb{R},$$

with $a_3 \in L^1(\Omega), \ 0 \leq s < p^*$. It can be checked that j is continuous. By using the arguments in [7] (see also [18]), we can prove the following result:

Theorem 4.7. *If u_1 and u_2 satisfy (2.9) with $J = j$, then $u = \max\{u_1, u_2\}$ also satisfies (2.9) with $J = j$.*

It follows that if u_1, u_2 are solutions of (1.1), then $\max\{u_1, u_2\}$ satisfies (2.9).

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