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ON HARMONIC FUNCTIONS CONSTRUCTED BY THE HADAMARD PRODUCT

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ABSTRACT. A function $f = u + iv$ defined in the domain $D \subset \mathbb{C}$ is harmonic in D if u, v are real harmonic. Such functions can be represented as $f = h + \bar{g}$ where h, g are analytic in D. In this paper the class of harmonic functions constructed by the Hadamard product in the unit disk, and properties of some of its subclasses are examined.

Key words and phrases: Harmonic functions, Hadamard product and extremal problems.

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1. INTRODUCTION

Let U denote the open unit disk in $\mathbb{C}$ and let $f = u + iv$ be a complex valued harmonic function on U. Since u and v are real parts of analytic functions, f admits a representation $f = h + \overline{g}$ for two functions h and g, analytic on U.

The Jacobian of f is given by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$. The necessary and sufficient conditions for f to be local univalent and sense-preserving is $J_f(z) > 0$, $z \in U$ [1].

Many mathematicians studied the class of harmonic univalent and sense-preserving functions on U and its subclasses [2, 5].

Here we discuss two classes obtained by the Hadamard product.

2. THE CLASS $\widetilde{P}_H^0(\alpha)$

Let P_H denote the class of all functions $f = h + \bar{g}$ so that $\operatorname{Re} f > 0$ and $f(0) = 1$ where h and g are analytic on U.

If the function $f_z + \overline{f_{\bar{z}}} = h' + \overline{g'}$ belongs to P_H for the analytic and normalized functions

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad \text{and} \qquad g(z) = \sum_{n=2}^{\infty} b_n z^n, \tag{2.1}$$

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then the class of functions $f = h + \overline{g}$ is denoted by $\widetilde{P}_H^0$ [5].

The function

$$t_\alpha(z) = z + \frac{1}{1+\alpha}z^2 + \cdots + \frac{1}{1+(n-1)\alpha}z^n + \cdots \tag{2.2}$$

is analytic on U when α is a complex number different from $-1, -\frac{1}{2}, -\frac{1}{3}, \dots$.

For $f \in \widetilde{P}_H^0$, we denote, by $\widetilde{P}_H^0(\alpha)$, the class of functions defined by

$$F = f * (t_\alpha + \overline{t_\alpha}). \tag{2.3}$$

Here $f * (t_\alpha + \overline{t_\alpha})$ is the Hadamard product of the functions f and $t_\alpha + \overline{t_\alpha}$. Therefore

$$\begin{aligned} F(z) &= H(z) + \overline{G(z)} \\ &= z + \sum_{n=2}^{\infty} \frac{a_n}{1+(n-1)\alpha} z^n + \sum_{n=2}^{\infty} \overline{\frac{b_n}{1+(n-1)\alpha} z^n} \\ &= z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=2}^{\infty} \overline{B_n z^n}, \qquad z \in U \end{aligned} \tag{2.4}$$

is in $\widetilde{P}_H^0(\alpha)$.

Conversely, if F is in the form (2.4), with a_n, b_n being the coefficients of $f \in \widetilde{P}_H^0$, then $F \in \widetilde{P}_H^0(\alpha)$.

Furthermore, if $\alpha = 0$, then as $F = f$, we have $\widetilde{P}_H^0(0) = \widetilde{P}_H^0$. Moreover $\widetilde{P}_H^0(\infty) = \{I : I(z) \equiv z,\ z \in U\}$ and since $I \in \widetilde{P}_H^0$, $\widetilde{P}_H^0 \cap \widetilde{P}_H^0(\alpha) \neq \phi$.

Theorem 2.1. *If $F \in \widetilde{P}_H^0(\alpha)$ then there exists $f \in \widetilde{P}_H^0$ so that*

$$\alpha[zF_z(z) + \overline{z}F_{\overline{z}}(z)] + (1-\alpha)F(z) = f(z). \tag{2.5}$$

Conversely, for any function $f \in \widetilde{P}_H^0$, there exists $F \in \widetilde{P}_H^0(\alpha)$ satisfying (2.5).

Proof. Let $F \in \widetilde{P}_H^0(\alpha)$. If $f \in \widetilde{P}_H^0$, then since

$$\alpha z t_\alpha'(z) + (1-\alpha)t_\alpha(z) = t_0(z),$$

as $F = f * (t_\alpha + \overline{t_\alpha})$ we obtain that

$$f(z) = \alpha[f(z) * (zt_\alpha'(z) + \overline{zt_\alpha'(z)})] + (1-\alpha)[f(z) * (t_\alpha(z) + \overline{t_\alpha}(z))].$$

Therefore,

$$f(z) = \alpha[zF_z(z) + \overline{z}F_{\overline{z}}(z)] + (1-\alpha)F(z).$$

Conversely, for $f \in \widetilde{P}_H^0$, from (2.1), (2.2) and (2.5),

$$z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=2}^{\infty} \overline{b_n z^n} = z + \sum_{n=2}^{\infty} [1+(n-1)\alpha]A_n z^n + \sum_{n=2}^{\infty} \overline{[1+(n-1)\alpha]B_n z^n}.$$

From these one obtains

$$A_n = \frac{a_n}{1+(n-1)\alpha} \qquad \text{and} \qquad B_n = \frac{b_n}{1+(n-1)\alpha}. \tag{2.6}$$

Therefore,

$$\begin{aligned} F(z) &= z + \sum_{n=2}^{\infty} \frac{a_n}{1+(n-1)\alpha} z^n + \sum_{n=2}^{\infty} \overline{\frac{b_n}{1+(n-1)\alpha} z^n} \\ &= f(z) * [t_\alpha(z) + \overline{t_\alpha}(z)]. \end{aligned}$$

□

Corollary 2.2. *A function $F = H + \overline{G}$ of the form (2.4) belongs to $\widetilde{P}^0_H(\alpha)$, if and only if*

(2.7) $$\operatorname{Re}\{z(\alpha H''(z) + \overline{\alpha}G''(z)) + H'(z) + G'(z)\} > 0, \quad z \in U.$$

Proof. If $F = H + \overline{G} \in \widetilde{P}^0_H(\alpha)$, then from Theorem 2.1

$$\alpha[zH'(z) + \overline{zG'(z)}] + (1-\alpha)[H(z) + \overline{G(z)}] = h(z) + \overline{g(z)} \in \widetilde{P}^0_H$$

and $h' + \overline{g'} \in P_H$. Hence

$$\begin{aligned} 0 &< \operatorname{Re}\{h'(z) + g'(z)\} \\ &= \operatorname{Re}\{\alpha z H''(z) + \alpha H'(z) + (1-\alpha)H'(z) \\ &\qquad + \overline{\alpha} z G''(z) + \overline{\alpha}G'(z) + (1-\overline{\alpha})G'(z)\} \\ &= \operatorname{Re}\{z(\alpha H''(z) + \overline{\alpha}G''(z)) + H'(z) + G'(z)\}. \end{aligned}$$

Conversely, if the function $F = H + \overline{G}$ of the form (2.4) satisfies (2.7), then by Theorem 2.1, $h' + \overline{g'} \in P_H$ and the function

$$f(z) = h(z) + \overline{g(z)} = \alpha[zH'(z) + \overline{zG'(z)}] + (1-\alpha)(H(z) + G(z))$$

is from the class $\widetilde{P}^0_H$. Hence by Theorem 2.1, $F = H + \overline{G} \in \widetilde{P}^0_H(\alpha)$. □

Proposition 2.3. $\widetilde{P}^0_H(\alpha)$ *is convex and compact.*

Proof. Let $F_1 = H_1 + \overline{G}_1,\ F_2 = H_2 + \overline{G}_2 \in \widetilde{P}^0_H(\alpha)$ and let $\lambda \in [0,1]$. Then

$$\begin{aligned} &\operatorname{Re}\{z[\alpha(\lambda H_1''(z) + (1-\lambda)H_2''(z)\bar{\alpha}(\lambda G_1''(z) + (1-\lambda)G_2''(z))] \\ &\qquad + \lambda[H_1'(z) + G_1'(z)] + (1-\lambda)[H_2'(z) + G_2'(z)]\} \\ &= \lambda \operatorname{Re}\{z[\alpha H_1''(z) + \bar{\alpha}G_1''(z)] + H_1'(z) + G_1'(z)\} \\ &\qquad + (1-\lambda)\operatorname{Re}\{z[\alpha H_2''(z) + \bar{\alpha}G_2''(z)] + H_2'(z) + G_2'(z)\} \\ &> 0. \end{aligned}$$

Hence, from Corollary 2.2, $\lambda\, F_1 + (1-\lambda)F_2 \in \widetilde{P}^0_H(\alpha)$. Therefore, $\widetilde{P}^0_H(\alpha)$ is convex.

On the other hand, let $F_n = H_n + \overline{G}_n \in \widetilde{P}^0_H(\alpha)$ and let $F_n \to F = H + \overline{G}$. By Corollary 2.2,

$$\alpha[zH_n^{'}(z) + \overline{zG_n'(z)}] + (1-\alpha)[H_n(z) + \overline{G_n(z)}] \in \widetilde{P}^0_H.$$

Since $\widetilde{P}^0_H$ is compact, [5],

$$\alpha[zH^{'}(z) + \overline{zG'(z)}] + (1-\alpha)[H(z) + \overline{G(z)}] \in \widetilde{P}^0_H.$$

Hence, by Theorem 2.1, $F = H + \bar{G} \in \widetilde{P}^0_H(\alpha)$. Therefore, $\widetilde{P}^0_H(\alpha)$ is compact. □

Proposition 2.4. *If $F = H + \overline{G} \in \widetilde{P}^0_H(\alpha)$ and $|z| = r < 1$ then*

$$\begin{aligned} -r + 2\ln(1+r) &\le \operatorname{Re}\{\alpha[zH'(z) + \overline{zG'(z)}] + (1-\alpha)[H(z) + \overline{G(z)}]\} \\ &\le -r - 2\ln(1-r). \end{aligned}$$

Equality is obtained for the function (2.3) where

$$f(z) = 2z + \ln(1-z) - 3\overline{z} - 3\ln(1-\overline{z}), \quad z \in U.$$

Proof. From Theorem 2.1, if $F = H + \overline{G} \in \widetilde{P}^0_H(\alpha)$, then there exists $f = h + \bar{g} \in \widetilde{P}^0_H$ so that

$$\alpha[zH^{'}(z) + \overline{zG'(z)}] + (1-\alpha)[H(z) + \overline{G(z)}] = f(z).$$

Since by [5, Proposition 2.2]

$$-r + 2\ln(1+r) \le \operatorname{Re} f(z) \le -r - 2\ln(1-r),$$

the proof is complete. □

Proposition 2.5. *If* $F = H + \overline{G} \in \widetilde{P}^0_H(\alpha)$ *and* $\operatorname{Re}\alpha > 0$, *then there exists an* $f \in \widetilde{P}^0_H$ *so that*

$$F(z) = \frac{1}{\alpha}\int_0^1 \zeta^{\frac{1}{\alpha}-2} f(z\zeta)\, d\zeta, \quad z \in U. \tag{2.8}$$

Proof. Since

$$t_\alpha(z) = \frac{1}{\alpha}\int_0^1 \zeta^{\frac{1}{\alpha}-1}\frac{z}{1-z\zeta}\, d\zeta, \quad |\zeta| \le 1, \quad \operatorname{Re}\alpha > 0,$$

and for $f = h + \overline{g} \in \widetilde{P}^0_H$

$$h(z) * \frac{z}{1-z\zeta} = \frac{h(z\zeta)}{\zeta}, \quad g(z) * \frac{z}{1-z\zeta} = \frac{g(z\zeta)}{\zeta},$$

we have

$$H(z) = h(z) * t_\alpha(z) = \frac{1}{\alpha}\int_0^1 \zeta^{\frac{1}{\alpha}-2} h(z\zeta)\, d\zeta$$

and

$$G(z) = g(z) * t_\alpha(z) = \frac{1}{\alpha}\int_0^1 \zeta^{\frac{1}{\alpha}-2} g(z\zeta)\, d\zeta.$$

Hence F is type (2.8). □

Theorem 2.6. *If* $\operatorname{Re}\alpha > 0$, *then* $\widetilde{P}^0_H(\alpha) \subset \widetilde{P}^0_H$. *Further, for any* $0 < \operatorname{Re}\alpha_1 \le \operatorname{Re}\alpha_2$, $\widetilde{P}^0_H(\alpha_2) \subset \widetilde{P}^0_H(\alpha_1)$.

Proof. Let $F \in \widetilde{P}^0_H(\alpha)$ and $\operatorname{Re}\ \alpha > 0$. Then there exists $f \in \widetilde{P}^0_H$ so that

$$F = H + \overline{G} = f * (t_\alpha + \overline{t_\alpha}) = (h * t_\alpha) + (\overline{g * t_\alpha}).$$

Hence, $0 < \operatorname{Re}\{h' + \overline{g'}\} = \operatorname{Re}\{h' + g'\}$ and since $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\{H' + G'\} > 0$, and $H(0) = 0,\ H'(0) = 1,\ G(0) = G'(0) = 0$ and hence $F = H + \overline{G} \in \widetilde{P}^0_H$.

For $0 < \operatorname{Re}\alpha_1 \le \operatorname{Re}\alpha_2$, if $F \in \widetilde{P}^0_H(\alpha_2)$, from Corollary 2.2

$$\begin{aligned} 0 &< \operatorname{Re}\{z(\alpha_2 H''(z) + \overline{\alpha_2} G''(z)) + H'(z) + G'(z)\} \\ &\le \operatorname{Re}\{z(\alpha_1 H''(z) + \overline{\alpha_1} G''(z)) + H'(z) + G'(z)\} \end{aligned}$$

we get $F \in \widetilde{P}^0_H(\alpha_1)$. □

Remark 2.7. For some values of α, $\widetilde{P}^0_H(\alpha) \subset \widetilde{P}^0_H$ is not true. It is known [5, Corollary 2.5] that the sharp inequalities

$$|a_n| \le \frac{2n-1}{n} \quad \text{and} \quad |b_n| \le \frac{2n-3}{n} \tag{2.9}$$

are true. Hence, for example, the function

$$f(z) = z + \sum_{n=2}^{\infty}\frac{2n-1}{n}z^n + \sum_{n=2}^{\infty}\frac{2n-3}{n}\overline{z^n}$$

belongs to $\widetilde{P}^0_H$. In this case

$$F(z) = z + \sum_{n=2}^{\infty} \frac{2n-1}{n[1+(n-1)\alpha]} z^n + \overline{\sum_{n=2}^{\infty} \frac{2n-3}{n[1+(n-1)\alpha]} z^n}$$

belongs to the class $\widetilde{P}_H^0(\alpha)$ for $\alpha \in \mathbb{C},\ \alpha \neq -1/n,\ n \in N$. However, for $\operatorname{Re}\alpha \in \left(-\frac{|\alpha|^2}{3}, 0\right),\ \alpha \neq -1, -\frac{1}{2}, \ldots$ as the coefficient conditions of $\widetilde{P}_H^0$ given in (2.9) are not satisfied, $F \notin \widetilde{P}_H^0$. Hence for each $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha \in \left(-\frac{|\alpha|^2}{3}, 0\right),\ \alpha \neq -1, -\frac{1}{2}, \ldots,\ \widetilde{P}_H^0(\alpha) - \widetilde{P}_H^0 \neq \phi$.

Theorem 2.8. *Let $F = H + \overline{G} \in \widetilde{P}_H^0(\alpha)$. Then*

(i) $||A_n| - |B_n|| \leq \dfrac{2}{n|1+(n-1)\alpha|}, \quad n \geq 1$

(ii) *If F is sense-preserving, then*

$$|A_n| \leq \frac{2n-1}{n} \frac{1}{|1+(n-1)\alpha|}, \quad n = 1, 2, \ldots$$

and

$$|B_n| \leq \frac{2n-3}{n} \frac{1}{|1+(n-1)\alpha|}, \quad n = 2, 3, \ldots$$

Equality occurs for the functions of type (2.3) where

$$f(z) = \frac{2z}{1-z} + \ln(1-z) - \frac{3\bar{z} - \bar{z}^2}{1-\bar{z}} - 3\ln(1-\bar{z}), \quad z \in U.$$

Proof. By (2.6),

$$||A_n| - |B_n|| = \frac{1}{|1+(n-1)\alpha|} \, ||a_n| - |b_n|| \, .$$

Also by [5, Theorem 2.3], we have

$$||a_n| - |b_n|| \leq \frac{2}{n}$$

the required results are obtained.

On the other hand, from (2.6) and from the coefficient relations in $\widetilde{P}_H^0$ given in (2.9), we obtain the coefficient inequalities for $\widetilde{P}_H^0(\alpha)$. □

3. THE CLASS $P_H(\beta, \alpha)$

Let $f = h + \overline{g}$ for analytic functions

$$h(z) = 1 + \sum_{n=1}^{\infty} a_n z^n \qquad \text{and} \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n$$

on U. The class $P_H(\beta)$ of all functions with $\operatorname{Re} f(z) > \beta,\ 0 \leq \beta < 1$ and $f(0) = 1$ is studied in [5].

Let us consider the function

(3.1) $$k_\alpha(z) = 1 + \frac{1}{1+\alpha} z + \cdots + \frac{1}{1+n\alpha} z^n + \cdots,\ \alpha \in \mathbb{C},\ \alpha \neq -1, -\frac{1}{2}, \ldots$$

which is analytic on U.

For $f \in P_H(\beta)$, let us denote the class of functions

(3.2) $$F = f * (k_\alpha + \overline{k_\alpha}) = (h * k_\alpha) + (\overline{g * k_\alpha}) = H + \overline{G},$$

by $P_H(\beta, \alpha)$. If $\alpha = 0$, then since $F = f$, $P_H(\beta, 0) = P_H(\beta)$.

Therefore,

$$F(z) = H(z) + \overline{G(z)} = 1 + \sum_{n=1}^{\infty} \frac{a_n}{1+n\alpha} z^n + \overline{\sum_{n=1}^{\infty} \frac{b_n}{1+n\alpha} z^n} = 1 + \sum_{n=1}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B_n z^n}, \qquad z \in U \tag{3.3}$$

Theorem 3.1. *If $F \in P_H(\beta, \alpha)$ then there exists an $f \in P_H(\beta)$, so that*

$$\alpha[zF_z(z) + \overline{z}F_{\overline{z}}(z)] + F(z) = f(z). \tag{3.4}$$

Conversely, for $f \in P_H(\beta)$, there is a solution of (3.4) belonging to $P_H(\beta, \alpha)$.

Proof. Since $k_0(z) = \alpha z k'_\alpha(z) + k_\alpha(z)$, for $f \in P_H(\beta)$, using the fact that, $f = f * (k_0 + \overline{k_0})$,

$$f(z) = \alpha[f(z) * (zk'_\alpha(z) + \overline{zk'_\alpha(z)})] + [f(z) * (k_\alpha(z) + \overline{k_\alpha(z)})]$$

is obtained. Hence, for $F \in P_H(\beta, \alpha)$

$$f(z) = \alpha[zF_z(z) + \overline{z}F_{\overline{z}}(z)] + F(z).$$

Conversely, let $f = h + \overline{g} \in P_H(\beta)$ be given by (3.4). Hence, we can write

$$h(z) = \alpha z H'(z) + H(z), \quad g(z) = \alpha z G'(z) + G(z). \tag{3.5}$$

From the system (3.5) the analytic functions H and G are in the form

$$H(z) = 1 + \sum_{n=1}^{\infty} \frac{a_n}{1+n\alpha} z^n = h(z) * k_\alpha(z),$$

$$G(z) = \sum_{n=1}^{\infty} \frac{b_n}{1+n\alpha} z^n = g(z) * k_\alpha(z).$$

Hence the function $F = H + \overline{G}$ belongs to the class $P_H(\beta, \alpha)$. □

Corollary 3.2. *The necessary and sufficient conditions for a function F of form (3.3) to belong to $P_H(\beta, \alpha)$ are*

$$\operatorname{Re}\{z(\alpha H'(z) + \overline{\alpha} G'(z)) + H(z) + G(z)\} > \beta, \quad z \in U. \tag{3.6}$$

Proof. If $F \in P_H(\beta, \alpha)$ then by Theorem 3.1,

$$\begin{aligned} \beta &< \operatorname{Re}\{f(z)\} \\ &= \operatorname{Re}\{\alpha[zF_z(z) + \overline{z}F_{\overline{z}}(z)] + F(z)\} \\ &= \operatorname{Re}\{z(\alpha H'(z) + \overline{\alpha} G'(z)) + H(z) + G(z)\}, \ z \in U. \end{aligned}$$

Conversely, if a function $F = H + \overline{G}$ of form (3.3) satisfies (3.6), then

$$z\alpha H'(z) + H(z) + \alpha\overline{zG'(z)} + \overline{G(z)} \in P_H(\beta).$$

Hence, from Theorem 3.1, we have $F = H + \bar{G} \in P_H(\beta, \alpha)$. □

Proposition 3.3. *If $F \in P_H(\beta, \alpha)$, $\operatorname{Re}\alpha > 0$ then there exists an $f \in P_H(\beta)$ so that*

$$F(z) = \frac{1}{\alpha} \int_0^1 t^{\frac{1}{\alpha}-1} f(zt)\,dt, \quad z \in U. \tag{3.7}$$

The converse is also true.

Proof. Since

$$k_\alpha(z) = \frac{1}{\alpha}\int_0^1 t^{\frac{1}{\alpha}-1}\frac{1}{1-zt}dt, \quad \text{Re } \alpha > 0,$$

and for $f = h + \overline{g} \in P_H(\beta)$,

$$h(z) * \frac{1}{1-zt} = h(zt) \quad \text{and} \quad g(z) * \frac{1}{1-zt} = g(zt),$$

we obtain

$$H(z) = h(z) * k_\alpha(z) = \frac{1}{\alpha}\int_0^1 t^{\frac{1}{\alpha}-1}h(zt)dt$$

and

$$G(z) = g(z) * k_\alpha(z) = \frac{1}{\alpha}\int_0^1 t^{\frac{1}{\alpha}-1}g(zt)dt.$$

Therefore, $F = H + \overline{G}$ is of type (3.7). □

Theorem 3.4. *Let $F \in P_H(\beta, \alpha)$. Then*

(i) $||A_n| - |B_n|| \leq \dfrac{2(1-\beta)}{|1+n\alpha|}, \quad n \geq 1$

(ii) *If F is sense- preserving, then for $n = 1, 2, \ldots$*

$$|A_n| \leq \frac{(1-\beta)(n+1)}{|1+n\alpha|} \quad \textit{and} \quad |B_n| \leq \frac{(1-\beta)(n-1)}{|1+n\alpha|}.$$

Equality is valid for the functions of type (3.2) where

$$f(z) = \text{Re}\left\{\frac{1+(1-2\beta)z}{1-z}\right\} + i\,\text{Im}\left\{\frac{1+z}{1-z}\right\}. \tag{3.8}$$

Proof. Let $F \in P_H(\beta, \alpha)$. Then from (3.3), as the coefficient relation for $P_H(\beta)$ is

$$||a_n| - |b_n|| \leq 2(1-\beta)$$

[5, Proposition 3.4], the required inequalities are obtained.

On the other hand, from (3.3), as the coefficient relations for $P_H(\beta)$ are

$$|a_n| \leq (1-\beta)(n+1) \quad \text{and} \quad |b_n| \leq (1-\beta)(n-1)$$

the required inequalities are obtained. □

Proposition 3.5. *If $F = H + \overline{G} \in P_H(\beta, \alpha)$, then for $X = \{\eta : |\eta| = 1\}$ and $z \in U$,*

$$H(z) + G(z) = 2(1-\beta)\int_{|\eta|=1} k_\alpha(\eta z)\, d\mu(\eta).$$

Here μ is the probability measure defined on the Borel sets on X.

Proof. From [5, Corollary 3.3] there exists a probability measure μ defined on the Borel sets on X so that

$$h(z) + g(z) = \int_{|\eta|=1} \frac{1+(1-2\beta)\, z\eta}{1-z\eta}\, d\mu(\eta).$$

Taking the Hadamard product of both sides by $k_\alpha(z)$, we get

$$\begin{aligned} H(z) + G(z) &= \int_{|\eta|=1}\left\{\left(k_\alpha(z) * \frac{1}{1-z\eta}\right) + (1-2\beta)\eta\left(k_\alpha(z) * \frac{z}{1-z\eta}\right)\right\}\, d\mu(\eta) \\ &= \int_{|\eta|=1}\left\{k_\alpha(\eta z) + (1-2\beta)\eta\frac{k_\alpha(\eta z)}{\eta}\right\}\, d\mu(\eta). \end{aligned}$$

□

Theorem 3.6. *If* $\operatorname{Re}\alpha \geq 0$, *then* $P_H(\beta,\alpha) \subset P_H(\beta)$. *Further if* $0 \leq \operatorname{Re}\alpha_1 \leq \operatorname{Re}\alpha_2$, *then* $P_H(\beta,\alpha_2) \subset P_H(\beta,\alpha_1)$.

Proof. Let $F \in P_H(\beta,\alpha)$ and $\operatorname{Re}\alpha \geq 0$. Then as $\operatorname{Re}\{h' + g'\} > \beta$, we have $\operatorname{Re}\{H' + G'\} > \beta$ and $F(0) = 1$. Hence $F \in P_H(\beta)$. Further as $0 \leq \operatorname{Re}\alpha_1 \leq \operatorname{Re}\alpha_2$, for $F \in P_H(\beta,\alpha_2)$

$$\begin{aligned}\beta &< \operatorname{Re}\{z(\alpha_2 H'(z) + \overline{\alpha}_2 G'(z)) + H(z) + G(z)\}\\ &< \operatorname{Re}\{z(\alpha_1 H'(z) + \overline{\alpha}_1 G'(z)) + H(z) + G(z)\}.\end{aligned}$$

Therefore, by Corollary 3.2, $F \in P_H(\beta,\alpha_1)$. □

For $f \in P_H$, the class $B_H(\alpha)$ consisting of the functions $F = f * (k_\alpha + \overline{k_\alpha})$ is studied in [2]. The relation between the classes $P_H(\beta,\alpha)$ and $B_H(\alpha)$ is given as follows.

Proposition 3.7. *For* $\operatorname{Re}\alpha \geq 0$, $P_H(\beta,\alpha) \subset B_H(\alpha)$.

Proof. If $F \in P_H(\beta,\alpha)$ then there exists an $f \in P_H(\beta)$ so that $F = f * (k_\alpha + \overline{k_\alpha})$. Since $\operatorname{Re} f(z) > \beta,\ f(0) = 1$ and $0 \leq \beta < 1$, $\operatorname{Re} f(z) > 0$. Hence, $f \in P_H$. By the definition of $B_H(\alpha)$, $F \in B_H(\alpha)$. □

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