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SOME INEQUALITIES BETWEEN MOMENTS OF PROBABILITY DISTRIBUTIONS

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Abstract

In this paper inequalities between univariate moments are obtained when the random variate, discrete or continuous, takes values on a finite interval. Further some inequalities are given for the moments of bivariate distributions.

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1. Introduction

The rth order moment μ'_r of a continuous random variate which takes values on the interval $[a, b]$ with pdf $\phi(x)$ is defined as

$$\mu'_r = \int_a^b x^r \phi(x) dx. \tag{1.1}$$

For a random variate which takes a discrete set of finite values x_i $(i = 1, 2, \ldots, n)$ with corresponding probabilities p_i $(i = 1, 2, \ldots, n)$, we define

$$\mu'_r = \sum_{i=1}^{n} p_i x_i^r. \tag{1.2}$$

The power mean of order r is defined as

$$M_r = (\mu'_r)^{1/r} \qquad \text{for } r \neq 0, \tag{1.3}$$

and

$$M_r = \lim_{r \to 0} (\mu'_r)^{1/r} \qquad \text{for } r = 0. \tag{1.4}$$

It may be noted here that M_{-1}, M_0 and M_1 respectively define harmonic mean, geometric mean and arithmetic mean.

Kapur [1] has reported the following bound for μ'_r when μ'_s is prescribed, $r > s$, and the random variate, discrete or continuous, takes values in the interval $[a, b]$ with $a \geq 0$,

$$(\mu'_s)^{r/s} \leq \mu'_r \leq \frac{(b^r - a^r)\ \mu'_s + a^r b^s - a^s b^r}{b^s - a^s}. \tag{1.5}$$

Inequality (1.5) gives the condition which the given moment values must necessarily satisfy in order to be the moments of a probability distribution in the given range $[a, b]$. Kapur [1] was motivated by the consideration of maximizing the entropy function subject to certain constraints. But before maximizing the entropy function one has to see whether the given moment values are consistent or not i.e whether there is any probability distribution which corresponds to the given values of moments. If there is no such distribution then the efforts of finding out the maximum entropy probability distribution will not produce any result and hence we should not proceed to apply Lagrange's or any other method to find the maximum entropy probability distribution, [2].

Here we try to obtain a generalization of inequality (1.5) for the case where r and s can assume any real value. This shall help us in deducing bounds between power means. This will also provide us with an alternate proof of inequality (1.5) and enable us to tighten it when the random variate takes a finite set of discrete values $x_1, x_2, \ldots, x_n$.

In addition some inequalities between the moments of bivariate distributions are also obtained.

2. Some Elementary Inequalities

We prove the following theorems:

Theorem 2.1. *If r is a positive real number and s is any non zero real number with $r > s$ then for $a \leq x \leq b$; with $a > 0$, we have*

$$x^r \leq \frac{(b^r - a^r)\ x^s + a^r b^s - a^s b^r}{b^s - a^s}, \tag{2.1}$$

and for x lying outside (a, b) we have

$$x^r \geq \frac{(b^r - a^r)\ x^s + a^r b^s - a^s b^r}{b^s - a^s}. \tag{2.2}$$

If r is a negative real number with $r > s$ then inequality (2.1) holds for x lying outside (a, b) and inequality (2.2) holds for $a \leq x \leq b$.

Proof. Consider the following function $f(x)$ for positive real values of x:

$$f(x) = x^r - \frac{b^r - a^r}{b^s - a^s} x^s + \frac{a^s b^r - a^r b^s}{b^s - a^s}, \tag{2.3}$$

where r and s are real numbers such that $r > s$ and $s \neq 0$. The function $f(x)$ is continuous in the interval $[a, b]$ with $a > 0$. Then $f'(x)$ is given by

$$f'(x) = x^{s-1} \left[r x^{r-s} - s \left(\frac{b^r - a^r}{b^s - a^s} \right) \right]. \tag{2.4}$$

$f'(x)$ vanishes at $x = 0$ and c, where

$$c = \left[\frac{s}{r}\left(\frac{b^r - a^r}{b^s - a^s}\right)\right]^{\frac{1}{r-s}}. \tag{2.5}$$

By Rolle's theorem we have that c lies in the interval (a, b).

If r is a positive real number and s is a negative real number with $r > s$ then $f'(x) \leq 0$ iff $x \leq c$. This means that $f(x)$ decreases in the interval $(0, c)$ and increases in the interval (c, ∞). Further, since c lies in the interval (a, b) and $f(a) = f(b) = 0$, it follows that

$$f(x) \leq 0 \qquad \text{for} \quad a \leq x \leq b, \tag{2.6}$$

and for x lying outside (a, b)

$$f(x) \geq 0. \tag{2.7}$$

On substituting the value of $f(x)$ from equation (2.3) in inequalities (2.6) and (2.7), we obtain inequalities (2.1) and (2.2) respectively.

If r is a negative real number with $r > s$ then $f'(x) \leq 0$ iff $x \geq c$. This means that $f(x)$ increases in the interval $(0, c)$ and decreases in the interval (c, ∞). Since c lies in the interval (a, b) and $f(a) = f(b) = 0$ it follows that inequality (2.7) holds for $a \leq x \leq b$ while inequality (2.6) holds for x lying outside (a, b) and thus we get inequalities for the case when r is negative real number. □

Theorem 2.2. *For $a \leq x \leq b$ with $a > 0$, we have*

$$x^r \leq \frac{(b^r - a^r)\log x + a^r \log b - b^r \log a}{\log b - \log a}, \tag{2.8}$$

and for x lying outside (a,b), we have

$$x^r \geq \frac{(b^r - a^r)\log x + a^r \log b - b^r \log a}{\log b - \log a}, \tag{2.9}$$

where r is a real number.

Proof. Consider the following function $f(x)$ defined for positive real values of x,

$$f(x) = x^r - \frac{(b^r - a^r)}{\log b - \log a}\log x + \frac{b^r \log a - a^r \log b}{\log b - \log a}. \tag{2.10}$$

The function $f(x)$ is continuous in the interval $[a,b]$ where $a > 0$. Then $f'(x)$ is given by

$$f'(x) = \frac{1}{x}\left[rx^r - \frac{b^r - a^r}{\log b - \log a}\right], \tag{2.11}$$

and we have $f'(x) = 0$ at $x = c$ where

$$c = \left[\frac{b^r - a^r}{r(\log b - \log a)}\right]^{\frac{1}{r}}. \tag{2.12}$$

By Rolle's Theorem we have that c lies in the interval (a,b). Also $f'(x) \leq 0$ iff $x \leq c$. This means that $f(x)$ decreases in the interval $(0,c)$ and increases in the interval (c,∞). Further, since c lies in the interval (a,b) and $f(a) = f(b) = 0$ it follows that

$$f(x) \leq 0 \qquad \text{for} \quad a \leq x \leq b, \tag{2.13}$$

and for x lying outside (a,b) we have

$$f(x) \geq 0. \tag{2.14}$$

On substituting the value of $f(x)$ from equation (2.10) in inequalities (2.13) and (2.14), we obtain inequalities (2.8) and (2.9) respectively. $\square$

Some Inequalities Between Moments of Probability Distributions

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3. Inequalities Between Moments

Theorem 3.1. *Let r be a positive real number and s be any non zero real number with $r > s$. If a positive random variate takes values x_i $(i = 1, 2, \ldots, n)$ in the interval $[a, b]$, with $a > 0$, then we have*

$$\mu'_r \leq \frac{(b^r - a^r)\ \mu'_s + a^r b^s - a^s b^r}{b^s - a^s}, \tag{3.1}$$

and

$$\mu'_r \geq \frac{\left(x_j^r - x_{j-1}^r\right)\ \mu'_s + x_{j-1}^r x_j^s - x_{j-1}^s x_j^r}{x_j^s - x_{j-1}^s}, \tag{3.2}$$

where $j = 2, 3, \ldots, n$.

If a continuous random variate takes values in the interval $[a, b]$, with $a > 0$, then the upper bound for μ'_r is given by the inequality (3.1) whereas the lower bound is given by following inequality

$$\mu'_r \geq \left(\mu'_s\right)^{r/s}. \tag{3.3}$$

Proof. It is seen that μ'_r can be expressed in terms of μ'_s in the following form :

$$\begin{aligned} \mu'_r = {} & \left(\frac{x_\beta^r - x_\alpha^r}{x_\beta^s - x_\alpha^s}\right) \mu'_s + \frac{x_\beta^s x_\alpha^r - x_\alpha^s x_\beta^r}{x_\beta^s - x_\alpha^s} \\ & + \sum_{i=1}^{n} p_i \left[x_i^r - \frac{x_\beta^r - x_\alpha^r}{x_\beta^s - x_\alpha^s} x_i^s + \frac{x_\beta^r x_\alpha^s - x_\alpha^r x_\beta^s}{x_\beta^s - x_\alpha^s}\right], \end{aligned} \tag{3.4}$$

where α and β take one of the values among $1, 2, \ldots, n$ with $\alpha \neq \beta$. Without loss of generality we can arrange values of the variate such that $a = x_1 \leq x_2 \leq \cdots \leq x_n = b$. If we take $\alpha = 1$ and $\beta = n$ then $x_1 \leq x_i \leq x_n$ for $i = 1, 2, \ldots, n$. It follows from (2.1) that the last term in equation (3.4) is negative and we conclude that the upper bound for μ'_r is given by inequality (3.1). Further if $x_\alpha = x_{j-1}$ and $x_\beta = x_j$, $j = 2, 3, \ldots, n$ then each x_i lies outside (x_{j-1}, x_j) and it follows from (2.2) that the last term in equation (3.4) is positive and we conclude that the lower bound for μ'_r is given by inequality (3.2). It is also clear that equality in the inequalities (3.1) and (3.2) holds iff $n = 2$.

If the value of μ'_s coincides with one of x^s_{j-1} or x^s_j, then from inequality (3.2) we have

$$\mu'_r \geq (\mu'_s)^{r/s}. \tag{3.5}$$

Also if x_{j-1} approaches $x_j$we get inequality (3.5) and we conclude that for a continuous random variate the lower bound for μ'_r is given by inequality (3.5). The upper bound for μ'_r can be deduced from Theorem 2.1. Multiplying both sides of inequality (2.1) by pdf $\phi(x)$ we get, on using the properties of definite integrals, inequality (3.1). □

Theorem 3.2. *Let r and s be negative real numbers with $r > s$. If a positive random variate takes values x_i $(i = 1, 2, \ldots, n)$ in the interval $[a, b]$, with $a > 0$, we have*

$$\mu'_r \geq \frac{(b^r - a^r)\ \mu'_s + a^r b^s - a^s b^r}{b^s - a^s}, \tag{3.6}$$

and

$$\mu'_r \leq \frac{\left(x_j^r - x_{j-1}^r\right)\mu'_s + x_{j-1}^r x_j^s - x_{j-1}^s x_j^r}{x_j^s - x_{j-1}^s}, \tag{3.7}$$

where $j = 2, 3, \ldots, n$.

If a continuous random variate takes values in the interval $[a, b]$, *with* $a > 0$, *the lower bound for* μ'_r *is given by inequality (3.6) whereas the upper bound for* μ'_r *is given by following inequality:*

$$\mu'_r \leq \left(\mu'_s\right)^{r/s}. \tag{3.8}$$

Proof. We again consider equation (3.4). If we take $\alpha = 1$ and $\beta = n$ then $x_1 \leq x_i \leq x_n$ for $i = 1, 2, \ldots, n$. It follows from Theorem 2.1 that the last term in equation (3.4) is positive and we conclude that the lower bound for μ'_r is given by inequality (3.6). Also if $x_\alpha = x_{j-1}$ and $x_\beta = x_j$, $j = 2, 3, \ldots, n$ then each x_i lies outside (x_{j-1}, x_j). It follows from Theorem 2.1 that the last term in equation (3.4) is negative and we conclude that the upper bound for μ'_r is given by inequality (3.7). Also if x_{j-1} approaches x_j we get inequality (3.8). The lower bound for μ'_rcan be deduced from Theorem 2.1. Multiplying both sides of inequality (2.2) by pdf $\phi(x)$ we get, on using the properties of definite integrals, inequality (3.6). □

Theorem 3.3. *For a random variate which takes values* x_i $(i = 1, 2, \ldots, n)$ *in the interval* $[a, b]$, *with* $a > 0$, *we have*

$$\mu'_r \leq \frac{\left(b^r - a^r\right)\log M_0 + a^r \log b - b^r \log a}{\log b - \log a}, \tag{3.9}$$

and

$$\mu'_r \geq \frac{\left(x_j^r - x_{j-1}^r\right) \log M_0 + x_{j-1}^r \log x_j - x_j^r \log x_{j-1}}{\log x_j - \log x_{j-1}}, \tag{3.10}$$

where $j = 2, 3, \ldots n$, r *is a real number and*

$$M_0 = x_1^{P_1} x_2^{P_2} \cdots x_n^{P_n}. \tag{3.11}$$

For a continuous random variate which takes values in the interval $[a, b]$ *with* $a > 0$ *the upper bound for* μ'_r *is given by inequality (3.9) whereas the lower bound for* μ'_r *is given by the following inequality*

$$\mu'_r \geq (M_0)^r. \tag{3.12}$$

Proof. It is seen that μ'_rcan be expressed in terms of $\log M_0$ in the following form:

$$\begin{aligned} \mu'_r = {} & \frac{x_\beta^r - x_\alpha^r}{\log x_\beta - \log x_\alpha} \log M_0 + \frac{x_\alpha^r \log x_\beta - x_\beta^r \log x_\alpha}{\log x_\beta - \log x_\alpha} \\ & + \sum_{i=1}^{n} P_i \left[x_i^r - \frac{x_\beta^r - x_\alpha^r}{\log x_\beta - \log x_\alpha} \log x_i + \frac{x_\beta^r \log x_\alpha - x_\alpha^r \log x_\beta}{\log x_\beta - \log x_\alpha} \right]. \end{aligned} \tag{3.13}$$

Without loss of generality we can arrange values of the variate such that $a = x_1 < x_2 < \cdots < x_n = b$. If we take $\alpha = 1$and $\beta = n$ then $x_1 \leq x_i \leq x_n$ for $i = 1, 2, \ldots, n$. It follows from Theorem 2.2 that last term in equation (3.13) is negative and we conclude that the upper bound for μ'_r is given by inequality

(3.9). Also if $x_\alpha = x_{j-1}$ and $x_\beta = x_j$, $j = 2, 3, \ldots, n$ then each x_i lies outside (x_{j-1}, x_j). It follows from Theorem 2.2 that the last term in equation (3.13) is positive and we conclude that the lower bound for μ'_r is given by inequality (3.10).

If the value of M_0 coincides with one of x_{j-1} or x_j then from inequality (3.10) we have

$$\mu'_r \geq (M_0)^r. \tag{3.14}$$

Also if x_{j-1} approaches x_j we get inequality (3.14) and we conclude that for the continuous random variate the lower bound for μ'_r is given by inequality (3.14). The upper bound for μ'_r can be deduced from Theorem 2.2. Multiplying both sides of inequality (2.8) by pdf $\phi(x)$ we get, on using the properties of definite integrals, inequality (3.9). $\square$

4. Inequalities Between Moments of Bivariate Distributions

The moments of a bivariate probability distribution are the generalizations of those of univariate one and are equally important in the theory of mathematical statistics. For a discrete probability distribution, if p_i is the probability of the occurrence of the pair of values (x_i, y_i) $i = 1, 2, \ldots, n$, the moment μ'_{rs} about the origin is given by

$$\mu'_{rs} = \sum_{i=1}^{n} P_i x_i^r y_i^s. \tag{4.1}$$

We obtain a bound on μ'_{rs} in the following theorem:

Theorem 4.1. *Let μ'_{rs} be the moment of order r in x and of order s in y, about the origin $(0,0)$, of a discrete bivariate probability distribution. The random variates x and y vary respectively over the finite positive real intervals $[a,b]$ and $[c,d]$. If $\mu'_{k\,m}$ is the corresponding moment of order k in x and m in y such that $r \geq k$, $s \geq m$ and $rm = ks$ then we must have by necessity,*

$$\left(\mu'_{k\,m}\right)^{\frac{r+s}{k+m}} \leq \mu'_{r\,s} \leq \frac{\left(b^r d^s - a^r c^s\right)\mu'_{k\,m} + a^r c^s b^k d^m - a^k c^m b^r d^s}{b^k d^m - a^k c^m}. \tag{4.2}$$

Proof. If u, v, α and β are positive real numbers with $\alpha + \beta = 1$ then from Hölder's inequality [3],

$$\sum_{i=1}^{n} u_i^{\alpha} v_i^{\beta} \leq \left(\sum_{i=1}^{n} u_i\right)^{\alpha} \left(\sum_{i=1}^{n} v_i\right)^{\beta}. \tag{4.3}$$

We make the following substitutions,

$$u_i = p_i x_i^r y_i^s\ ,\ v_i = p_i \quad \text{and} \quad \alpha = \frac{k+m}{r+s}. \tag{4.4}$$

This gives,

$$u_i^{\alpha} v_i^{\beta} = p_i x_i^k y_i^m. \tag{4.5}$$

Also,

$$\left(\sum_{i=1}^{n} u_i\right)^{\alpha} = \left(\sum_{i=1}^{n} p_i x_i^r y_i^s\right)^{\frac{k+m}{r+s}}, \tag{4.6}$$

and

$$\left(\sum_{i=1}^{n} v_i\right)^{\beta} = 1. \tag{4.7}$$

From (4.3), (4.5), (4.6) and (4.7), we get

$$\mu'_{r\,s} \geq \left(\mu'_{k\,m}\right)^{\frac{r+s}{k+m}}. \tag{4.8}$$

For $a \leq x \leq b$, $c \leq y \leq d$, $r \geq k$, $s \geq m$ and $rm = ks$, inequality (4.3) will remain valid if we substitute $n = 2$, $u_1 = p_1 a^r c^s$, $u_2 = p_2 b^r d^s$, $v_1 = p_1$, $v_2 = p_2$, $\alpha = \frac{k+m}{r+s}$,

$$p_1 = \frac{b^k d^m - x^k y^m}{b^k d^m - a^k c^m},$$

and

$$p_2 = \frac{x^k y^m - a^k c^m}{b^k d^m - a^k c^m}.$$

These substitutions give

$$x^r y^s \leq \frac{\left(b^r d^s - a^r c^s\right) x^k y^m + a^r c^s b^k d^m - a^k c^m b^r d^s}{b^k d^m - a^k c^m}. \tag{4.9}$$

Without loss of generality we can have that the random variate take values $a = x_1 < x_2 < \cdots < x_n = b$ and $c = y_1 < y_2 < \cdots < y_n = d$ therefore $a \leq x_i \leq b$ and $c \leq y_i \leq d$, $i = 1, 2, \ldots, n$. From inequality (4.9), it follows that

$$x_i^r y_i^s \leq \frac{\left(b^r d^s - a^r c^s\right) x_i^k y_i^m + a^r c^s b^k d^m - a^k c^m b^r d^s}{b^k d^m - a^k c^m},$$

or

$$\sum_{i=1}^{n} P_i x_i^r y_i^s \leq \frac{\left(b^r d^s - a^r c^s\right) \sum_{i=1}^{n} P_i x_i^k y_i^m + \left(a^r c^s b^k d^m - a^k c^m b^r d^s\right) \sum_{i=1}^{n} P_i}{b^k d^m - a^k c^m},$$

or

$$\mu'_{r\,s} \leq \frac{\left(b^r d^s - a^r c^s\right) \mu'_{k\,m} + a^r c^s b^k d^m - a^k c^m b^r d^s}{b^k d^m - a^k c^m}.$$

Inequality (4.2) also holds for the continuous bivariate distributions. The upper bound in inequality (4.2) is a consequence of inequality (4.9). Multiplying both sides of inequality (4.9) by joint pdf $\phi(x, y)$ and integrating over the corresponding limits, we get the maximum value of μ'_{rs} where

$$\mu'_{rs} = \int_a^b \int_c^d x^r y^s \phi(x, y) dx dy \qquad \text{and} \qquad \int_a^b \int_c^d \phi(x, y) dx dy = 1.$$

Now consider,

$$\begin{aligned}
\frac{\int_a^b \int_c^d f^\alpha g^\beta dxdy}{\left(\int_a^b \int_c^d f\,dxdy\right)^\alpha \left(\int_a^b \int_c^d g\,dxdy\right)^\beta}
&= \int_a^b \int_c^d \left(\frac{f}{\int_a^b \int_c^d f\,dxdy}\right)^\alpha \left(\frac{g}{\int_a^b \int_c^d g\,dxdy}\right)^\beta dxdy \\
&\le \int_a^b \int_c^d \left[\frac{\alpha f}{\int_a^b \int_c^d f\,dxdy} + \frac{\beta g}{\int_a^b \int_c^d g\,dxdy}\right] dx\,dy \\
&= 1,
\end{aligned}$$

where $\alpha + \beta = 1$ and f and g are positive functions. We therefore have

$$\int_a^b \int_c^d f^\alpha g^\beta dx\,dy \le \left(\int_a^b \int_c^d f\,dxdy\right)^\alpha \left(\int_a^b \int_c^d g\,dxdy\right)^\beta, \tag{4.10}$$

and make the following substitutions,

$$f = x^r y^s \quad \phi(x,y), g = \phi(x,y) \qquad \text{and} \qquad \alpha = \frac{k+m}{r+s}.$$

Inequality (4.10) then yields the minimum value of μ'_{rs}. $\square$

5. Applications of Results

On using the results derived in Section 3 and giving particular values to r and s it is possible to derive a host of results connecting the Harmonic mean (H), Geometric mean (G), Arithmetic mean (A) and Root mean square (R) when one of the means is given and the random variate takes the prescribed set of positive values $x_1, x_2, \ldots, x_n$.

If we put $r = +1$ and $s = -1$ we get inequalities between A and H, if we put $r = 0$ and $s = -1$ we get inequalities between G and H, and so on. Root mean square R corresponds to $r = 2$. In particular the following inequalities are obtained from the general result,

$$\left[\left(x_{j-1} + x_j\right) A - x_{j-1}x_j\right]^{\frac{1}{2}} \leq R \leq \left[\left(a + b\right) A - ab\right]^{\frac{1}{2}}, \tag{5.1}$$

$$\frac{ab}{a + b - A} \leq H \leq \frac{x_{j-1}x_j}{x_{j-1} + x_j - A}, \tag{5.2}$$

$$\left(b^{A-a}a^{b-A}\right)^{\frac{1}{b-a}} \leq G \leq \left(x_j^{A-x_{j-1}}x_{j-1}^{x_j-A}\right)^{\frac{1}{x_j-x_{j-1}}}, \tag{5.3}$$

$$\begin{aligned}\left[x_{j-1}^2 + x_{j-1}x_j + x_j^2 - \frac{x_{j-1}x_j\left(x_{j-1} + x_j\right)}{H}\right]^{\frac{1}{2}} \\ \leq R \leq \left[a^2 + ab + b^2 - \frac{ab\left(a + b\right)}{H}\right]^{\frac{1}{2}},\end{aligned} \tag{5.4}$$

$$(x_{j-1}+x_j)-\frac{x_{j-1}x_j}{H}\le A\le a+b-\frac{ab}{H}, \tag{5.5}$$

$$\left[x_j^{x_j(H-x_{j-1})}x_{j-1}^{x_{j-1}(x_j-H)}\right]^{\frac{1}{H(x_j-x_{j-1})}}\le G\le\left[b^{b(H-a)}a^{a(b-H)}\right]^{\frac{1}{H(b-a)}}, \tag{5.6}$$

$$\frac{\log\left(\frac{G}{x_{j-1}}\right)^{x_j^2}\left(\frac{x_j}{G}\right)^{x_{j-1}^2}}{\log\frac{x_j}{x_{j-1}}}\le R^2\le\frac{\log\left(\frac{G}{a}\right)^{b^2}\left(\frac{b}{G}\right)^{a^2}}{\log\frac{b}{a}}, \tag{5.7}$$

$$\frac{\log\left(\frac{b}{a}\right)^{ab}}{\log\left(\frac{G}{a}\right)^{a}\left(\frac{b}{G}\right)^{b}}\le H\le\frac{\log\left(\frac{x_j}{x_{j-1}}\right)^{x_{j-1}x_j}}{\log\left(\frac{G}{x_{j-1}}\right)^{x_{j-1}}\left(\frac{x_j}{G}\right)^{x_j}}, \tag{5.8}$$

$$\frac{\log\left(\frac{G}{x_{j-1}}\right)^{x_j}\left(\frac{x_j}{G}\right)^{x_{j-1}}}{\log\frac{x_j}{x_{j-1}}}\le A\le\frac{\log\left(\frac{G}{a}\right)^{b}\left(\frac{b}{G}\right)^{a}}{\log\frac{b}{a}}, \tag{5.9}$$

$$\frac{R^2+ab}{a+b}\le A\le\frac{R^2+x_{j-1}x_j}{x_{j-1}+x_j}, \tag{5.10}$$

$$\frac{ab\,(a+b)}{a^2+ab+b^2-R^2} \le H \le \frac{x_{j-1}x_j\,(x_{j-1}+x_j)}{x_{j-1}^2+x_{j-1}x_j+x_j^2-R^2}, \tag{5.11}$$

and

$$b^{\frac{R^2-a^2}{b^2-a^2}}\,a^{\frac{b^2-R^2}{b^2-a^2}} \le G \le x_j^{\frac{R^2-x_{j-1}^2}{x_j^2-x_{j-1}^2}}\,x_{j-1}^{\frac{x_j^2-R^2}{x_j^2-x_{j-1}^2}}, \tag{5.12}$$

where $j = 2, 3, \ldots, n$.

We now deduce the result that the power mean M_r is an increasing function of r. If r is positive and s is any real number with $r > s$ then from inequality (3.3) we have

$$\left(\mu_r'\right)^{\frac{1}{r}} \ge \left(\mu_s'\right)^{\frac{1}{s}}, \tag{5.13}$$

or $M_r \ge M_s$.

If r is a negative real number with $r > s$ we again get inequality (5.13) from inequality (3.8). From inequality (3.12) we have $M_r \ge M_0$ for $r > 0$, and $M_r \le M_0$ for $r < 0$. Hence we conclude that the power mean of order r is an increasing function of r. In particular, we get that

$$M_{-1} \le M_0 \le M_1 \le M_2.$$

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