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SOME COMPANIONS OF AN OSTROWSKI TYPE INEQUALITY AND APPLICATIONS

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ABSTRACT. We establish some companions of an Ostrowski type integral inequality for functions whose derivatives are absolutely continuous. Applications for composite quadrature rules are also given.

Key words and phrases: Ostrowski type inequality, Perturbed trapezoid rule, Midpoint rule, Composite quadrature rule.

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1. INTRODUCTION

Motivated by [1], Dragomir in [2] has proved the following companion of the Ostrowski inequality:

$$\left|\frac{1}{2}\left[f(x)+f(a+b-x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \begin{cases} \left[\frac{1}{8}+2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a)\left\|f'\right\|_{[a,b],\infty} & \text{if } f'\in L_\infty[a,b];\\ \frac{2^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\left[\left(\frac{x-a}{b-a}\right)^{q+1}+\left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\left\|f'\right\|_{[a,b],p}, & \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ & \text{and } f'\in L_p[a,b];\\ \left[\frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right]\left\|f'\right\|_{[a,b],1} & \text{if } f'\in L_1[a,b], \end{cases} \tag{1.1}$$

for all $x\in\left[a,\frac{a+b}{2}\right]$, where $f:[a,b]\to\mathbb{R}$ is an absolutely continuous function.

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In particular, the best result in (1.1) is obtained for $x=\frac{a+3b}{4}$, giving the following trapezoid type inequalities:

$$\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \begin{cases} \frac{1}{8}(b-a)\|f'\|_{[a,b],\infty} & \text{if } f'\in L_\infty[a,b]; \\ \frac{1}{4}\cdot\frac{(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\|f'\|_{[a,b],p}, & \text{if } f'\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{4}\|f'\|_{[a,b],1} & \text{if } f'\in L_1[a,b]. \end{cases} \tag{1.2}$$

Some natural applications of (1.1) and (1.2) are also provided in [2].

In [3], Dedić et al. have derived the following trapezoid type inequality:

$$\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{(b-a)^2}{32}\|f''\|_\infty, \tag{1.3}$$

for a function $f:[a,b]\to\mathbb{R}$ whose derivative f' is absolutely continuous and $f''\in L_\infty[a,b]$.

In [4], we find that for a function $f:[a,b]\to\mathbb{R}$ whose derivative f' is absolutely continuous, the following perturbed trapezoid inequalities hold:

$$\left|\int_a^b f(t)\,dt-\frac{b-a}{2}[f(a)+f(b)]+\frac{(b-a)^2}{8}[f'(b)-f'(a)]\right| \leq \begin{cases} \frac{(b-a)^3}{24}\|f''\|_\infty & \text{if } f''\in L_\infty[a,b]; \\ \frac{(b-a)^{2+\frac{1}{q}}}{8(2q+1)^{\frac{1}{q}}}\|f''\|_p, & \text{if } f''\in L_P[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(b-a)^2}{8}\|f''\|_1 & \text{if } f''\in L_1[a,b]. \end{cases} \tag{1.4}$$

In this paper, we provide some companions of Ostrowski type inequalities for functions whose first derivatives are absolutely continuous and whose second derivatives belong to the Lebesgue spaces $L_p[a,b]$, $1\leq p\leq\infty$. These improve (1.3) and recapture (1.4). Applications for composite quadrature rules are also given.

2. Some Integral Inequalities

Lemma 2.1. *Let $f:[a,b]\to\mathbb{R}$ be such that the derivative f' is absolutely continuous on $[a,b]$. Then we have the equality*

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{2}[f(x)+f(a+b-x)]+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)[f'(x)-f'(a+b-x)] \\ =\frac{1}{2(b-a)}\left[\int_a^x (t-a)^2 f''(t)\,dt+\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2 f''(t)\,dt \right. \\ \left. +\int_{a+b-x}^b (t-b)^2 f''(t)\,dt\right] \end{aligned} \tag{2.1}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$.

Proof. Using the integration by parts formula for Lebesgue integrals, we have

$$\int_a^x (t-a)^2 f''(t)\ dt = (x-a)^2 f'(x) - 2(x-a) f(x) + 2\int_a^x f(t)\ dt,$$

$$\int_x^{a+b-x} \left(t-\frac{a+b}{2}\right)^2 f''(t)\, dt = \left(x-\frac{a+b}{2}\right)^2 \left[f'(a+b-x) - f'(x)\right] + 2\left(x-\frac{a+b}{2}\right)\left[f(x)+f(a+b-x)\right] + 2\int_x^{a+b-x} f(t)\, dt,$$

and

$$\int_{a+b-x}^b (t-b)^2 f''(t)\, dt = -(x-a)^2 f'(a+b-x) - 2(x-a) f(a+b-x) + 2\int_{a+b-x}^b f(t)\, dt.$$

Summing the above equalities, we deduce the desired identity (2.1). □

Theorem 2.2. *Let $f : [a,b] \to \mathbb{R}$ be such that the derivative f' is absolutely continuous on $[a,b]$. Then we have the inequality*

$$\begin{aligned}(2.2)\qquad &\left|\frac{1}{b-a}\int_a^b f(t)\ dt - \frac{1}{2}\left[f(x)+f(a+b-x)\right] + \frac{1}{2}\left(x-\frac{3a+b}{4}\right)\left[f'(x)-f'(a+b-x)\right]\right| \\ &\le \frac{1}{2(b-a)}\left[\int_a^x (t-a)^2 \left|f''(t)\right| dt + \int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2 \left|f''(t)\right| dt + \int_{a+b-x}^b (t-b)^2\left|f''(t)\right| dt\right] \\ &:= M(x)\end{aligned}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

If $f'' \in L_\infty[a,b]$, then we have the inequalities

$$\begin{aligned}(2.3)\qquad M(x) &\le \frac{1}{2(b-a)}\left[\frac{(x-a)^3}{3}\|f''\|_{[a,x],\infty} + \frac{2}{3}\left(\frac{a+b}{2}-x\right)^3 \|f''\|_{[x,a+b-x],\infty} + \frac{(x-a)^3}{3}\|f''\|_{[a+b-x,b]}\right] \\ &\le \begin{cases} \left[\frac{1}{96}+\frac{1}{2}\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a)^2\|f''\|_{[a,b],\infty}; \\ \left[\frac{1}{2^{\alpha-1}}\left(\frac{x-a}{b-a}\right)^{3\alpha} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^{3\alpha}\right]^{\frac{1}{\alpha}} \\ \quad\times\left[\|f''\|^\beta_{[a,x],\infty} + \|f''\|^\beta_{[x,a+b-x],\infty} + \|f''\|^\beta_{[a+b-x,b],\infty}\right]^{\frac{1}{\beta}}\frac{(b-a)^2}{3} \\ \qquad\qquad \text{if } \alpha>1, \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \max\left\{\frac{1}{2}\left(\frac{x-a}{b-a}\right)^3, \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^3\right\} \\ \quad\times\left[\|f''\|_{[a,x],\infty} + \|f''\|_{[x,a+b-x],\infty} + \|f''\|_{[a+b-x,b],\infty}\right]\frac{(b-a)^2}{3}; \end{cases}\end{aligned}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

The inequality (2.2), the first inequality in (2.3) and the constant $\frac{1}{96}$ are sharp.

Proof. The inequality (2.2) follows by Lemma 2.1 on taking the modulus and using its properties.

If $f'' \in L_\infty [a,b]$, then

$$\int_a^x (t-a)^2 |f''(t)| \, dt \le \frac{(x-a)^3}{3} \|f''\|_{[a,x],\infty},$$

$$\int_x^{a+b-x} \left(t - \frac{a+b}{2}\right)^2 |f''(t)| \, dt \le \frac{2}{3} \left(\frac{a+b}{2} - x\right)^3 \|f''\|_{[x,a+b-x],\infty},$$

$$\int_{a+b-x}^b (t-b)^2 |f''(t)| \, dt \le \frac{(x-a)^3}{3} f''\|_{[a+b-x,b],\infty}$$

and the first inequality in (2.3) is proved.

Denote

$$\bar{M}(x) := \frac{(x-a)^3}{6} \|f''\|_{[a,x],\infty} + \frac{1}{3} \left(\frac{a+b}{2} - x\right)^3 \|f''\|_{[x,a+b-x],\infty} + \frac{(x-a)^3}{6} \|f''\|_{[a+b-x,b]}$$

for $x \in \left[a, \frac{a+b}{2}\right]$.

Firstly, observe that

$$\begin{aligned} \bar{M}(x) &\le \max\left\{\|f''\|_{[a,x],\infty}, \|f''\|_{[x,a+b-x],\infty}, \|f''\|_{[a+b-x,b],\infty}\right\} \\ &\qquad \times \left[\frac{(x-a)^3}{6} + \frac{1}{3}\left(\frac{a+b}{2} - x\right)^3 + \frac{(x-a)^3}{6}\right] \\ &= \|f''\|_{[a,b],\infty} \left[\frac{(b-a)^2}{96} + \frac{1}{2}\left(x - \frac{3a+b}{4}\right)^2\right](b-a) \end{aligned}$$

and the first part of the second inequality in (2.3) is proved.

Using the Hölder inequality for $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, we also have

$$\begin{aligned} \bar{M}(x) &\le \frac{1}{3} \left\{\left[\frac{(x-a)^3}{2}\right]^\alpha + \left(x - \frac{a+b}{2}\right)^{3\alpha} + \left[\frac{(x-a)^3}{2}\right]^\alpha\right\}^{\frac{1}{\alpha}} \\ &\qquad \times \left[\|f''\|^\beta_{[a,x],\infty} + \|f''\|^\beta_{[x,a+b-x],\infty} + \|f''\|^\beta_{[a+b-x,b],\infty}\right]^{\frac{1}{\beta}} \end{aligned}$$

giving the second part of the second inequality in (2.3)

Finally, we also observe that

$$\begin{aligned} \bar{M}(x) &\le \frac{1}{3} \max\left\{\frac{(x-a)^3}{2}, \left(x - \frac{a+b}{2}\right)^3\right\} \\ &\qquad \times \left[\|f''\|_{[a,x],\infty} + \|f''\|_{[x,a+b-x],\infty} + \|f''\|_{[a+b-x,b],\infty}\right]. \end{aligned}$$

The sharpness of the inequalities mentioned follows from the fact that we can choose a function $f : [a,b] \to \mathbb{R}$, $f(t) = t^2$ for any $x \in \left[a, \frac{a+b}{2}\right]$ to obtain the corresponding equalities. ☐

Remark 1. If in Theorem 2.2 we choose $x = a$, then we recapture the first part of the inequality (1.4), i.e.,

$$\left|\frac{1}{b-a} \int_a^b f(t) \, dt - \frac{1}{2} [f(a) + f(b)] + \frac{b-a}{8} [f'(b) - f'(a)]\right| \le \frac{1}{24} (b-a)^2 \|f''\|_\infty$$

with $\frac{1}{24}$ as a sharp constant. If we choose $x = \frac{a+b}{2}$, then we get

$$\left| \frac{1}{b-a} \int_a^b f(t)\, dt - f\left(\frac{a+b}{2}\right) \right| \le \frac{1}{48} \left[\|f''\|_{\left[a,\frac{a+b}{2}\right],\infty} + \|f''\|_{\left[\frac{a+b}{2},b\right],\infty} \right]$$
$$\le \frac{1}{24} (b-a)^2 \|f''\|_{[a,b],\infty}$$

with the constants $\frac{1}{48}$ and $\frac{1}{24}$ being sharp.

Corollary 2.3. *With the assumptions in Theorem 2.2, one has the inequality*

$$\left| \frac{1}{b-a} \int_a^b f(t)\, dt - \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} \right| \le \frac{1}{96} (b-a)^2 \|f''\|_{[a,b],\infty}. \tag{2.4}$$

The constant $\frac{1}{96}$ is best possible in the sense that it cannot be replaced by a smaller constant. Clearly (2.4) is an improvement of (1.3).

Theorem 2.4. *Let $f : [a,b] \to \mathbb{R}$ be such that the derivative f' is absolutely continuous on $[a,b]$ and $f'' \in L_p[a,b]$, $p > 1$. If $M(x)$ is as defined in (2.2), then we have the bounds:*

$$M(x) \le \frac{1}{2(2q+1)^{\frac{1}{q}}} \left[\left(\frac{x-a}{b-a} \right)^{2+\frac{1}{q}} \|f''\|_{[a,x],p} \right. \tag{2.5}$$
$$\left. + 2^{\frac{1}{q}} \left(\frac{\frac{a+b}{2} - x}{b-a} \right)^{2+\frac{1}{q}} \|f''\|_{[x,a+b-x],p} \left(\frac{x-a}{b-a} \right)^{2+\frac{1}{q}} \|f''\|_{[a+b-x,b],p} \right] (b-a)^{1+\frac{1}{q}}$$

$$\le \frac{1}{2(2q+1)^{\frac{1}{q}}} \times \begin{cases} \left[2\left(\frac{x-a}{b-a}\right)^{2+\frac{1}{q}} + 2^{\frac{1}{q}} \left(\frac{\frac{a+b}{2}-x}{b-a} \right)^{2+\frac{1}{q}} \right] \\ \times \max\left\{ \|f''\|_{[a,x],p}, \|f''\|_{[x,a+b-x],p}, \|f''\|_{[a+b-x,b],p} \right\} (b-a)^{1+\frac{1}{q}}; \\ \left[2\left(\frac{x-a}{b-a}\right)^{2\alpha+\frac{\alpha}{q}} + 2^{\frac{\alpha}{q}} \left(\frac{\frac{a+b}{2}-x}{b-a} \right)^{2\alpha+\frac{\alpha}{q}} \right]^{\frac{1}{\alpha}} \\ \times \left[\|f''\|_{[a,x],p}^{\beta} + \|f''\|_{[x,a+b-x],p}^{\beta} + \|f''\|_{[a+b-x,b],p}^{\beta} \right]^{\frac{1}{\beta}} (b-a)^{1+\frac{1}{q}} \\ \qquad \text{if } \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ \max\left\{ \left(\frac{x-a}{b-a}\right)^{2+\frac{1}{q}}, 2^{\frac{1}{q}} \left(\frac{\frac{a+b}{2}-x}{b-a} \right)^{2+\frac{1}{q}} \right\} \\ \times \left[\|f''\|_{[a,x],p} + \|f''\|_{[x,a+b-x],p} + \|f''\|_{[a+b-x,b],p} \right] (b-a)^{1+\frac{1}{q}}; \end{cases}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

Proof. Using Hölder's integral inequality for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\int_a^x (t-a)^2 |f''(t)|\, dt \le \left(\int_a^x (t-a)^{2q}\, dt \right)^{\frac{1}{q}} \|f''\|_{[a,x],p}$$
$$= \frac{(x-a)^{2+\frac{1}{q}}}{(2q+1)^{\frac{1}{q}}} \|f''\|_{[a,x],p},$$

$$\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2|f''(t)|\,dt \le \left(\int_x^{a+b-x}|t-\frac{a+b}{2}|^{2q}\,dt\right)^{\frac{1}{q}}\|f''\|_{[x,a+b-x],p}$$
$$= \frac{2^{\frac{1}{q}}\left(\frac{a+b}{2}-x\right)^{2+\frac{1}{q}}}{(2q+1)^{\frac{1}{q}}}\|f''\|_{[x,a+b-x],p},$$

and

$$\int_{a+b-x}^{b}(t-b)^2|f''(t)|\,dt \le \left(\int_{a+b-x}^{b}(b-t)^{2q}\,dt\right)^{\frac{1}{q}}\|f''\|_{[a+b-x,b],p}$$
$$= \frac{(x-a)^{2+\frac{1}{q}}}{(2q+1)^{\frac{1}{q}}}\|f''\|_{[a+b-x,b],p}.$$

Summing the above inequalities, we deduce the first bound in (2.5).

The last part may be proved in a similar fashion to the one in Theorem 2.2, and we omit the details. □

Remark 2. If in (2.5) we choose $\alpha = q$, $\beta = p$, $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, then we get the inequality

$$M(x) \le \frac{2^{\frac{1}{q}}}{2(2q+1)^{\frac{1}{q}}}\left[\left(\frac{x-a}{b-a}\right)^{2q+1}+\left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{2q+1}\right]^{\frac{1}{q}}(b-a)^{1+\frac{1}{q}}\|f''\|_{[a,b],p}, \tag{2.6}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

Remark 3. If in Theorem 2.4 we choose $x = a$, then we recapture the second part of the inequality (1.4), i.e.,

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{2}[f(a)+f(b)] + \frac{b-a}{8}[f'(b)-f'(a)]\right| \le \frac{1}{8}\cdot\frac{(b-a)^{1+\frac{1}{q}}\|f''\|_{[a,b],p}}{(2q+1)^{\frac{1}{q}}}. \tag{2.7}$$

The constant $\frac{1}{8}$ is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. Indeed, if we assume that (2.7) holds with a constant $C>0$, instead of $\frac{1}{8}$, i.e.,

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{2}[f(a)+f(b)] + \frac{b-a}{8}[f'(b)-f'(a)]\right| \le C\cdot\frac{(b-a)^{1+\frac{1}{q}}\|f''\|_{[a,b],p}}{(2q+1)^{\frac{1}{q}}}, \tag{2.8}$$

then for the function $f:[a,b]\to\mathbb{R}$, $f(x)=k\left(x-\frac{a+b}{2}\right)^2$, $k>0$, we have

$$\frac{f(a)+f(b)}{2} = k\cdot\frac{(b-a)^2}{4},$$
$$f'(b)-f'(a) = 2k(b-a),$$
$$\frac{1}{b-a}\int_a^b f(t)\,dt = k\cdot\frac{(b-a)^2}{12},$$
$$\|f''\|_{[a,b],p} = 2k(b-a)^{\frac{1}{p}};$$

and by (2.8) we deduce

$$\left|\frac{k\left(b-a\right)^{2}}{12}-\frac{k\left(b-a\right)^{2}}{4}+\frac{k\left(b-a\right)^{2}}{4}\right|\leq\frac{2C\cdot k\left(b-a\right)^{2}}{\left(2q+1\right)^{\frac{1}{q}}},$$

giving $C\geq\frac{(2q+1)^{\frac{1}{q}}}{24}$. Letting $q\to 1+$, we deduce $C\geq\frac{1}{8}$, and the sharpness of the constant is proved. ☐

Remark 4. If in Theorem 2.4 we choose $x=\frac{a+b}{2}$, then we get the midpoint inequality

$$\begin{aligned}(2.9)\qquad &\left|\frac{1}{b-a}\int_{a}^{b}f\left(t\right)\,dt-f\left(\frac{a+b}{2}\right)\right|\\ &\leq\frac{1}{8}\cdot\frac{\left(b-a\right)^{1+\frac{1}{q}}}{2^{\frac{1}{q}}\left(2q+1\right)^{\frac{1}{q}}}\left[\left\|f''\right\|_{\left[a,\frac{a+b}{2}\right],p}+\left\|f''\right\|_{\left[\frac{a+b}{2},b\right],p}\right]\\ &\leq\frac{1}{8}\cdot\frac{\left(b-a\right)^{1+\frac{1}{q}}}{\left(2q+1\right)^{\frac{1}{q}}}\left\|f''\right\|_{[a,b],p},\quad p>1,\frac{1}{p}+\frac{1}{q}=1.\end{aligned}$$

In both inequalities the constant $\frac{1}{8}$ is sharp in the sense that it cannot be replaced by a smaller constant.

To show this fact, assume that (2.9) holds with $C,D>0$, i.e.,

$$\begin{aligned}(2.10)\qquad &\left|\frac{1}{b-a}\int_{a}^{b}f\left(t\right)\,dt-f\left(\frac{a+b}{2}\right)\right|\\ &\leq C\cdot\frac{\left(b-a\right)^{1+\frac{1}{q}}}{2^{\frac{1}{q}}\left(2q+1\right)^{\frac{1}{q}}}\left[\left\|f''\right\|_{\left[a,\frac{a+b}{2}\right],p}+\left\|f''\right\|_{\left[\frac{a+b}{2},b\right],p}\right]\\ &\leq D\cdot\frac{\left(b-a\right)^{1+\frac{1}{q}}}{\left(2q+1\right)^{\frac{1}{q}}}\left\|f''\right\|_{[a,b],p}.\end{aligned}$$

For the function $f:[a,b]\to\mathbb{R}$, $f\left(x\right)=k\left(x-\frac{a+b}{2}\right)^{2}$, $k>0$, we have

$$f\left(\frac{a+b}{2}\right)=0,\qquad\frac{1}{b-a}\int_{a}^{b}f\left(t\right)\,dt=\frac{k\left(b-a\right)^{2}}{12},$$

$$\left\|f''\right\|_{\left[a,\frac{a+b}{2}\right],p}+\left\|f''\right\|_{\left[\frac{a+b}{2},b\right],p}=4k\left(\frac{b-a}{2}\right)^{\frac{1}{p}}=2^{1+\frac{1}{q}}\left(b-a\right)^{\frac{1}{p}}k,$$

$$\left\|f''\right\|_{[a,b],p}=2\left(b-a\right)^{\frac{1}{p}}k;$$

and then by (2.10) we deduce

$$\frac{k\left(b-a\right)^{2}}{12}\leq C\cdot\frac{2k\left(b-a\right)^{2}}{\left(2q+1\right)^{\frac{1}{q}}}\leq D\cdot\frac{2k\left(b-a\right)^{2}}{\left(2q+1\right)^{\frac{1}{q}}},$$

giving $C,D\geq\frac{(2q+1)^{\frac{1}{q}}}{24}$ for any $q>1$. Letting $q\to 1+$, we deduce $C,D\geq\frac{1}{8}$ and the sharpness of the constants in (2.9) is proved.

The following result is useful in providing the best quadrature rule in the class for approximating the integral of a function $f:[a,b]\to\mathbb{R}$ whose first derivative is absolutely continuous on $[a,b]$ and whose second derivative is in $L_{p}\left[a,b\right]$.

Corollary 2.5. *With the assumptions in Theorem 2.4, one has the inequality*

$$\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{32}\cdot\frac{(b-a)^{1+\frac{1}{q}}}{(2q+1)^{\frac{1}{q}}}\|f''\|_{[a,b],p}, \tag{2.11}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

The constant $\frac{1}{32}$ *is the best possible in the sense that it cannot be replaced by a smaller constant.*

Proof. The inequality follows by Theorem 2.4 and (2.6) on choosing $x=\frac{3a+b}{4}$.

To prove the sharpness of the constant, assume that (2.11) holds with a constant $E>0$, i.e.,

$$\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq E\cdot\frac{(b-a)^{1+\frac{1}{q}}}{(2q+1)^{\frac{1}{q}}}\|f''\|_{[a,b],p}. \tag{2.12}$$

Consider the function $f:[a,b]\to\mathbb{R}$,

$$f(x)=\begin{cases} -\frac{1}{2}\left(x-\frac{3a+b}{4}\right)^2 & \text{if} \quad x\in\left[a,\frac{3a+b}{4}\right], \\ \frac{1}{2}\left(x-\frac{3a+b}{4}\right)^2 & \text{if} \quad x\in\left(\frac{3a+b}{4},\frac{a+b}{2}\right], \\ -\frac{1}{2}\left(x-\frac{a+3b}{4}\right)^2 & \text{if} \quad x\in\left(\frac{a+b}{2},\frac{a+3b}{4}\right], \\ \frac{1}{2}\left(x-\frac{a+3b}{4}\right)^2 & \text{if} \quad x\in\left(\frac{a+3b}{4},b\right]. \end{cases}$$

We have

$$f'(x)=\begin{cases} \left|x-\frac{3a+b}{4}\right| & \text{if} \quad x\in\left[a,\frac{a+b}{2}\right], \\ \left|x-\frac{a+3b}{4}\right| & \text{if} \quad x\in\left(\frac{a+b}{2},b\right]. \end{cases}$$

Then f' is absolutely continuous and $f''\in L_p[a,b]$, $p>1$. We also have

$$\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]=0,$$
$$\frac{1}{b-a}\int_a^b f(t)\,dt=\frac{(b-a)^2}{96},$$
$$\|f''\|_{[a,b],p}=(b-a)^{\frac{1}{p}},$$

and then, by (2.12), we obtain

$$\frac{(b-a)^2}{96}\leq E\cdot\frac{(b-a)^2}{(2q+1)^{\frac{1}{q}}},$$

giving $E\geq\frac{(2q+1)^{\frac{1}{q}}}{96}$ for any $q>1$, i.e., $E\geq\frac{1}{32}$, and the corollary is proved. □

Theorem 2.6. *Let $f : [a,b] \to \mathbb{R}$ be such that the derivative f' is absolutely continuous on $[a,b]$ and $f'' \in L_1[a,b]$. If $M(x)$ is as defined in (2.2), then we have the bounds:*

$$M(x) \leq \frac{b-a}{2}\left[\left(\frac{x-a}{b-a}\right)^2 \|f''\|_{[a,x],1} + \left(\frac{\frac{a+b}{2}-x}{b-a}\right)^2 \|f''\|_{[x,a+b-x],1} + \left(\frac{x-a}{b-a}\right)^2 \|f''\|_{[a+b-x,b],1}\right] \tag{2.13}$$

$$\leq \begin{cases} \frac{b-a}{2}\left[2\left(\frac{x-a}{b-a}\right)^2 + \left(\frac{\frac{a+b}{2}-x}{b-a}\right)^2\right] \\ \times \max\left[\|f''\|_{[a,x],1}, \|f''\|_{[x,a+b-x],1}, \|f''\|_{[a+b-x,b],1}\right]; \\ \frac{b-a}{2}\left[2\left(\frac{x-a}{b-a}\right)^{2\alpha} + \left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{2\alpha}\right]^{\frac{1}{\alpha}} \\ \times\left[\|f''\|^{\beta}_{[a,x],1} + \|f''\|^{\beta}_{[x,a+b-x],1} + \|f''\|^{\beta}_{[a+b-x,b],1}\right]^{\frac{1}{\beta}} \\ \qquad \textit{if} \quad \alpha > 1, \frac{1}{\alpha}+\frac{1}{\beta} = 1; \\ \frac{b-a}{2}\left[\left|\frac{x-\frac{3a+b}{4}}{b-a}\right| + \frac{1}{4}\right]^2 \|f''\|_{[a,b],1}; \end{cases}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

The proof is as in Theorem 2.2 and we need only to prove the third inequality of the last part as

$$\begin{aligned} M(x) &\leq \frac{b-a}{2}\max\left\{\left(\frac{x-a}{b-a}\right)^2, \left(\frac{\frac{a+b}{2}-x}{b-a}\right)^2\right\} \\ &\qquad \times\left[\|f''\|_{[a,x],1} + \|f''\|_{[x,a+b-x],1} + \|f''\|_{[a+b-x,b],1}\right] \\ &= \frac{b-a}{2}\left[\left|\frac{x-\frac{3a+b}{4}}{b-a}\right| + \frac{1}{4}\right]^2 \|f''\|_{[a,b],1}. \end{aligned}$$

Remark 5. By the use of Theorem 2.6, for $x = a$, we recapture the third part of the inequality (1.4), i.e.,

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{2}[f(a)+f(b)] + \frac{b-a}{8}[f'(b)-f'(a)]\right| \leq \frac{1}{8}(b-a)\|f''\|_{[a,b],1}.$$

If in (2.13) we choose $x = \frac{a+b}{2}$, then we get the mid-point inequality

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\right| \leq \frac{1}{8}(b-a)\|f''\|_{[a,b],1}.$$

Corollary 2.7. *With the assumptions in Theorem 2.6, one has the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}\right| \leq \frac{1}{32}(b-a)\|f''\|_{[a,b],1}.$$

3. A COMPOSITE QUADRATURE FORMULA

We use the following inequalities obtained in the previous section:

$$\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \begin{cases} \frac{1}{96}(b-a)^2\|f''\|_{[a,b],\infty} & \text{if } f''\in L_\infty[a,b]; \\ \frac{1}{32}\cdot\frac{(b-a)^{1+\frac{1}{q}}}{(2q+1)^{\frac{1}{q}}}\|f''\|_{[a,b],p} & \text{if } f''\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{32}(b-a)\|f''\|_{[a,b],1} & \text{if } f''\in L_1[a,b]. \end{cases} \tag{3.1}$$

Let $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$ and $h_i := x_{i+1} - x_i$ $(i = 0,\ldots,n-1)$ and $\nu(I_n) := \max\{h_i | i = 0,\ldots,n-1\}$.

Consider the composite quadrature rule

$$Q_n(I_n,f) := \frac{1}{2}\sum_{i=0}^{n-1}\left[f\left(\frac{3x_i+x_{i+1}}{4}\right)+f\left(\frac{x_i+3x_{i+1}}{4}\right)\right]h_i. \tag{3.2}$$

The following result holds.

Theorem 3.1. *Let $f : [a,b] \to \mathbb{R}$ be such that the derivative f' is absolutely continuous on $[a,b]$. Then we have*

$$\int_a^b f(t)\,dt = Q_n(I_n,f) + R_n(I_n,f),$$

where $Q_n(I_n,f)$ is defined by the formula (3.2), and the remainder satisfies the estimates

$$|R_n(I_n,f)| \leq \begin{cases} \frac{1}{96}\|f''\|_{[a,b],\infty}\sum_{i=0}^{n-1}h_i^3 & \text{if } f''\in L_\infty[a,b]; \\ \frac{1}{32(2q+1)^{\frac{1}{q}}}\|f''\|_{[a,b],p}\left(\sum_{i=0}^{n-1}h_i^{2q+1}\right)^{\frac{1}{q}} & \text{if } f''\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{32}\|f''\|_{[a,b],1}[\nu(I_n)]^2 & \text{if } f''\in L_1[a,b]. \end{cases} \tag{3.3}$$

Proof. Applying inequality (3.1) on the interval $[x_i, x_{i+1}]$, we may state that

$$\left|\int_{x_i}^{x_{i+1}} f(t)\,dt - \frac{1}{2}\left[f\left(\frac{3x_i+x_{i+1}}{4}\right)+f\left(\frac{x_i+3x_{i+1}}{4}\right)\right]h_i\right| \leq \begin{cases} \frac{1}{96}h_i^3\|f''\|_{[x_i,x_{i+1}],\infty}; \\ \frac{1}{32(2q+1)^{\frac{1}{q}}}h_i^{2+\frac{1}{q}}\|f''\|_{[x_i,x_{i+1}],p}, & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{32}h_i^2\|f''\|_{[x_i,x_{i+1}],1}; \end{cases} \tag{3.4}$$

for each $i \in \{0,\ldots,n-1\}$.

Summing the inequality (3.4) over i from 0 to $n-1$ and using the generalized triangle inequality, we get

$$|R_n(I_n,f)| \leq \begin{cases} \frac{1}{96}\sum_{i=0}^{n-1}h_i^3\|f''\|_{[x_i,x_{i+1}],\infty}; \\ \frac{1}{32(2q+1)^{\frac{1}{q}}}\sum_{i=0}^{n-1}h_i^{2+\frac{1}{q}}\|f''\|_{[x_i,x_{i+1}],p}, & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{32}\sum_{i=0}^{n-1}h_i^2\|f''\|_{[x_i,x_{i+1}],1}. \end{cases} \tag{3.5}$$

Now, we observe that

$$\sum_{i=0}^{n-1} h_i^3 \|f''\|_{[x_i,x_{i+1}],\infty} \leq \|f''\|_{[a,b],\infty} \sum_{i=0}^{n-1} h_i^3.$$

Using Hölder's discrete inequality, we may write that

$$\begin{aligned}
\sum_{i=0}^{n-1} h_i^{2+\frac{1}{q}} \|f''\|_{[x_i,x_{i+1}],p} &\leq \left(\sum_{i=0}^{n-1} h_i^{\left(2+\frac{1}{q}\right)q}\right)^{\frac{1}{q}} \left(\sum_{i=0}^{n-1} \|f''\|_{[x_i,x_{i+1}],p}^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{i=0}^{n-1} h_i^{2q+1}\right)^{\frac{1}{q}} \left(\sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} |f''(t)|^p \, dt\right)^{\frac{1}{p}} \\
&= \left(\sum_{i=0}^{n-1} h_i^{2q+1}\right)^{\frac{1}{q}} \|f''\|_{[a,b],p}.
\end{aligned}$$

Also, we note that

$$\begin{aligned}
\sum_{i=0}^{n-1} h_i^2 \|f''\|_{[x_i,x_{i+1}],1} &\leq \max_{0\leq i\leq n-1} \left\{h_i^2\right\} \sum_{i=0}^{n-1} \|f''\|_{[x_i,x_{i+1}],1} \\
&= \left[\nu(I_n)\right]^2 \|f''\|_{[a,b],1}.
\end{aligned}$$

Consequently, by the use of (3.5), we deduce the desired result (3.3). □

For the particular case where the division I_n is equidistant, i.e.,

$$I_n := x_i = a + i \cdot \frac{b-a}{n}, \qquad i = 0, \ldots, n,$$

we may consider the quadrature rule:

$$Q_n(f) := \frac{b-a}{2n} \sum_{i=0}^{n-1} \left\{ f\left[a + \left(\frac{4i+1}{4n}\right)(b-a)\right] + f\left[a + \left(\frac{4i+3}{4n}\right)(b-a)\right] \right\}. \tag{3.6}$$

The following corollary will be more useful in practice.

Corollary 3.2. *With the assumption of Theorem 3.1, we have*

$$\int_a^b f(t)\, dt = Q_n(f) + R_n(f),$$

where $Q_n(f)$ is defined by (3.6) and the remainder $R_n(f)$ satisfies the estimate:

$$|R_n(I_n, f)| \leq \begin{cases} \frac{1}{96} \|f''\|_{[a,b],\infty} \frac{(b-a)^3}{n^2}; \\[2ex] \frac{1}{32(2q+1)^{\frac{1}{q}}} \|f''\|_{[a,b],p} \frac{(b-a)^{2+\frac{1}{q}}}{n^2}, \quad p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\[2ex] \frac{1}{32} \|f''\|_{[a,b],1} \frac{(b-a)^2}{n^2}. \end{cases}$$

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