

# Journal of Inequalities in Pure and Applied Mathematics

## OSTROWSKI TYPE INEQUALITIES FOR ISOTONIC LINEAR FUNCTIONALS

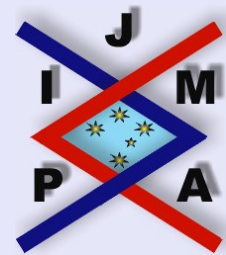
S.S. DRAGOMIR

School of Communications and Informatics  
Victoria University of Technology  
PO Box 14428  
Melbourne City MC  
8001 Victoria, Australia

*E*Mail: [sever@matilda.vu.edu.au](mailto:sever@matilda.vu.edu.au)

*U*RL: <http://rgmia.vu.edu.au/SSDragomirWeb.html>

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Abstract

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## Abstract

Some inequalities of Ostrowski type for isotonic linear functionals defined on a linear class of function  $L := \{f : [a, b] \rightarrow \mathbb{R}\}$  are established. Applications for integral and discrete inequalities are also given.

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*Key words:* Ostrowski Type Inequalities, Isotonic Linear Functionals.

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# 1. Introduction

The following result is known in the literature as Ostrowski's inequality [13].

**Theorem 1.1.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(a, b)$  with the property that  $|f'(t)| \leq M$  for all  $t \in (a, b)$ . Then*

$$(1.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[ \frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) M$$

for all  $x \in [a, b]$ .

The constant  $\frac{1}{4}$  is the best possible in the sense that it cannot be replaced by a smaller constant.

The following Ostrowski type result for absolutely continuous functions whose derivatives belong to the Lebesgue spaces  $L_p[a, b]$  also holds (see [9], [10] and [11]).

**Theorem 1.2.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be absolutely continuous on  $[a, b]$ . Then, for all  $x \in [a, b]$ , we have:*

$$(1.2) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \begin{cases} \left[ \frac{1}{4} + \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \|f'\|_\infty & \text{if } f' \in L_\infty[a, b]; \\ \frac{(b-a)^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}} \left[ \left( \frac{x-a}{b-a} \right)^{p+1} + \left( \frac{b-x}{b-a} \right)^{p+1} \right]^{\frac{1}{p}} \|f'\|_q & \text{if } f' \in L_q[a, b], \frac{1}{p} + \frac{1}{q} = 1, p > 1; \\ \left[ \frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \|f'\|_1 & \end{cases}$$



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where  $\|\cdot\|_r$  ( $r \in [1, \infty)$ ) are the usual Lebesgue norms on  $L_r[a, b]$ , i.e.,

$$\|g\|_\infty := \operatorname{ess\,sup}_{t \in [a, b]} |g(t)|$$

and

$$\|g\|_r := \left( \int_a^b |g(t)|^r dt \right)^{\frac{1}{r}}, \quad r \in [1, \infty).$$

The constants  $\frac{1}{4}$ ,  $\frac{1}{(p+1)^{\frac{1}{p}}}$  and  $\frac{1}{2}$  respectively are sharp in the sense presented in Theorem 1.1.

The above inequalities can also be obtained from Fink's result in [12] on choosing  $n = 1$  and performing some appropriate computations.

If one drops the condition of absolute continuity and assumes that  $f$  is Hölder continuous, then one may state the result (see [7]):

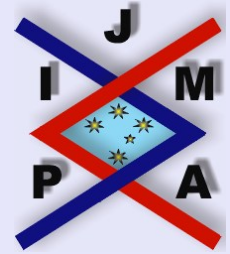
**Theorem 1.3.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be of  $r - H$ -Hölder type, i.e.,

$$(1.3) \quad |f(x) - f(y)| \leq H |x - y|^r, \quad \text{for all } x, y \in [a, b],$$

where  $r \in (0, 1]$  and  $H > 0$  are fixed. Then for all  $x \in [a, b]$  we have the inequality:

$$(1.4) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{H}{r+1} \left[ \left( \frac{b-x}{b-a} \right)^{r+1} + \left( \frac{x-a}{b-a} \right)^{r+1} \right] (b-a)^r.$$

The constant  $\frac{1}{r+1}$  is also sharp in the above sense.



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Note that if  $r = 1$ , i.e.,  $f$  is Lipschitz continuous, then we get the following version of Ostrowski's inequality for Lipschitzian functions (with  $L$  instead of  $H$ ) (see [3])

$$(1.5) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[ \frac{1}{4} + \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a)L.$$

Here the constant  $\frac{1}{4}$  is also best.

Moreover, if one drops the continuity condition of the function, and assumes that it is of bounded variation, then the following result may be stated (see [4]).

**Theorem 1.4.** *Assume that  $f : [a, b] \rightarrow \mathbb{R}$  is of bounded variation and denote by  $\bigvee_a^b(f)$  its total variation. Then*

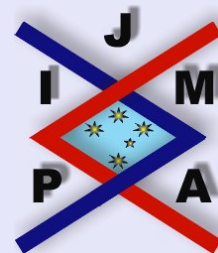
$$(1.6) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[ \frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b(f)$$

for all  $x \in [a, b]$ .

The constant  $\frac{1}{2}$  is the best possible.

If we assume more about  $f$ , i.e.,  $f$  is monotonically increasing, then the inequality (1.6) may be improved in the following manner [5] (see also [2]).

**Theorem 1.5.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be monotonic nondecreasing. Then for all*




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$x \in [a, b]$ , we have the inequality:

$$\begin{aligned}
 (1.7) \quad & \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\
 & \leq \frac{1}{b-a} \left\{ [2x - (a+b)] f(x) + \int_a^b \operatorname{sgn}(t-x) f(t) dt \right\} \\
 & \leq \frac{1}{b-a} \{ (x-a)[f(x) - f(a)] + (b-x)[f(b) - f(x)] \} \\
 & \leq \left[ \frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] [f(b) - f(a)].
 \end{aligned}$$

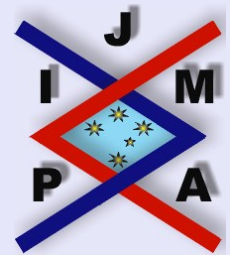
All the inequalities in (1.7) are sharp and the constant  $\frac{1}{2}$  is the best possible.

The version of Ostrowski's inequality for convex functions was obtained in [6] and is incorporated in the following theorem:

**Theorem 1.6.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a convex function on  $[a, b]$ . Then for any  $x \in (a, b)$  we have the inequality

$$\begin{aligned}
 (1.8) \quad & \frac{1}{2} [(b-x)^2 f'_+(x) - (x-a)^2 f'_-(x)] \\
 & \leq \int_a^b f(t) dt - (b-a) f(x) \\
 & \leq \frac{1}{2} [(b-x)^2 f'_-(b) - (x-a)^2 f'_+(a)].
 \end{aligned}$$

In both parts of the inequality (1.8) the constant  $\frac{1}{2}$  is sharp.




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For other Ostrowski type inequalities, see [8].

In this paper we extend Ostrowski's inequality for arbitrary isotonic linear functionals  $A : L \rightarrow \mathbb{R}$ , where  $L$  is a linear class of absolutely continuous functions defined on  $[a, b]$ . Some applications for particular instances of linear functionals  $A$  are also provided.



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## 2. Preliminaries

Let  $L$  be a *linear class* of real-valued functions,  $g : E \rightarrow \mathbb{R}$  having the properties

(L1)  $f, g \in L$  imply  $(\alpha f + \beta g) \in L$  for all  $\alpha, \beta \in \mathbb{R}$ ;

(L2)  $\mathbf{1} \in L$ , i.e., if  $f(t) = 1, t \in E$ , then  $f \in L$ .

An *isotonic linear functional*  $A : L \rightarrow \mathbb{R}$  is a functional satisfying

(A1)  $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$  for all  $f, g \in L$  and  $\alpha, \beta \in \mathbb{R}$ ;

(A2) If  $f \in L$  and  $f \geq 0$ , then  $A(f) \geq 0$ .

The mapping  $A$  is said to be *normalised* if

(A3)  $A(\mathbf{1}) = 1$ .

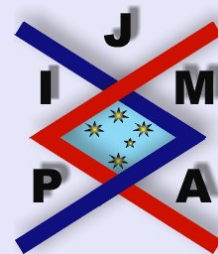
Usual examples of isotonic linear functional that are normalised are the following ones

$$A(f) := \frac{1}{\mu(X)} \int_X f(x) d\mu(x), \quad \text{if } \mu(X) < \infty$$

or

$$A_w(f) := \frac{1}{\int_X w(x) d\mu(x)} \int_X w(x) f(x) d\mu(x),$$

where  $w(x) \geq 0, \int_X w(x) d\mu(x) > 0, X$  is a measurable space and  $\mu$  is a positive measure on  $X$ .



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In particular, for  $\bar{x} = (x_1, \dots, x_n)$ ,  $\bar{w} := (w_1, \dots, w_n) \in \mathbb{R}^n$  with  $w_i \geq 0$ ,  $W_n := \sum_{i=1}^n w_i > 0$  we have

$$A(\bar{x}) := \frac{1}{n} \sum_{i=1}^n x_i$$

and

$$A_{\bar{w}}(\bar{x}) := \frac{1}{W_n} \sum_{i=1}^n w_i x_i,$$

are normalised isotonic linear functionals on  $\mathbb{R}^n$ .

The following representation result for absolutely continuous functions holds.

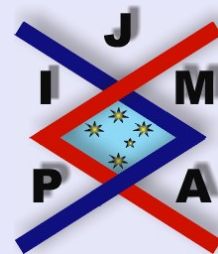
**Lemma 2.1.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be an absolutely continuous function on  $[a, b]$  and define  $e(t) = t$ ,  $t \in [a, b]$ ,  $g(t, x) = \int_0^1 f'[(1 - \lambda)x + \lambda t] d\lambda$ ,  $t \in [a, b]$  and  $x \in [a, b]$ . If  $A : L \rightarrow \mathbb{R}$  is a normalised linear functional on a linear class  $L$  of absolutely continuous functions defined on  $[a, b]$  and  $(x - e) \cdot g(\cdot, x) \in L$ , then we have the representation*

$$(2.1) \quad f(x) = A(f) + A[(x - e) \cdot g(\cdot, x)],$$

for  $x \in [a, b]$ .

*Proof.* For any  $x, t \in [a, b]$  with  $t \neq x$ , one has

$$\frac{f(x) - f(t)}{x - t} = \frac{\int_t^x f'(u) du}{x - t} = \int_0^1 f'[(1 - \lambda)x + \lambda t] d\lambda = g(t, x),$$



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giving the equality

$$(2.2) \quad f(x) = f(t) + (x - t)g(t, x)$$

for any  $t, x \in [a, b]$ .

Applying the functional  $A$ , we get

$$A(f(x) \cdot \mathbf{1}) = A(f + (x - e)g(\cdot, x)),$$

for any  $x \in [a, b]$ .

Since

$$A(f(x) \cdot \mathbf{1}) = f(x)A(\mathbf{1}) = f(x)$$

and

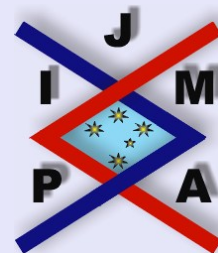
$$A(f + (x - e) \cdot g(\cdot, x)) = A(f) + A((x - e) \cdot g(\cdot, x)),$$

the equality (2.1) is obtained.  $\square$

The following particular cases are of interest:

**Corollary 2.2.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be an absolutely continuous function on  $[a, b]$ . Then we have the representation:*

$$(2.3) \quad f(x) = \frac{1}{\int_a^b w(t) dt} \int_a^b w(t) f(t) dt + \frac{1}{\int_a^b w(t) dt} \int_a^b w(t) (x - t) \left( \int_0^1 f'[(1 - \lambda)x + \lambda t] d\lambda \right) dt$$



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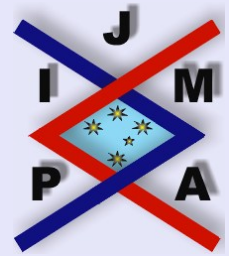


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for any  $x \in [a, b]$ , where  $p : [a, b] \rightarrow \mathbb{R}$  is a Lebesgue integrable function with  $\int_a^b w(t) dt \neq 0$ .

In particular, we have

$$(2.4) \quad f(x) = \frac{1}{b-a} \int_a^b f(t) dt + \frac{1}{b-a} \int_a^b (x-t) \left( \int_0^1 f'[(1-\lambda)x + \lambda t] d\lambda \right) dt$$

for each  $x \in [a, b]$ .

The proof is obvious by Lemma 2.1 applied for the normalised linear functionals

$$A_w(f) := \frac{1}{\int_a^b w(t) dt} \int_a^b w(t) f(t) dt, \quad A(f) := \frac{1}{b-a} \int_a^b f(t) dt$$

defined on

$$L := \{f : [a, b] \rightarrow \mathbb{R}, f \text{ is absolutely continuous on } [a, b]\}.$$

The following discrete case also holds.

**Corollary 2.3.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be an absolutely continuous function on  $[a, b]$ . Then we have the representation:

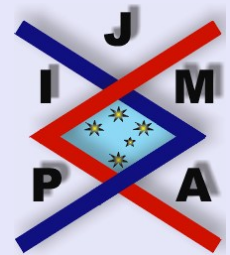
$$(2.5) \quad f(x) = \frac{1}{W_n} \sum_{i=1}^n w_i f(x_i) + \frac{1}{W_n} \sum_{i=1}^n w_i (x - x_i) \left( \int_0^1 f'[(1-\lambda)x + \lambda x_i] d\lambda \right)$$

for any  $x \in [a, b]$ , where  $x_i \in [a, b]$ ,  $w_i \in \mathbb{R}$  ( $i = \{1, \dots, n\}$ ) with  $W_n := \sum_{i=1}^n w_i \neq 0$ .

In particular, we have

$$(2.6) \quad f(x) = \frac{1}{n} \sum_{i=1}^n f(x_i) + \frac{1}{n} \sum_{i=1}^n (x - x_i) \left( \int_0^1 f'[(1 - \lambda)x + \lambda x_i] d\lambda \right)$$

for any  $x \in [a, b]$ .




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### 3. Ostrowski Type Inequalities

The following theorem holds.

**Theorem 3.1.** *With the assumptions of Lemma 2.1, and assuming that  $A : L \rightarrow \mathbb{R}$  is isotonic, then we have the inequalities*

$$(3.1) \quad |f(x) - A(f)| \leq \begin{cases} A(|x - e| \|f'\|_{[x,\cdot],\infty}) & \text{if } |x - e| \|f'\|_{[x,\cdot],\infty} \in L, f' \in L_\infty[a, b]; \\ A(|x - e|^{\frac{1}{q}} \|f'\|_{[x,\cdot],p}) & \text{if } |x - e|^{\frac{1}{q}} \|f'\|_{[x,\cdot],p} \in L, f' \in L_p[a, b], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ A(\|f'\|_{[x,\cdot],1}) & \text{if } \|f'\|_{[x,\cdot],1} \in L, \end{cases}$$

where

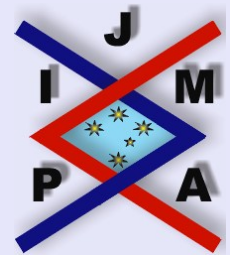
$$\|h\|_{[m,n],\infty} := \operatorname{ess\,sup}_{\substack{t \in [m,n] \\ (t \in [n,m])}} |h(t)| \quad \text{and}$$

$$\|h\|_{[m,n],p} := \left| \int_m^n |h(t)|^p dt \right|^{\frac{1}{p}}, \quad p \geq 1.$$

If we denote

$$M_\infty(x) := A(|x - e| \|f'\|_{[x,\cdot],\infty}),$$

$$M_p(x) := A(|x - e|^{\frac{1}{q}} \|f'\|_{[x,\cdot],p}), \quad \text{and} \quad M_1(x) := A(\|f'\|_{[x,\cdot],1}),$$



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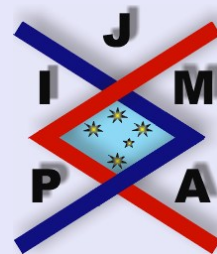
then we have the inequalities:

$$(3.2) \quad M_{\infty}(x) \leq \begin{cases} \|f'\|_{[a,b],\infty} A(|x-e|) \text{ if } |x-e| \in L, f' \in L_{\infty}[a,b]; \\ \left[ A\left(\|f'\|_{[x,\cdot],\infty}^{\beta}\right) \right]^{\frac{1}{\beta}} \left[ A(|x-e|^{\alpha}) \right]^{\frac{1}{\alpha}} \\ \text{if } \|f'\|_{[x,\cdot],\infty}^{\beta}, |x-e|^{\alpha} \in L, f' \in L_{\infty}[a,b], \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \left[ \frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right| \right] A\left(\|f'\|_{[x,\cdot],\infty}\right) \text{ if } \|f'\|_{[x,\cdot],\infty} \in L, f' \in L_{\infty}[a,b]. \end{cases}$$

$$(3.3) \quad M_p(x) \leq \begin{cases} \max\left\{\|f'\|_{[a,x],p}, \|f'\|_{[x,b],p}\right\} A\left(|x-e|^{\frac{1}{q}}\right) \text{ if } |x-e|^{\frac{1}{q}} \in L, f' \in L_p[a,b]; \\ \left[ A\left(\|f'\|_{[x,\cdot],p}^{\beta}\right) \right]^{\frac{1}{\beta}} \left[ A\left(|x-e|^{\frac{\alpha}{q}}\right) \right]^{\frac{1}{\alpha}} \\ \text{if } \|f'\|_{[x,\cdot],p}^{\beta}, |x-e|^{\frac{\alpha}{q}} \in L, f' \in L_p[a,b], \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \left[ \frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right| \right]^{\frac{1}{q}} A\left(\|f'\|_{[x,\cdot],p}\right) \text{ if } \|f'\|_{[x,\cdot],p} \in L, f' \in L_p[a,b] \end{cases}$$

and

$$(3.4) \quad M_1(x) \leq \begin{cases} \frac{1}{2} \|f'\|_{[a,b],1} + \frac{1}{2} \left| \|f'\|_{[a,x],1} - \|f'\|_{[x,b],1} \right|, \\ \left[ A\left(\|f'\|_{[x,\cdot],1}^{\beta}\right) \right]^{\frac{1}{\beta}}, \quad \beta > 1. \end{cases}$$



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*Proof.* Using (2.1) and taking the modulus, we have

$$\begin{aligned}
 (3.5) \quad |f(x) - A(f)| &= |A((x - e) \cdot g(\cdot, x))| \\
 &\leq A(|(x - e) \cdot g(\cdot, x)|) \\
 &= A(|x - e| |g(\cdot, x)|).
 \end{aligned}$$

For  $t \neq x$  ( $t, x \in [a, b]$ ) we may state

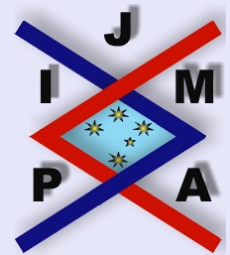
$$\begin{aligned}
 |g(t, x)| &\leq \int_0^1 |f'((1 - \lambda)x + \lambda t)| d\lambda \\
 &\leq \text{ess sup}_{\lambda \in [0,1]} |f'((1 - \lambda)x + \lambda t)| \\
 &= \|f'\|_{[x,t],\infty}.
 \end{aligned}$$

Hölder's inequality will produce

$$\begin{aligned}
 |g(t, x)| &\leq \int_0^1 |f'((1 - \lambda)x + \lambda t)| d\lambda \\
 &\leq \left[ \int_0^1 |f'((1 - \lambda)x + \lambda t)|^p d\lambda \right]^{\frac{1}{p}} \\
 &= \left( \frac{1}{x - t} \int_t^x |f'(u)|^p du \right)^{\frac{1}{p}} \\
 &= |x - t|^{-\frac{1}{p}} \|f'\|_{[x,t],p}, \quad p > 1, \quad \frac{1}{p} + \frac{1}{q} = 1;
 \end{aligned}$$

and finally

$$|g(t, x)| \leq \int_0^1 |f'((1 - \lambda)x + \lambda t)| d\lambda = \frac{1}{t - x} \|f'\|_{[x,t],1}.$$



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Consequently

$$(3.6) \quad |(x - e)| |g(\cdot, x)| \leq \begin{cases} |x - e| \|f'\|_{[x, \cdot], \infty} & \text{if } f' \in L_\infty [a, b]; \\ |x - e|^{\frac{1}{q}} \|f'\|_{[x, \cdot], p} & \text{if } f' \in L_p [a, b], \\ \|f'\|_{[x, \cdot], 1} \end{cases}$$

for any  $x \in [a, b]$ .

Applying the functional  $A$  to (3.6) and using (3.5) we deduce the inequality (3.1).

We have

$$\begin{aligned} M_\infty(x) &\leq \sup_{t \in [a, b]} \left\{ \|f'\|_{[x, t], \infty} \right\} A(|x - e|) \\ &= \max \left\{ \|f'\|_{[a, x], \infty}, \|f'\|_{[x, b], \infty} \right\} A(|x - e|) \\ &= \|f'\|_{[a, b], \infty} A(|x - e|) \end{aligned}$$

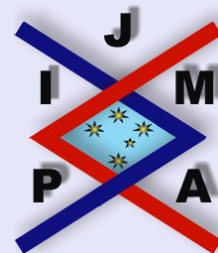
and the first inequality in (3.2) is proved.

Using Hölder's inequality for the functional  $A$ , i.e.,

$$(3.7) \quad |A(hg)| \leq [A(|h|^\alpha)]^{\frac{1}{\alpha}} \left[ A(|g|^\beta) \right]^{\frac{1}{\beta}}, \quad \alpha > 1, \quad \frac{1}{\alpha} + \frac{1}{\beta} = 1,$$

where  $hg, |h|^\alpha, |g|^\beta \in L$ , we have

$$M_\infty(x) \leq [A(|x - e|^\alpha)]^{\frac{1}{\alpha}} \left[ A\left(\|f'\|_{[x, \cdot], \infty}^\beta\right) \right]^{\frac{1}{\beta}}$$



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and the second part of (3.2) is proved.

In addition,

$$\begin{aligned} M_\infty(x) &\leq \sup_{t \in [a,b]} |x - t| A \left( \|f'\|_{[x,\cdot],\infty} \right) \\ &= \max \{x - a, b - x\} A \left( \|f'\|_{[x,\cdot],\infty} \right) \\ &= \left[ \frac{1}{2} (b - a) + \left| x - \frac{a + b}{2} \right| \right] A \left( \|f'\|_{[x,\cdot],\infty} \right) \end{aligned}$$

and the inequality (3.2) is completely proved.

We also have

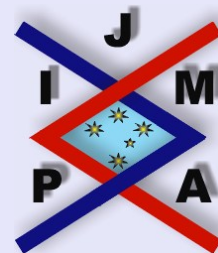
$$\begin{aligned} M_p(x) &\leq \sup_{t \in [a,b]} \left\{ \|f'\|_{[x,t],p} \right\} A \left( |x - e|^{\frac{1}{q}} \right) \\ &= \max \left\{ \|f'\|_{[a,x],p}, \|f'\|_{[x,b],p} \right\} A \left( |x - e|^{\frac{1}{q}} \right). \end{aligned}$$

Using Hölder's inequality (3.7) one has

$$M_p(x) \leq \left[ A \left( |x - e|^{\frac{\alpha}{q}} \right) \right]^{\frac{1}{\alpha}} \left[ A \left( \|f'\|_{[x,\cdot],p}^\beta \right) \right]^{\frac{1}{\beta}}, \quad \alpha > 1, \quad \frac{1}{\alpha} + \frac{1}{\beta} = 1$$

and

$$\begin{aligned} M_p(x) &\leq \sup_{t \in [a,b]} \left\{ |x - t|^{\frac{1}{q}} \right\} A \left( \|f'\|_{[x,\cdot],p} \right) \\ &= \max \left\{ (x - a)^{\frac{1}{q}}, (b - x)^{\frac{1}{q}} \right\} A \left( \|f'\|_{[x,\cdot],p} \right) \\ &= \left[ \frac{1}{2} (b - a) + \left| x - \frac{a + b}{2} \right| \right]^{\frac{1}{q}} A \left( \|f'\|_{[x,\cdot],p} \right), \end{aligned}$$



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proving the inequality (3.3).

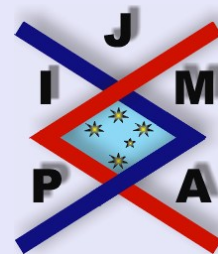
Finally,

$$\begin{aligned} A\left(\|f'\|_{[x,\cdot],1}\right) &\leq \sup_{t \in [a,b]} \left\{ \|f'\|_{[x,t],1} \right\} A(1) \\ &= \max \left\{ \|f'\|_{[a,x],1}, \|f'\|_{[x,b],1} \right\} \\ &= \frac{1}{2} \|f'\|_{[a,b],1} + \frac{1}{2} \left| \|f'\|_{[a,x],1} - \|f'\|_{[x,b],1} \right|. \end{aligned}$$

By Hölder's inequality, we have

$$A\left(\|f'\|_{[x,\cdot],1}\right) \leq \left[ A\left(\|f'\|_{[x,\cdot],\beta}^\beta\right) \right]^{\frac{1}{\beta}}, \quad \beta > 1,$$

and the last part of (3.4) is also proved. □



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## 4. The Case where $|f'|$ is Convex

The following theorem also holds.

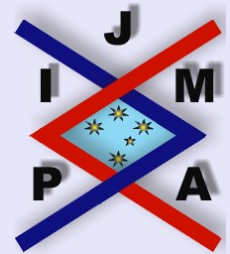
**Theorem 4.1.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be an absolutely continuous function such that  $f' : (a, b) \rightarrow \mathbb{R}$  is convex in absolute value, i.e.,  $|f'|$  is convex on  $(a, b)$ . If  $A : L \rightarrow \mathbb{R}$  is a normalised isotonic linear functional and  $|x - e|, |x - e| |f'| \in L$ , then*

$$(4.1) \quad |f(x) - A(f)| \leq \frac{1}{2} [ |f'(x)| A(|x - e|) + A(|x - e| |f'|) ]$$

$$\leq \begin{cases} \frac{1}{2} [ \|f'\|_{[a,b],\infty} + |f'(x)| ] A(|x - e|), & \text{if } f' \in L_\infty[a, b]; \\ \frac{1}{2} \left[ |f'(x)| A(|x - e|) + [A(|x - e|^\alpha)]^{\frac{1}{\alpha}} \left[ A(|f'|^\beta) \right]^{\frac{1}{\beta}} \right] \\ \quad \text{if } |x - e|^\alpha, |f'|^\beta \in L, \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \frac{1}{2} [ |f'(x)| A(|x - e|) + \left[ \frac{1}{2}(b - a) + \left| x - \frac{a+b}{2} \right| \right] A(|f'|) ] & \text{if } |f'| \in L. \end{cases}$$

*Proof.* Since  $|f'|$  is convex, we have

$$\begin{aligned} |g(t, x)| &\leq \int_0^1 |f'((1 - \lambda)x + \lambda t)| d\lambda \\ &= |f'(x)| \int_0^1 (1 - \lambda) d\lambda + |f'(t)| \int_0^1 \lambda d\lambda \\ &= \frac{|f'(x)| + |f'(t)|}{2}. \end{aligned}$$



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Thus,

$$\begin{aligned} |f(x) - A(f)| &\leq A\left(|x - e| \cdot \frac{|f'(x)| + |f'(t)|}{2}\right) \\ &= \frac{1}{2} [|f'(x)| A(|x - e|) + A(|x - e| |f'|)] \end{aligned}$$

and the first part of (4.1) is proved.

We have

$$\begin{aligned} A(|x - e| |f'|) &\leq \operatorname{ess\,sup}_{t \in [a, b]} \{|f'(t)|\} \cdot A(|x - e|) \\ &= \|f'\|_{[a, b], \infty} A(|x - e|). \end{aligned}$$

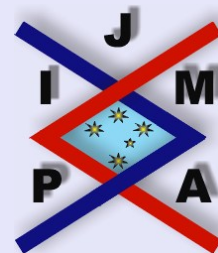
By Hölder's inequality for isotonic linear functionals, we have

$$A(|x - e| |f'|) \leq [A(|x - e|^\alpha)]^{\frac{1}{\alpha}} \left[ A(|f'|^\beta) \right]^{\frac{1}{\beta}}, \quad \alpha > 1, \quad \frac{1}{\alpha} + \frac{1}{\beta} = 1$$

and finally,

$$\begin{aligned} A(|x - e| |f'|) &\leq \sup_{t \in [a, b]} |x - t| \cdot A(|f'|) \\ &= \max(x - a, b - x) \cdot A(|f'|) \\ &= \left( \frac{1}{2}(b - a) + \left| x - \frac{a + b}{2} \right| \right) A(|f'|). \end{aligned}$$

The theorem is thus proved. □



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## 5. Some Integral Inequalities

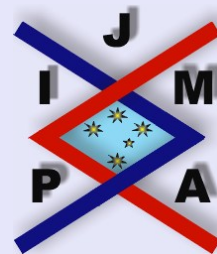
If we consider the normalised isotonic linear functional  $A(f) = \frac{1}{b-a} \int_a^b f$ , then by Theorem 3.1 for  $f : [a, b] \rightarrow \mathbb{R}$  an absolutely continuous function, we may state the following integral inequalities

$$(5.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{b-a} \int_a^b |x-t| \|f'\|_{[x,t],\infty} dt$$

$$\leq \begin{cases} \|f'\|_{[a,b],\infty} \left[ \frac{1}{4} + \left( \frac{x-\frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) & \text{(Ostrowski's inequality)} \\ & \text{provided } f' \in L_\infty[a, b]; \\ \left[ \frac{1}{b-a} \int_a^b \|f'\|_{[x,t],\infty}^\beta dt \right]^{\frac{1}{\beta}} \left[ \frac{(b-x)^{\alpha+1} + (x-a)^{\alpha+1}}{(\alpha+1)(b-a)} \right]^{\frac{1}{\alpha}} \\ & \text{if } f' \in L_\infty[a, b], \|f'\|_{[x,\cdot],\infty} \in L_\beta[a, b], \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \left[ \frac{1}{2} + \frac{|x-\frac{a+b}{2}|}{b-a} \right] \int_a^b \|f'\|_{[x,t],\infty} dt \\ & \text{if } f' \in L_\infty[a, b], \text{ and if } \|f'\|_{[x,\cdot],\infty} \in L_1[a, b], \end{cases}$$

for each  $x \in [a, b]$ ;

$$(5.2) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{b-a} \int_a^b |x-t|^{\frac{1}{q}} \|f'\|_{[x,t],p} dt$$



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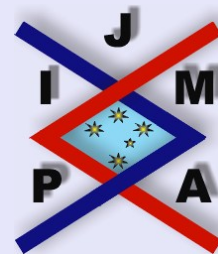
$$\leq \left\{ \begin{array}{l} q \max \left\{ \|f'\|_{[a,x],p}, \|f'\|_{[x,b],p} \right\} \left[ \frac{(b-x)^{\frac{1}{q}+1} + (x-a)^{\frac{1}{q}+1}}{(b-a)(q+1)} \right], \\ \quad p > 1, \frac{1}{p} + \frac{1}{q} = 1 \text{ and } f' \in L_p[a, b]; \\ \\ q^{\frac{1}{\alpha}} \left( \frac{1}{b-a} \int_a^b \|f'\|_{[x,t],p}^{\beta} dt \right)^{\frac{1}{\beta}} \left[ \frac{(b-x)^{\frac{\alpha}{q}+1} + (x-a)^{\frac{\alpha}{q}+1}}{(b-a)(q+\alpha)} \right]^{\frac{1}{\alpha}} \text{ if } f' \in L_p[a, b], \\ \quad \text{and } \|f'\|_{[x,\cdot],p} \in L_{\beta}[a, b], \text{ where } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1 \\ \\ \left[ \frac{1}{2} + \frac{|x - \frac{a+b}{2}|}{b-a} \right]^{\frac{1}{q}} \frac{1}{b-a} \int_a^b \|f'\|_{[x,t],p} dt \\ \quad \text{if } f' \in L_p[a, b], \text{ and } \|f'\|_{[x,\cdot],p} \in L_1[a, b], \end{array} \right.$$

for each  $x \in [a, b]$  and

$$(5.3) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ \leq \frac{1}{b-a} \int_a^b \|f'\|_{[x,t],1} dt \\ \leq \left\{ \begin{array}{l} \frac{1}{2} \|f'\|_{[a,b],1} + \frac{1}{2} \left| \|f'\|_{[a,x],1} - \|f'\|_{[x,b],1} \right| \text{ if } f' \in L_1[a, b]; \\ \\ \left( \frac{1}{b-a} \int_a^b \|f'\|_{[x,t],1}^{\beta} dt \right)^{\frac{1}{\beta}} \\ \quad \text{if } f' \in L_1[a, b], \|f'\|_{[x,\cdot],1} \in L_{\beta}[a, b], \text{ where } \beta > 1, \end{array} \right.$$

for each  $x \in [a, b]$ .

If we assume now that  $f : [a, b] \rightarrow \mathbb{R}$  is absolutely continuous and such that  $|f'|$  is convex on  $(a, b)$ , then by Theorem 4.1 we obtain the following integral



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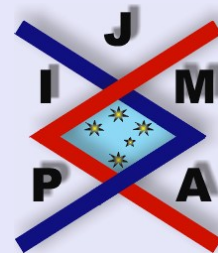
inequalities established in [1]

$$(5.4) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right|$$

$$\leq \frac{1}{2} \left[ |f'(x)| \left[ \frac{1}{4} + \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) + \frac{1}{b-a} \int_a^b |x-t| |f'(t)| dt \right]$$

$$\leq \begin{cases} \frac{1}{2} \left[ \|f'\|_{[a,b],\infty} + |f'(x)| \right] \left[ \frac{1}{4} + \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) & \text{if } f' \in L_\infty[a,b]; \\ \frac{1}{2} \left\{ |f'(x)| \left[ \frac{1}{4} + \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \right. \\ \quad \left. + \left[ \frac{(b-x)^{\alpha+1} + (x-a)^{\alpha+1}}{(\alpha+1)(b-a)} \right]^{\frac{1}{\alpha}} \left[ \frac{1}{b-a} \int_a^b |f'(t)|^\beta dt \right]^{\frac{1}{\beta}} \right\} & \text{if } f' \in L_\beta[a,b], \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \frac{1}{2} \left\{ |f'(x)| \left[ \frac{1}{4} + \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) + \left[ \frac{1}{2} + \frac{|x - \frac{a+b}{2}|}{b-a} \right] \int_a^b |f'(t)| dt \right\} & \text{if } f' \in L_1[a,b], \end{cases}$$

for each  $x \in [a, b]$ .



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## 6. Some Discrete Inequalities

For a given interval  $[a, b]$ , consider the division

$$I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

and the intermediate points  $\xi_i \in [x_i, x_{i+1}]$ ,  $i = \overline{0, n-1}$ . If  $h_i := x_{i+1} - x_i > 0$  ( $i = \overline{0, n-1}$ ) we may define the following functionals

$$A(f; I_n, \xi) := \frac{1}{b-a} \sum_{i=0}^{n-1} f(\xi_i) h_i \quad (\text{Riemann Rule})$$

$$A_T(f; I_n) := \frac{1}{b-a} \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} \cdot h_i \quad (\text{Trapezoid Rule})$$

$$A_M(f; I_n) := \frac{1}{b-a} \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) \cdot h_i \quad (\text{Mid-point Rule})$$

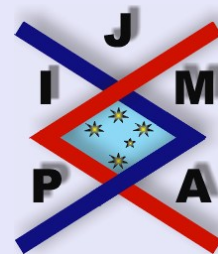
$$A_S(f; I_n) := \frac{1}{3} A_T(f; I_n) + \frac{2}{3} A_M(f; I_n). \quad (\text{Simpson Rule})$$

We observe that, all the above functionals are obviously linear, isotonic and normalised.

Consequently, all the inequalities obtained in Sections 2 and 3 may be applied for these functionals.

If, for example, we use the following inequality (see Theorem 3.1)

$$(6.1) \quad |f(x) - A(f)| \leq \|f'\|_{[a,b]} A(|x - e|), \quad x \in [a, b],$$



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provided  $f : [a, b] \rightarrow \mathbb{R}$  is absolutely continuous and  $f' \in L_\infty [a, b]$ , then we get the inequalities

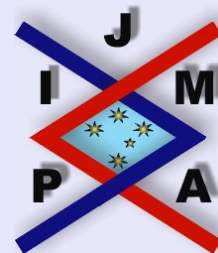
$$(6.2) \quad \left| f(x) - \frac{1}{b-a} \sum_{i=0}^{n-1} f(\xi_i) h_i \right| \leq \|f'\|_{[a,b],\infty} \frac{1}{b-a} \sum_{i=0}^{n-1} |x - \xi_i| h_i,$$

$$(6.3) \quad \left| f(x) - \frac{1}{b-a} \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} \cdot h_i \right| \leq \|f'\|_{[a,b],\infty} \cdot \frac{1}{b-a} \sum_{i=0}^{n-1} \frac{|x - x_i| + |x - x_{i+1}|}{2} h_i,$$

$$(6.4) \quad \left| f(x) - \frac{1}{b-a} \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) \cdot h_i \right| \leq \|f'\|_{[a,b],\infty} \frac{1}{b-a} \sum_{i=0}^{n-1} \left| x - \frac{x_i + x_{i+1}}{2} \right| h_i,$$

for each  $x \in [a, b]$ .

Similar results may be stated if one uses for example Theorem 4.1. We omit the details.




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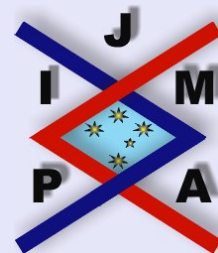
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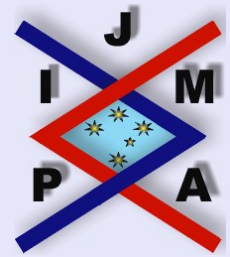
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