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ON WEIGHTED OSTROWSKI TYPE INEQUALITIES FOR OPERATORS AND VECTOR-VALUED FUNCTIONS

[1]N.S. BARNETT, [2]C. BUSE, [1]P. CERONE AND [1]S.S. DRAGOMIR

[1]School of Communications and Informatics
Victoria University of Technology
PO Box 14428
Melbourne City MC
8001 Victoria, Australia
EMail: neil@matilda.vu.edu.au
URL: http://sci.vu.edu.au/staff/neilb.html

[2]Department of Mathematics
West University of Timişoara
Bd. V. Pârvan 4
1900 Timişoara, România.
EMail:buse@hilbert.math.uvt.ro
URL: http://rgmia.vu.edu.au/BuseCVhtml/

EMail:pc@matilda.vu.edu.au
URL: http://rgmia.vu.edu.au/cerone

EMail:sever@matilda.vu.edu.au
URL: http://rgmia.vu.edu.au/SSDragomirWeb.html

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Abstract

Some weighted Ostrowski type integral inequalities for operators and vector-valued functions in Banach spaces are given. Applications for linear operators in Banach spaces and differential equations are also provided.

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Contents

1. Introduction

In [12], Pečarić and Savić obtained the following Ostrowski type inequality for weighted integrals (see also [7, Theorem 3]):

Theorem 1.1. *Let* $w : [a,b] \to [0,\infty)$ *be a weight function on* $[a,b]$ *. Suppose that* $f : [a,b] \to \mathbb{R}$ *satisfies*

$$|f(t) - f(s)| \leq N|t-s|^{\alpha}, \text{ for all } t,s \in [a,b], \tag{1.1}$$

where $N > 0$ *and* $0 < \alpha \leq 1$ *are some constants. Then for any* $x \in [a,b]$

$$\left| f(x) - \frac{\int_a^b w(t) f(t)\,dt}{\int_a^b w(t)\,dt} \right| \leq N \cdot \frac{\int_a^b |t-x|^{\alpha} w(t)\,dt}{\int_a^b w(t)\,dt}. \tag{1.2}$$

Further, if for some constants c *and* λ

$$0 < c \leq w(t) \leq \lambda c, \text{ for all } t \in [a,b],$$

then for any $x \in [a,b]$*, we have*

$$\left| f(x) - \frac{\int_a^b w(t) f(t)\,dt}{\int_a^b w(t)\,dt} \right| \leq N \cdot \frac{\lambda L(x) J(x)}{L(x) - J(x) + \lambda J(x)}, \tag{1.3}$$

where

$$L(x) := \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^{\alpha}$$

and

$$J(x) := \frac{(x-a)^{1+\alpha} + (b-x)^{1+\alpha}}{(1+\alpha)(b-a)}.$$

The inequality (1.2) was rediscovered in [4] where further applications for different weights and in Numerical Analysis were given.

For other results in connection to weighted Ostrowski inequalities, see [3], [8] and [10].

In the present paper we extend the weighted Ostrowski's inequality for vector-valued functions and Bochner integrals and apply the obtained results to operatorial inequalities and linear differential equations in Banach spaces. Some numerical experiments are also conducted.

2. Weighted Inequalities

Let X be a Banach space and $-\infty < a < b < \infty$. We denote by $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators acting on X. The norms of vectors or operators acting on X will be denoted by $\|\cdot\|$.

A function $f : [a,b] \to X$ is called *measurable* if there exists a sequence of simple functions $f_n : [a,b] \to X$ which converges punctually almost everywhere on $[a,b]$ at f. We recall also that a measurable function $f : [a,b] \to X$ is *Bochner integrable* if and only if its norm function (i.e. the function $t \mapsto \|f(t)\| : [a,b] \to \mathbb{R}_+$) is Lebesgue integrable on $[a,b]$.

The following theorem holds.

Theorem 2.1. *Assume that $B : [a,b] \to \mathcal{L}(X)$ is Hölder continuous on $[a,b]$, i.e.,*

$$\|B(t) - B(s)\| \leq H |t-s|^{\alpha} \quad \text{for all } t,s \in [a,b], \tag{2.1}$$

where $H > 0$ and $\alpha \in (0,1]$.

If $f : [a,b] \to X$ is Bochner integrable on $[a,b]$, then we have the inequality:

$$\left\| B(t)\int_a^b f(s)\,ds - \int_a^b B(s) f(s)\,ds \right\| \leq H \int_a^b |t-s|^{\alpha} \|f(s)\|\,ds \tag{2.2}$$

$$\leq H \times \begin{cases} \frac{(b-t)^{\alpha+1}+(t-a)^{\alpha+1}}{\alpha+1} |||f|||_{[a,b],\infty} & \text{if} \quad f \in L_{\infty}([a,b];X); \\ \left[\frac{(b-t)^{q\alpha+1}+(t-a)^{q\alpha+1}}{q\alpha+1}\right]^{\frac{1}{q}} |||f|||_{[a,b],p} & \text{if} \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ & \text{and} \quad f \in L_p([a,b];X); \\ \left[\frac{1}{2}(b-a) + \left|t - \frac{a+b}{2}\right|\right]^{\alpha} |||f|||_{[a,b],1} & \end{cases}$$

for any $t \in [a,b]$, *where*

$$|||f|||_{[a,b],\infty} := ess \sup_{t\in[a,b]} \|f(t)\|$$

and

$$|||f|||_{[a,b],p} := \left(\int_a^b \|f(t)\|^p \, dt\right)^{\frac{1}{p}}, \quad p \geq 1.$$

Proof. Firstly, we prove that the $X-$valued function $s \mapsto B(s) f(s)$ is Bochner integrable on $[a,b]$. Indeed, let (f_n) be a sequence of $X-$valued, simple functions which converge almost everywhere on $[a,b]$ at the function f. The maps $s \mapsto B(s) f_n(s)$ are measurable (because they are continuous with the exception of a finite number of points s in $[a,b]$). Then

$$\|B(s) f_n(s) - B(s) f(s)\| \leq \|B(s)\| \, \|f_n(s) - f(s)\| \to 0 \text{ a.e. on } [a,b]$$

when $n \to \infty$ so that the function $s \mapsto B(s) f(s) : [a,b] \to X$ is measurable. Now, using the estimate

$$\|B(s) f(s)\| \leq \sup_{\xi\in[a,b]} \|B(\xi)\| \cdot \|f(s)\|, \quad \text{for all } s \in [a,b],$$

it is easy to see that the function $s \mapsto B(s) f(s)$ is Bochner integrable on $[a,b]$.

We have successively

$$\begin{aligned}\left\|B(t)\int_a^b f(s)\,ds - \int_a^b B(s) f(s)\,ds\right\| &= \left\|\int_a^b (B(t) - B(s)) f(s)\,ds\right\| \\ &\leq \int_a^b \|(B(t) - B(s)) f(s)\|\,ds\end{aligned}$$

$$\leq \int_a^b \|(B(t)-B(s))\| \|f(s)\| \, ds \leq H \int_a^b |t-s|^\alpha \|f(s)\| \, ds =: M(t)$$

for any $t \in [a,b]$, proving the first inequality in (2.2).

Now, observe that

$$\begin{aligned} M(t) &\leq H \, |||f|||_{[a,b],\infty} \int_a^b |t-s|^\alpha \, ds \\ &= H \, |||f|||_{[a,b],\infty} \cdot \frac{(b-t)^{\alpha+1}+(t-a)^{\alpha+1}}{\alpha+1} \end{aligned}$$

and the first part of the second inequality is proved.

Using Hölder's integral inequality, we may state that

$$\begin{aligned} M(t) &\leq H \left(\int_a^b |t-s|^{q\alpha} \, ds \right)^{\frac{1}{q}} \left(\int_a^b \|f(s)\|^p \, ds \right)^{\frac{1}{p}} \\ &= H \left[\frac{(b-t)^{q\alpha+1}+(t-a)^{q\alpha+1}}{q\alpha+1} \right]^{\frac{1}{q}} |||f|||_{[a,b],p}, \end{aligned}$$

proving the second part of the second inequality.

Finally, we observe that

$$\begin{aligned} M(t) &\leq H \sup_{s\in[a,b]} |t-s|^\alpha \int_a^b \|f(s)\| \, ds \\ &= H \max\left\{ (b-t)^\alpha, (t-a)^\alpha \right\} |||f|||_{[a,b],1} \\ &= H \left[\frac{1}{2}(b-a) + \left| t - \frac{a+b}{2} \right| \right]^\alpha |||f|||_{[a,b],1} \end{aligned}$$

and the theorem is proved. $\square$

The following corollary holds.

Corollary 2.2. *Assume that* $B : [a, b] \to \mathcal{L}(X)$ *is Lipschitzian with the constant* $L > 0$. *Then we have the inequality*

$$\left\| B(t) \int_a^b f(s)\,ds - \int_a^b B(s) f(s)\,ds \right\| \tag{2.3}$$

$$\leq L \int_a^b |t - s| \, \|f(s)\| \, ds$$

$$\leq L \times \begin{cases} \left[\frac{1}{4}(b-a)^2 + \left(t - \frac{a+b}{2}\right)^2\right] |||f|||_{[a,b],\infty} & \textit{if} \quad f \in L_\infty([a,b];X); \\ \left[\frac{(b-t)^{q+1}+(t-a)^{q+1}}{q+1}\right]^{\frac{1}{q}} |||f|||_{[a,b],p} & \textit{if} \quad p > 1,\ \frac{1}{p}+\frac{1}{q}=1 \\ & \textit{and} \quad f \in L_p([a,b];X); \\ \left[\frac{1}{2}(b-a) + \left|t - \frac{a+b}{2}\right|\right] |||f|||_{[a,b],1} \end{cases}$$

for any $t \in [a, b]$.

Remark 2.1. If we choose $t = \frac{a+b}{2}$ in (2.2) and (2.3), then we get the following midpoint inequalities:

$$\left\| B\left(\frac{a+b}{2}\right) \int_a^b f(s)\,ds - \int_a^b B(s) f(s)\,ds \right\| \tag{2.4}$$

$$\leq H \int_a^b \left| s - \frac{a+b}{2} \right|^\alpha \|f(s)\| \, ds$$

$$\leq H \times \begin{cases} \frac{1}{2^{\alpha}(\alpha+1)} (b-a)^{\alpha+1} \||f|\|_{[a,b],\infty} & \text{if} \quad f \in L_{\infty}([a,b];X); \\ \frac{1}{2^{\alpha}(q\alpha+1)^{\frac{1}{q}}} (b-a)^{\alpha+\frac{1}{q}} \||f|\|_{[a,b],p} & \text{if} \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ & \text{and} \quad f \in L_p([a,b];X); \\ \frac{1}{2^{\alpha}} (b-a)^{\alpha} \||f|\|_{[a,b],1} \end{cases}$$

and

$$\left\| B\left(\frac{a+b}{2}\right) \int_a^b f(s)\,ds - \int_a^b B(s) f(s)\,ds \right\| \tag{2.5}$$

$$\leq L \int_a^b \left| s - \frac{a+b}{2} \right| \|f(s)\|\,ds$$

$$\leq L \times \begin{cases} \frac{1}{4} (b-a)^2 \||f|\|_{[a,b],\infty} & \text{if} \quad f \in L_{\infty}([a,b];X); \\ \frac{1}{2(q+1)^{\frac{1}{q}}} (b-a)^{1+\frac{1}{q}} \||f|\|_{[a,b],p} & \text{if} \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ & \text{and} \quad f \in L_p([a,b];X); \\ \frac{1}{2} (b-a) \||f|\|_{[a,b],1} \end{cases}$$

respectively.

Remark 2.2. Consider the function $\Psi_{\alpha} : [a,b] \to \mathbb{R}$, $\Psi_{\alpha}(t) := \int_a^b |t-s|^{\alpha} \|f(s)\|\,ds$, $\alpha \in (0,1)$. If f is continuous on $[a,b]$, then Ψ_{α} is differentiable and

$$\begin{aligned} \frac{d\Psi_{\alpha}(t)}{dt} &= \frac{d}{dt}\left[\int_a^t (t-s)^{\alpha} \|f(s)\|\,ds + \int_t^b (s-t)^{\alpha} \|f(s)\|\,ds\right] \\ &= \alpha\left[\int_a^t \frac{\|f(s)\|}{(t-s)^{1-\alpha}} ds - \int_t^b \frac{\|f(s)\|}{(s-t)^{1-\alpha}} ds\right]. \end{aligned}$$

If $t_0 \in (a,b)$ is such that

$$\int_a^{t_0} \frac{\|f(s)\|}{(t_0-s)^{1-\alpha}} ds = \int_{t_0}^b \frac{\|f(s)\|}{(s-t_0)^{1-\alpha}} ds$$

and $\Psi'(\cdot)$ is negative on (a,t_0) and positive on (t_0,b), then the best inequality we can get in the first part of (2.2) is the following one

$$\left\| B(t_0) \int_a^b f(s)\, ds - \int_a^b B(s) f(s)\, ds \right\| \leq H \int_a^b |t_0 - s|^\alpha \|f(s)\|\, ds. \tag{2.6}$$

If $\alpha = 1$, then, for

$$\Psi(t) := \int_a^b |t-s| \|f(s)\|\, ds,$$

we have

$$\begin{aligned} \frac{d\Psi(t)}{dt} &= \int_a^t \|f(s)\|\, ds - \int_t^b \|f(s)\|\, ds, \quad t \in (a,b), \\ \frac{d^2\Psi(t)}{dt^2} &= 2\|f(t)\| \geq 0, \quad t \in (a,b), \end{aligned}$$

which shows that Ψ is convex on (a,b).

If $t_m \in (a,b)$ is such that

$$\int_a^{t_m} \|f(s)\|\, ds = \int_{t_m}^b \|f(s)\|\, ds,$$

then the best inequality we can get from the first part of (2.3) is

$$\left\| B\left(t_m\right) \int_a^b f\left(s\right) ds - \int_a^b B\left(s\right) f\left(s\right) ds \right\| \leq L \int_a^b sgn\left(s - t_m\right) s \left\| f\left(s\right) \right\| ds. \tag{2.7}$$

Indeed, as

$$\begin{aligned}
&\inf_{t\in[a,b]} \int_a^b |t-s| \left\| f\left(s\right) \right\| ds \\
&\quad = \int_a^b |t_m - s| \left\| f\left(s\right) \right\| ds \\
&\quad = \int_a^{t_m} \left(t_m - s\right) \left\| f\left(s\right) \right\| ds + \int_{t_m}^b \left(s - t_m\right) \left\| f\left(s\right) \right\| ds \\
&\quad = t_m \left(\int_a^{t_m} \left\| f\left(s\right) \right\| ds - \int_{t_m}^b \left\| f\left(s\right) \right\| ds \right) \\
&\qquad\qquad + \int_{t_m}^b s \left\| f\left(s\right) \right\| ds - \int_a^{t_m} s \left\| f\left(s\right) \right\| ds \\
&\quad = \int_{t_m}^b s \left\| f\left(s\right) \right\| ds - \int_a^{t_m} s \left\| f\left(s\right) \right\| ds \\
&\quad = \int_a^b sgn\left(s - t_m\right) s \left\| f\left(s\right) \right\| ds,
\end{aligned}$$

then the best inequality we can get from the first part of (2.3) is obtained for $t = t_m \in (a, b)$.

We recall that a function $F:[a,b]\to\mathcal{L}(X)$ is said to be *strongly continuous* if for all $x\in X$, the maps $s\mapsto F(s)x:[a,b]\to X$ are continuous on $[a,b]$. In this case the function $s\mapsto\|B(s)\|:[a,b]\to\mathbb{R}_+$ is (Lebesgue) measurable and bounded ([6]). The linear operator $L=\int_a^b F(s)\,ds$ (defined by $Lx:=\int_a^b F(s)\,xds$ for all $x\in X$) is bounded, because

$$\|Lx\|\le\left(\int_a^b\|F(s)\|\,ds\right)\cdot\|x\|\quad\text{for all }\; x\in X.$$

In a similar manner to Theorem 2.1, we may prove the following result as well.

Theorem 2.3. *Assume that $f:[a,b]\to X$ is Hölder continuous, i.e.,*

$$\|f(t)-f(s)\|\le K|t-s|^\beta\quad\textit{for all}\;\; t,s\in[a,b],\tag{2.8}$$

where $K>0$ and $\beta\in(0,1]$.

If $B:[a,b]\to\mathcal{L}(X)$ is strongly continuous on $[a,b]$, then we have the inequality:

$$\left\|\left(\int_a^b B(s)\,ds\right)f(t)-\int_a^b B(s)\,f(s)\,ds\right\| \le K\int_a^b|t-s|^\beta\,\|B(s)\|\,ds\tag{2.9}$$

$$\leq K \times \begin{cases} \frac{(b-t)^{\beta+1}+(t-a)^{\beta+1}}{\beta+1} \left|\left|\left|B\right|\right|\right|_{[a,b],\infty} & \textit{if} \quad \left\|B(\cdot)\right\| \in L_\infty([a,b];\mathbb{R}_+); \\ \left[\frac{(b-t)^{q\beta+1}+(t-a)^{q\beta+1}}{q\beta+1}\right]^{\frac{1}{q}} \left|\left|\left|B\right|\right|\right|_{[a,b],p} & \textit{if} \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ & \textit{and} \quad \left\|B(\cdot)\right\| \in L_p([a,b];\mathbb{R}_+); \\ \left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]^{\beta} \left|\left|\left|B\right|\right|\right|_{[a,b],1} \end{cases}$$

for any $t \in [a,b]$.

The following corollary holds.

Corollary 2.4. *Assume that* f *and* B *are as in Theorem 2.3. If, in addition,* $\int_a^b B(s)\,ds$ *is invertible in* $\mathcal{L}(X)$, *then we have the inequality:*

$$\left\| f(t) - \left(\int_a^b B(s)\,ds\right)^{-1} \int_a^b B(s) f(s)\,ds \right\| \leq K \left\| \left(\int_a^b B(s)\,ds\right)^{-1} \right\| \int_a^b |t-s|^{\beta} \left\|B(s)\right\| ds \tag{2.10}$$

for any $t \in [a,b]$.

Remark 2.3. It is obvious that the inequality (2.10) contains as a particular case what is the so called Ostrowski's inequality for weighted integrals (see (1.2)).

3. Inequalities for Linear Operators

Let $0 \leq a < b < \infty$ and $A \in \mathcal{L}(X)$. We recall that the operatorial norm of A is given by

$$\|A\| = \sup \{\|Ax\| : \|x\| \leq 1\}.$$

The *resolvent set* of A (denoted by $\rho(A)$) is the set of all complex scalars λ for which $\lambda I - A$ is an invertible operator. Here I is the identity operator in $\mathcal{L}(X)$. The complementary set of $\rho(A)$ in the complex plane, denoted by $\sigma(A)$, is the *spectrum* of A. It is known that $\sigma(A)$ is a compact set in $\mathbb{C}$. The series $\left(\sum_{n\geq 0} \frac{(tA)^n}{n!}\right)$ converges absolutely and locally uniformly for $t \in \mathbb{R}$. If we denote by e^{tA} its sum, then

$$\left\|e^{tA}\right\| \leq e^{|t|\|A\|}, \quad t \in \mathbb{R}.$$

Proposition 3.1. *Let X be a real or complex Banach space, $A \in \mathcal{L}(X)$ and β be a non-null real number such that $-\beta \in \rho(A)$. Then for all $0 \leq a < b < \infty$ and each $s \in [a, b]$, we have*

$$\left\|\frac{e^{\beta b} - e^{\beta a}}{\beta} \cdot e^{sA} - (\beta I + A)^{-1}\left[e^{b(\beta I + A)} - e^{a(\beta I + A)}\right]\right\| \leq \|A\| e^{b\|A\|} \cdot \left[\frac{1}{4}(b-a)^2 + \left(s - \frac{a+b}{2}\right)^2\right] \cdot \max\left\{e^{\beta b}, e^{\beta a}\right\}. \tag{3.1}$$

Proof. We apply the second inequality from Corollary 2.2 in the following particular case.

$$B(\tau) := e^{\tau A}, \quad f(\tau) = e^{\beta\tau} x, \quad \tau \in [a, b], \; x \in X.$$

For all $\xi,\eta\in[a,b]$ there exists an α between ξ and η such that

$$\begin{aligned}\left\|B(\xi)-B(\eta)\right\| &= \left\|\sum_{n=1}^{\infty}\frac{(\xi^n-\eta^n)}{n!}A^n\right\| \\ &= \left\|(\xi-\eta)A\sum_{n=0}^{\infty}\frac{(\alpha A)^n}{n!}\right\| \\ &\le \|A\|\left\|e^{\alpha A}\right\|\cdot|\xi-\eta| \\ &\le \|A\|\,e^{b\|A\|}\cdot|\xi-\eta|\,.\end{aligned}$$

The function $\tau\mapsto e^{\tau A}$ is thus Lipschitzian on $[a,b]$ with the constant $L:=\|A\|\,e^{b\cdot\|A\|}$. On the other hand we have

$$\begin{aligned}\int_a^b e^{\tau A}\left(e^{\beta\tau}x\right)d\tau &= \int_a^b e^{\tau A}\left(e^{\beta\tau}Ix\right)d\tau \\ &= \int_a^b e^{\tau(A+\beta I)}x d\tau \\ &= (A+\beta I)^{-1}\left[e^{b(A+\beta I)}-e^{a(A+\beta I)}\right]x,\end{aligned}$$

and

$$|||f|||_{[a,b],\infty}=\sup_{\tau\in[a,b]}\left\|e^{\tau\beta}x\right\|=\max\left\{e^{\beta b},e^{\beta a}\right\}\cdot\|x\|\,.$$

Placing all the above results in the second inequality from (2.3) and taking the supremum for all $x\in X$, we will obtain the desired inequality (3.1). □

Remark 3.1. Let $A \in \mathcal{L}(X)$ such that $0 \in \rho(A)$. Taking the limit as $\beta \to 0$ in (3.1), we get the inequality

$$\left\|(b-a)e^{sA} - A^{-1}\left[e^{bA} - e^{aA}\right]\right\| \le \|A\| e^{b\|A\|} \cdot \left[\frac{1}{4}(b-a)^2 + \left(s - \frac{a+b}{2}\right)^2\right],$$

where a, b and s are as in Proposition 3.1.

Proposition 3.2. *Let $A \in \mathcal{L}(X)$ be an invertible operator, $t \ge 0$ and $0 \le s \le t$. Then the following inequality holds:*

$$\left\|\frac{t^2}{2}\sin(sA) - A^{-2}\left[\sin(tA) - tA\cos(tA)\right]\right\| \le \frac{2s^3 + 2t^3 - 3st^2}{6}\|A\|. \tag{3.2}$$

In particular, if $X = \mathbb{R}$, $A = 1$ and $s = 0$ it follows the scalar inequality

$$|\sin t - t\cos t| \le \frac{t^3}{3}, \quad \textit{for all } t \ge 0.$$

Proof. We apply the inequality from (2.3) in the following particular case:

$$B(\tau) = \sin(\tau A) := \sum_{n=0}^{\infty}(-1)^n \frac{(\tau A)^{2n+1}}{(2n+1)!}, \quad \tau \ge 0,$$

and

$$f(\tau)=\tau\cdot x,\quad \text{for fixed } x\in X. \tag{3.3}$$

For each $\xi,\eta\in[0,t]$, we have

$$\begin{aligned}\|B(\xi)-B(\eta)\| &= \left\|A\left(\sum_{n=0}^{\infty}(-1)^n\frac{(\xi-\eta)\,\alpha^{2n}}{(2n)!}A^{2n}\right)\right\| \\ &\le \|A\|\,|\xi-\eta|\cdot\|\cos(\alpha A)\| \\ &\le \|A\|\,|\xi-\eta|,\end{aligned}$$

where α is a real number between ξ and η, i.e., the function $\tau\longmapsto B(\tau):\mathbb{R}_+\to\mathcal{L}(X)$ is $\|A\|$ –Lipschitzian.

Moreover, it is easy to see that

$$\int_0^t B(\tau)\,f(\tau)\,d\tau = A^{-2}\left[\sin(tA)-tA\cos(tA)\right]x$$

and

$$\int_0^t |s-\tau|\,|f(\tau)|\,d\tau=\frac{2s^3+2t^3-3st^2}{6}\|x\|. \tag{3.4}$$

Applying the first inequality from (2.3) and taking the supremum for $x\in X$ with $\|x\|\le 1$, we get (3.2). $\square$

4. Quadrature Formulae

Consider the division of the interval $[a,b]$ given by

$$I_n : a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b \tag{4.1}$$

and $h_i := t_{i+1} - t_i$, $\nu(h) := \max_{i=\overline{0,n-1}} h_i$. For the intermediate points $\xi := (\xi_0, \dots, \xi_{n-1})$ with $\xi_i \in [t_i, t_{i+1}]$, $i = \overline{0, n-1}$, define the sum

$$S_n^{(1)}(B, f; I_n, \xi) := \sum_{i=0}^{n-1} B(\xi_i) \int_{t_i}^{t_{i+1}} f(s)\,ds. \tag{4.2}$$

Then we may state the following result in approximating the integral

$$\int_a^b B(s) f(s)\,ds,$$

based on Theorem 2.1.

Theorem 4.1. *Assume that $B : [a,b] \to \mathcal{L}(X)$ is Hölder continuous on $[a,b]$, i.e., it satisfies the condition (2.1) and $f : [a,b] \to X$ is Bochner integrable on $[a,b]$. Then we have the representation*

$$\int_a^b B(s) f(s)\,ds = S_n^{(1)}(B, f; I_n, \xi) + R_n^{(1)}(B, f; I_n, \xi), \tag{4.3}$$

where $S_n^{(1)}(B, f; I_n, \xi)$ is as given by (4.2) and the remainder $R_n^{(1)}(B, f; I_n, \xi)$

satisfies the estimate

$$
\begin{aligned}
&\left\|R_n^{(1)}(B,f;I_n,\xi)\right\| \\
&\leq H\times\begin{cases}
\dfrac{1}{\alpha+1}\left|\left|\left|f\right|\right|\right|_{[a,b],\infty}\displaystyle\sum_{i=0}^{n-1}\left[(t_{i+1}-\xi_i)^{\alpha+1}+(\xi_i-t_i)^{\alpha+1}\right] \\
\dfrac{1}{(q\alpha+1)^{\frac{1}{q}}}\left|\left|\left|f\right|\right|\right|_{[a,b],p}\left\{\displaystyle\sum_{i=0}^{n-1}\left[(t_{i+1}-\xi_i)^{q\alpha+1}+(\xi_i-t_i)^{q\alpha+1}\right]\right\}^{\frac{1}{q}}, \\
\qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\
\left[\dfrac{1}{2}\nu(h)+\max\limits_{i=0,n-1}\left|\xi_i-\dfrac{t_{i+1}+t_i}{2}\right|\right]^{\alpha}\left|\left|\left|f\right|\right|\right|_{[a,b],1}
\end{cases} \\
&\leq H\times\begin{cases}
\dfrac{1}{\alpha+1}\left|\left|\left|f\right|\right|\right|_{[a,b],\infty}\displaystyle\sum_{i=0}^{n-1}h_i^{\alpha+1} \\
\dfrac{1}{(q\alpha+1)^{\frac{1}{q}}}\left|\left|\left|f\right|\right|\right|_{[a,b],p}\left(\displaystyle\sum_{i=0}^{n-1}h_i^{q\alpha+1}\right)^{\frac{1}{q}}, \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\
\left[\dfrac{1}{2}\nu(h)+\max\limits_{i=0,n-1}\left|\xi_i-\dfrac{t_{i+1}+t_i}{2}\right|\right]^{\alpha}\left|\left|\left|f\right|\right|\right|_{[a,b],1}
\end{cases} \\
&\leq H\times\begin{cases}
\dfrac{1}{\alpha+1}\left|\left|\left|f\right|\right|\right|_{[a,b],\infty}\left[\nu(h)\right]^{\alpha} \\
\dfrac{(b-a)^{\frac{1}{q}}}{(q\alpha+1)^{\frac{1}{q}}}\left|\left|\left|f\right|\right|\right|_{[a,b],p}\left[\nu(h)\right]^{\alpha} \\
\left|\left|\left|f\right|\right|\right|_{[a,b],1}\left[\nu(h)\right]^{\alpha}.
\end{cases}
\end{aligned}
$$

Proof. Applying Theorem 4.1 on $[x_i, x_{i+1}]$ $\left(i = \overline{0, n-1}\right)$, we may write that

$$\left\| \int_{t_i}^{t_{i+1}} B(s) f(s)\, ds - B(\xi_i) \int_{t_i}^{t_{i+1}} f(s)\, ds \right\| \leq H \times \begin{cases} \left[\frac{(t_{i+1}-\xi_i)^{\alpha+1} + (\xi_i - t_i)^{\alpha+1}}{\alpha+1}\right] |||f|||_{[t_i,t_{i+1}],\infty} \\ \\ \left[\frac{(t_{i+1}-\xi_i)^{q\alpha+1} + (\xi_i - t_i)^{q\alpha+1}}{q\alpha+1}\right]^{\frac{1}{q}} |||f|||_{[t_i,t_{i+1}],p} \\ \\ \left[\frac{1}{2}(t_{i+1} - t_i) + \left|\xi_i - \frac{t_{i+1}+t_i}{2}\right|\right]^{\alpha} |||f|||_{[t_i,t_{i+1}],1} . \end{cases}$$

Summing over i from 0 to $n-1$ and using the generalised triangle inequality we get

$$\begin{aligned} &\left\| R_n^{(1)}(B, f; I_n, \xi) \right\| \\ &\quad \leq \sum_{i=0}^{n-1} \left\| \int_{t_i}^{t_{i+1}} B(s) f(s)\, ds - B(\xi_i) \int_{t_i}^{t_{i+1}} f(s)\, ds \right\| \\ &\quad \leq H \times \begin{cases} \frac{1}{\alpha+1} \sum_{i=0}^{n-1} \left[(t_{i+1} - \xi_i)^{\alpha+1} + (\xi_i - t_i)^{\alpha+1}\right] |||f|||_{[t_i,t_{i+1}],\infty} \\ \\ \frac{1}{(q\alpha+1)^{\frac{1}{q}}} \left[\sum_{i=0}^{n-1} (t_{i+1} - \xi_i)^{q\alpha+1} + (\xi_i - t_i)^{q\alpha+1}\right]^{\frac{1}{q}} |||f|||_{[t_i,t_{i+1}],p} \\ \sum_{i=0}^{n-1} \left[\frac{1}{2} h_i + \left|\xi_i - \frac{t_{i+1}+t_i}{2}\right|\right]^{\alpha} |||f|||_{[t_i,t_{i+1}],1} . \end{cases} \end{aligned}$$

Now, observe that

$$\sum_{i=0}^{n-1}\left[(t_{i+1}-\xi_i)^{\alpha+1}+(\xi_i-t_i)^{\alpha+1}\right]|||f|||_{[t_i,t_{i+1}],\infty}$$
$$\leq |||f|||_{[a,b],\infty}\sum_{i=0}^{n-1}\left[(t_{i+1}-\xi_i)^{\alpha+1}+(\xi_i-t_i)^{\alpha+1}\right]$$
$$\leq |||f|||_{[a,b],\infty}\sum_{i=0}^{n-1}h_i^{\alpha+1}$$
$$\leq |||f|||_{[a,b],\infty}(b-a)\left[\nu(h)\right]^{\alpha}.$$

Using the discrete Hölder inequality, we may write that

$$\left[\sum_{i=0}^{n-1}(t_{i+1}-\xi_i)^{q\alpha+1}+(\xi_i-t_i)^{q\alpha+1}\right]^{\frac{1}{q}}|||f|||_{[t_i,t_{i+1}],p}$$
$$\leq \left[\sum_{i=0}^{n-1}\left(\left[(t_{i+1}-\xi_i)^{q\alpha+1}+(\xi_i-t_i)^{q\alpha+1}\right]^{\frac{1}{q}}\right)^{q}\right]^{\frac{1}{q}}\times\left[\sum_{i=0}^{n-1}|||f|||^{p}_{[t_i,t_{i+1}],p}\right]^{\frac{1}{p}}$$
$$=\left\{\sum_{i=0}^{n-1}\left[(t_{i+1}-\xi_i)^{q\alpha+1}+(\xi_i-t_i)^{q\alpha+1}\right]\right\}^{\frac{1}{q}}\left(\int_a^b\|f(t)\|^p\,ds\right)^{\frac{1}{p}}$$
$$\leq\left(\sum_{i=0}^{n-1}h_i^{q\alpha+1}\right)^{\frac{1}{q}}|||f|||^{p}_{[a,b],p}$$
$$\leq (b-a)^{\frac{1}{q}}|||f|||_{[a,b],p}\left[\nu(h)\right]^{\alpha}.$$

Finally, we have

$$\sum_{i=0}^{n-1}\left[\frac{1}{2}h_i+\left|\xi_i-\frac{t_{i+1}+t_i}{2}\right|\right]^{\alpha}|||f|||_{[t_i,t_{i+1}],1}$$
$$\leq\left[\frac{1}{2}\max_{i=\overline{0,n-1}}h_i+\max_{i=\overline{0,n-1}}\left|\xi_i-\frac{t_{i+1}+t_i}{2}\right|\right]^{\alpha}|||f|||_{[a,b],1}$$
$$\leq[\nu(h)]^{\alpha}|||f|||_{[a,b],1}$$

and the theorem is proved. $\square$

The following corollary holds.

Corollary 4.2. *If B is Lipschitzian with the constant L, then we have the representation (4.3) and the remainder $R_n^{(1)}(B,f;I_n,\xi)$ satisfies the estimates:*

$$\left\|R_n^{(1)}(B,f;I_n,\xi)\right\| \tag{4.4}$$
$$\leq L\times\begin{cases}|||f|||_{[a,b],\infty}\left[\frac{1}{4}\sum_{i=0}^{n-1}h_i^2+\sum_{i=0}^{n-1}\left(\xi_i-\frac{t_{i+1}+t_i}{2}\right)^2\right]\\[2ex] \frac{1}{(q+1)^{\frac{1}{q}}}|||f|||_{[a,b],p}\left\{\sum_{i=0}^{n-1}\left[(t_{i+1}-\xi_i)^{q+1}+(\xi_i-t_i)^{q+1}\right]\right\}^{\frac{1}{q}}, \\ \qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1\\[2ex] \left[\frac{1}{2}\nu(h)+\max_{i=\overline{0,n-1}}\left|\xi_i-\frac{t_{i+1}+t_i}{2}\right|\right]|||f|||_{[a,b],1}\end{cases}$$

$$\leq L \times \begin{cases} \frac{1}{2} \left|\left|\left|f\right|\right|\right|_{[a,b],\infty} \sum_{i=0}^{n-1} h_i^2 \\ \frac{1}{(q+1)^{\frac{1}{q}}} \left|\left|\left|f\right|\right|\right|_{[a,b],p} \left(\sum_{i=0}^{n-1} h_i^{q+1}\right)^{\frac{1}{q}} \\ \left[\frac{1}{2}\nu(h) + \max_{i=\overline{0,n-1}} \left|\xi_i - \frac{t_{i+1}+t_i}{2}\right|\right] \left|\left|\left|f\right|\right|\right|_{[a,b],1} \end{cases}$$

$$\leq L \times \begin{cases} \frac{1}{2} \left|\left|\left|f\right|\right|\right|_{[a,b],\infty} (b-a)\,\nu(h) \\ \frac{(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \left|\left|\left|f\right|\right|\right|_{[a,b],p} \nu(h) \\ \left|\left|\left|f\right|\right|\right|_{[a,b],1} \nu(h). \end{cases}$$

The second possibility we have for approximating the integral $\int_a^b B(s) f(s)\,ds$ is embodied in the following theorem based on Theorem 2.3.

Theorem 4.3. *Assume that $f : [a,b] \to X$ is Hölder continuous, i.e., the condition (2.8) holds. If $B : [a,b] \to \mathcal{L}(X)$ is strongly continuous on $[a,b]$, then we have the representation:*

$$\int_a^b B(s) f(s)\,ds = S_n^{(2)}(B, f; I_n, \xi) + R_n^{(2)}(B, f; I_n, \xi), \tag{4.5}$$

where

$$S_n^{(2)}(B, f; I_n, \xi) := \sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} B(s)\,ds\right) f(\xi_i) \tag{4.6}$$

and the remainder $R_n^{(2)}(B,f;I_n,\xi)$ *satisfies the estimate:*

$$\left\|R_n^{(2)}(B,f;I_n,\xi)\right\| \tag{4.7}$$

$$\leq K\times\begin{cases}\frac{1}{\beta+1}\left|\left|\left|B\right|\right|\right|_{[a,b],\infty}\sum_{i=0}^{n-1}\left[(t_{i+1}-\xi_i)^{\beta+1}+(\xi_i-t_i)^{\beta+1}\right]\\ \frac{1}{(q\beta+1)^{\frac{1}{q}}}\left|\left|\left|B\right|\right|\right|_{[a,b],p}\left\{\sum_{i=0}^{n-1}\left[(t_{i+1}-\xi_i)^{q\beta+1}+(\xi_i-t_i)^{q\beta+1}\right]\right\}^{\frac{1}{q}}, \\ \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1\\ \left[\frac{1}{2}\nu(h)+\max_{i=\overline{0,n-1}}\left|\xi_i-\frac{t_{i+1}+t_i}{2}\right|\right]^{\beta}\left|\left|\left|B\right|\right|\right|_{[a,b],1}\end{cases}$$

$$\leq K\times\begin{cases}\frac{1}{\beta+1}\left|\left|\left|B\right|\right|\right|_{[a,b],\infty}\sum_{i=0}^{n-1}h_i^{\beta+1}\\ \frac{1}{(q\beta+1)^{\frac{1}{q}}}\left|\left|\left|B\right|\right|\right|_{[a,b],p}\left\{\sum_{i=0}^{n-1}h_i^{q\beta+1}\right\}^{\frac{1}{q}}, \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \left[\frac{1}{2}\nu(h)+\max_{i=\overline{0,n-1}}\left|\xi_i-\frac{t_{i+1}+t_i}{2}\right|\right]^{\beta}\left|\left|\left|B\right|\right|\right|_{[a,b],1}\end{cases}$$

$$\leq K\times\begin{cases}\frac{1}{\beta+1}\left|\left|\left|B\right|\right|\right|_{[a,b],\infty}(b-a)\left[\nu(h)\right]^{\beta}\\ \frac{(b-a)^{\frac{1}{q}}}{(q\beta+1)^{\frac{1}{q}}}\left|\left|\left|B\right|\right|\right|_{[a,b],p}\left[\nu(h)\right]^{\beta}, \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1\\ \left|\left|\left|B\right|\right|\right|_{[a,b],1}\left[\nu(h)\right]^{\beta}\ .\end{cases}$$

If we consider the quadrature

$$M_{n}^{(1)}\left(B,f;I_{n}\right):=\sum_{i=0}^{n-1}B\left(\frac{t_{i}+t_{i+1}}{2}\right)\int_{t_{i}}^{t_{i+1}}f\left(s\right)ds, \tag{4.8}$$

then we have the representation

$$\int_{a}^{b}B\left(s\right)f\left(s\right)ds=M_{n}^{(1)}\left(B,f;I_{n}\right)+R_{n}^{(1)}\left(B,f;I_{n}\right), \tag{4.9}$$

and the remainder $R_{n}^{(1)}\left(B,f;I_{n}\right)$ satisfies the estimate:

$$\left\|R_{n}^{(1)}\left(B,f;I_{n}\right)\right\| \tag{4.10}$$

$$\leq H\times\begin{cases}\frac{1}{2^{\alpha}(\alpha+1)}\left|\left|\left|f\right|\right|\right|_{[a,b],\infty}\sum_{i=0}^{n-1}h_{i}^{\alpha+1}\\ \\ \frac{1}{2^{\alpha}(q\alpha+1)^{\frac{1}{q}}}\left|\left|\left|f\right|\right|\right|_{[a,b],p}\left[\sum_{i=0}^{n-1}h_{i}^{q\alpha+1}\right]^{\frac{1}{q}},\ p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \\ \frac{1}{2^{\alpha}}\left[\nu\left(h\right)\right]^{\alpha}\left|\left|\left|f\right|\right|\right|_{[a,b],1}\end{cases}$$

$$\leq H\times\begin{cases}\frac{1}{2^{\alpha}(\alpha+1)}\left(b-a\right)\left|\left|\left|f\right|\right|\right|_{[a,b],\infty}\left[\nu\left(h\right)\right]^{\alpha}\\ \\ \frac{(b-a)^{\frac{1}{q}}}{2^{\alpha}(q\alpha+1)^{\frac{1}{q}}}\left|\left|\left|f\right|\right|\right|_{[a,b],p}\left[\nu\left(h\right)\right]^{\alpha},\quad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \\ \frac{1}{2^{\alpha}}\left|\left|\left|f\right|\right|\right|_{[a,b],1}\left[\nu\left(h\right)\right]^{\alpha},\end{cases}$$

provided that B and f are as in Theorem 4.1.

Now, if we consider the quadrature

$$M_n^{(2)}(B,f;I_n) := \sum_{i=0}^{n-1}\left(\int_{t_i}^{t_{i+1}} B(s)\,ds\right) f\left(\frac{t_i+t_{i+1}}{2}\right), \tag{4.11}$$

then we also have

$$\int_a^b B(s) f(s)\,ds = M_n^{(2)}(B,f;I_n) + R_n^{(2)}(B,f;I_n), \tag{4.12}$$

and in this case the remainder satisfies the bound

$$\left\|R_n^{(2)}(B,f;I_n)\right\| \tag{4.13}$$

$$\leq K \times \begin{cases} \frac{1}{2^\beta(\beta+1)} \left|\left|\left|B\right|\right|\right|_{[a,b],\infty} \sum_{i=0}^{n-1} h_i^{\beta+1} \\ \frac{1}{2^\beta(q\beta+1)^{\frac{1}{q}}} \left|\left|\left|B\right|\right|\right|_{[a,b],p} \left(\sum_{i=0}^{n-1} h_i^{q\beta+1}\right)^{\frac{1}{q}}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2^\beta}\left[\nu(h)\right]^\beta \left|\left|\left|B\right|\right|\right|_{[a,b],1} \end{cases}$$

$$\leq K \times \begin{cases} \frac{1}{2^\beta(\beta+1)} (b-a) \left|\left|\left|B\right|\right|\right|_{[a,b],\infty} \left[\nu(h)\right]^\beta \\ \frac{(b-a)^{\frac{1}{q}}}{2^\beta(q\beta+1)^{\frac{1}{q}}} \left|\left|\left|B\right|\right|\right|_{[a,b],p} \left[\nu(h)\right]^\beta, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2^\beta} \left|\left|\left|B\right|\right|\right|_{[a,b],1} \left[\nu(h)\right]^\beta, \end{cases}$$

provided B and f satisfy the hypothesis of Theorem 4.3.

Now, if we consider the equidistant partitioning of $[a,b]$,

$$E_n : t_i := a + \left(\frac{b-a}{n}\right) \cdot i, \quad i = \overline{0,n},$$

then $M_n^{(1)}(B,f;E_n)$ becomes

$$M_n^{(1)}(B,f) := \sum_{i=0}^{n-1} B\left(a + \left(i+\frac{1}{2}\right) \cdot \frac{b-a}{n}\right) \int_{a+\frac{b-a}{n}\cdot i}^{a+\frac{b-a}{n}\cdot(i+1)} f(s)\,ds \tag{4.14}$$

and then

$$\int_a^b B(s) f(s)\,ds = M_n^{(1)}(B,f) + R_n^{(1)}(B,f), \tag{4.15}$$

where the remainder satisfies the bound

$$\left\| R_n^{(1)}(B,f) \right\| \leq H \times \begin{cases} \frac{(b-a)^{\alpha+1}}{2^\alpha(\alpha+1)n^\alpha} \left|\left|\left| f \right|\right|\right|_{[a,b],\infty} \\ \frac{(b-a)^{\alpha+\frac{1}{q}}}{2^\alpha(\alpha+1)n^\alpha} \left|\left|\left| f \right|\right|\right|_{[a,b],p}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(b-a)^\alpha}{2^\alpha n^\alpha} \left|\left|\left| f \right|\right|\right|_{[a,b],1}. \end{cases} \tag{4.16}$$

Also, we have

$$\int_a^b B(s) f(s)\,ds = M_n^{(2)}(B,f) + R_n^{(2)}(B,f), \tag{4.17}$$

where

$$M_n^{(2)}(B,f) := \sum_{i=0}^{n-1} \left(\int_{a+\frac{b-a}{n}\cdot i}^{a+\frac{b-a}{n}\cdot(i+1)} B(s)\,ds \right) f\left(a + \left(i + \frac{1}{2} \right) \cdot \frac{b-a}{n} \right),$$

and the remainder $R_n^{(2)}(B,f)$ satisfies the estimate

$$\left\| R_n^{(2)}(B,f) \right\| \leq K \times \begin{cases} \frac{(b-a)^{\beta+1}}{2^\beta(\beta+1)n^\beta} \|B\|_{[a,b],\infty} \\ \frac{(b-a)^{\beta+\frac{1}{q}}}{2^\beta(\beta+1)n^\beta} |||B|||_{[a,b],p}, \quad p > 1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(b-a)^{\beta}}{2^\beta n^\beta} |||B|||_{[a,b],1}. \end{cases} \tag{4.18}$$

5. Application for Differential Equations in Banach Spaces

We recall that a family of operators $\mathcal{U} = \{U(t,s) : t \geq s\} \subset \mathcal{L}(X)$ with $t, s \in \mathbb{R}$ or $t, s \in \mathbb{R}_+$ is called an *evolution family* if:

(i) $U(t,t) = I$ and $U(t,s)\,U(s,\tau) = U(t,\tau)$ for all $t \geq s \geq \tau$; and

(ii) for each $x \in X$, the function $(t,s) \longmapsto U(t,s)\,x$ is continuous for $t \geq s$.

Here I is the identity operator in $\mathcal{L}(X)$.

An evolution family $\{U(t,s) : t \geq s\}$ is said to be *exponentially bounded* if, in addition,

(iii) there exist the constants $M \geq 1$ and $\omega > 0$ such that

$$\|U(t,s)\| \leq Me^{\omega(t-s)}, \quad t \geq s. \tag{5.1}$$

Evolution families appear as solutions for abstract Cauchy problems of the form

$$\dot{u}(t) = A(t)\,u(t)\,,\ u(s) = x_s,\ x_s \in \mathcal{D}(A(s))\,,\ t \geq s,\ t,s \in \mathbb{R} \text{ (or } \mathbb{R}_+\text{)}, \tag{5.2}$$

where the domain $\mathcal{D}(A(s))$ of the linear operator $A(s)$ is assumed to be dense in X. An evolution family is said to solve the abstract Cauchy problem (5.2) if for each $s \in \mathbb{R}$ there exists a dense subset $Y_s \subseteq D(A(s))$ such that for each $x_s \in Y_s$ the function

$$t \longmapsto u(t) := U(t,s)\,x_s : [s, \infty) \to X,$$

is differentiable, $u(t) \in D(A(t))$ for all $t \geq s$ and

$$\frac{d}{dt} u(t) = A(t) u(t), \quad t \geq s.$$

This later definition can be found in [15]. In this definition the operators $A(t)$ can be unbounded. The Cauchy problem (5.2) is called *well-posed* if there exists an evolution family $\{U(t,s) : t \geq s\}$ which solves it.

It is known that the well-posedness of (5.2) can be destroyed by a bounded and continuous perturbation [13]. Let $f : \mathbb{R} \to X$ be a locally integrable function. Consider the inhomogeneous Cauchy problem:

$$\text{(5.3)} \quad \dot{u}(t) = A(t) u(t) + f(t), \ u(s) = x_s \in X, \ t \geq s, \ t, s \in \mathbb{R} \text{ (or } \mathbb{R}_+\text{)}.$$

A continuous function $t \longmapsto u(t) : [s, \infty) \to X$ is said to a *mild solution* of the Cauchy problem (5.3) if $u(s) = x_s$ and there exists an evolution family $\{U(t,\tau) : t \geq \tau\}$ such that

(5.4)

$$u(t) = U(t,s) x_s + \int_s^t U(t,\tau) f(\tau) \, d\tau, \ t \geq s, \ x_s \in X, \ t, s \in \mathbb{R} \text{ (or } \mathbb{R}_+\text{)}.$$

The following theorem holds.

Theorem 5.1. *Let $\mathcal{U} = \{U(\nu,\eta) : \nu \geq \eta\} \subset \mathcal{L}(X)$ be an evolution family and $f : \mathbb{R} \to X$ be a locally Bochner integrable and locally bounded function. We assume that for all $\nu \in \mathbb{R}$ (or $\mathbb{R}_+$) the function $\eta \longmapsto U(\nu,\eta) : [\nu, \infty) \to \mathcal{L}(X)$ is locally Hölder continuous (i.e. for all $a, b \geq \nu$, $a < b$, there exist $\alpha \in (0,1]$ and $H > 0$ such that*

$$\|U(\nu,t) - U(\nu,s)\| \leq H |t-s|^{\alpha}, \quad \textit{for all } t, s \in [a,b] \textit{).}$$

We use the notations in Section 4 for $a = 0$ and $b = t > 0$. The map $u(\cdot)$ from (5.4) can be represented as

$$u(t) = U(t,0)\,x_0 + \sum_{i=0}^{n-1} U(t,\xi_i) \int_{t_i}^{t_{i+1}} f(s)\,ds + R_n^{(1)}(\mathcal{U}, f, I_n, \xi) \tag{5.5}$$

where the remainder $R_n^{(1)}(\mathcal{U}, f, I_n, \xi)$ satisfies the estimate

$$\left\| R_n^{(1)}(\mathcal{U}, f, I_n, \xi) \right\| \le \frac{H}{\alpha+1} \left|\left|\left| f \right|\right|\right|_{[0,t],\infty} \sum_{i=0}^{n-1} \left[(t_{i+1} - \xi_i)^{\alpha+1} + (\xi_i - t_i)^{\alpha+1} \right].$$

Proof. It follows by representation (4.3) and the first estimate after it. □

Moreover, if n is a natural number, $i \in \{0, \dots, n\}$, $t_i := \frac{t \cdot i}{n}$ and $\xi_i := \frac{(2i+1)t}{2n}$, then

$$u(t) = U(t,0)\,x_0 + \sum_{i=0}^{n-1} U\left(t, \frac{(2i+1)\,t}{2n}\right) \int_{\frac{t \cdot i}{n}}^{\frac{t\cdot(i+1)}{n}} f(s)\,ds + R_n^{(1)} \tag{5.6}$$

and the remainder $R_n^{(1)}$ satisfies the estimate

$$\left\| R_n^{(1)} \right\| \le \frac{H}{\alpha+1} \cdot \frac{t^{\alpha+1}}{2^\alpha \cdot n^\alpha} \left|\left|\left| f \right|\right|\right|_{[0,t],\infty}. \tag{5.7}$$

The following theorem also holds.

Theorem 5.2. *Let $\mathcal{U} = \{U(\nu,\eta) : \nu \geq \eta\} \subset \mathcal{L}(X)$ be an exponentially bounded evolution family of bounded linear operators acting on the Banach space X and $f : \mathbb{R} \to X$ be a locally Hölder continuous function, i.e., for all $a, b \in \mathbb{R}$, $a < b$ there exist $\beta \in (0,1]$ and $K > 0$ such that (2.8) holds. We use the notations of Section 4 for $a = 0$ and $b = t > 0$. The map $u(\cdot)$ from (5.4) can be represented as*

$$u(t) = U(t,0)x_0 + \sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} U(t,\tau)\,d\tau \right) f(\xi_i) + R_n^{(2)}(\mathcal{U}, f, I_n, \xi) \tag{5.8}$$

where the remainder $R_n^{(2)}(\mathcal{U}, f, I_n, \xi)$ satisfies the estimate

$$\left\| R_n^{(2)}(\mathcal{U}, f, I_n, \xi) \right\| \leq \frac{KM}{\beta+1} e^{\omega t} \sum_{i=0}^{n-1} \left[(t_{i+1} - \xi_i)^{\beta+1} + (\xi_i - t_i)^{\beta+1} \right].$$

Proof. It follows from the first estimate in (4.7) for $B(s) := U(t,s)$, using the fact that

$$|||B(\cdot)|||_{[0,t],\infty} = \sup_{\tau\in[0,t]} \|U(t,\tau)\| \leq \sup_{\tau\in[0,t]} Me^{\omega(t-\tau)} \leq Me^{\omega t}.$$

$\square$

Moreover, if n is a natural number, $i \in \{0, \dots, n\}$, $t_i := \frac{t\cdot i}{n}$ and $\xi_i := \frac{(2i+1)t}{2n}$ then

$$u(t) = U(t,0)x_0 + \sum_{i=0}^{n-1} \left(\int_{\frac{t\cdot i}{n}}^{\frac{t\cdot(i+1)}{n}} U(t,\tau)\,d\tau \right) f\left(\frac{(2i+1)t}{2n} \right) + R_n^{(2)} \tag{5.9}$$

and the remainder $R_n^{(2)}$ satisfies the estimate

$$\left\|R_n^{(2)}\right\| \leq \frac{KM}{\beta+1} e^{\omega t} \cdot \frac{t^{\beta+1}}{2^{\beta} \cdot n^{\beta}}. \tag{5.10}$$

6. Some Numerical Examples

1. Let $X = \mathbb{R}^2$, $x = (\xi, \eta) \in \mathbb{R}^2$, $\|x\|_2 = \sqrt{\xi^r + \eta^2}$. We consider the linear 2-dimensional system

$$\text{(6.1)} \quad \begin{cases} \dot{u}_1(t) = \left(-1 - \sin^2 t\right) u_1(t) + \left(-1 + \sin t \cos t\right) u_2(t) + e^{-t}; \\ \dot{u}_2(t) = \left(1 + \sin t \cos t\right) u_1(t) + \left(-1 - \cos^2 t\right) u_2(t) + e^{-2t}; \\ u_1(0) = u_2(0) = 0. \end{cases}$$

If we denote

$$A(t) := \begin{pmatrix} -1 - \sin^2 t & -1 + \sin t \cos t \\ 1 + \sin t \cos t & -1 - \cos^2 t \end{pmatrix},$$

$$f(t) = \left(e^{-t}, e^{-2t}\right), \quad x = (0,0)$$

and we identify (ξ, η) with $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$, then the above system is a Cauchy problem. The evolution family associated with $A(t)$ is

$$U(t,s) = P(t) P^{-1}(s), \quad t \geq s, \; t, s \in \mathbb{R},$$

where

$$\text{(6.2)} \quad P(t) = \begin{pmatrix} e^{-t} \cos t & e^{-2t} \sin t \\ -e^{-t} \sin t & e^{-2t} \cos t \end{pmatrix}, \quad t \in \mathbb{R}.$$

The exact solution of the system (6.1) is $u = (u_1, u_2)$, where

$$u_1(t) = \left(e^{-t}\cos t\right) E_1(t) + \left(e^{-2t}\sin t\right) E_2(t)$$
$$u_2(t) = -\left(e^{-t}\sin t\right) E_1(t) + \left(e^{-2t}\cos t\right) E_2(t), \quad t \in \mathbb{R},$$

and

$$\begin{aligned} E_1(t) &= \sin t + \frac{1}{2}e^{-t}(\cos t + \sin t) - \frac{1}{2}, \\ E_2(t) &= \sin t + \frac{1}{2}e^{t}(\sin t - \cos t) + \frac{1}{2}, \end{aligned}$$

see [2, Section 4] for details. The function $t \mapsto A(t)$ is bounded on $\mathbb{R}$ and therefore there exist $M \geq 1$ and $\omega > 0$

$$\|U(t,s)\| \leq Me^{\omega|t-s|}, \quad \text{for all } t, s \in \mathbb{R}.$$

Let $\xi \geq 0$ be fixed and $t, s \geq \xi$. Then there exists a real number μ between t and s such that

$$\|U(\xi,t) - U(\xi,s)\| = |t-s| \, \|U(\xi,\mu) A(\mu)\| \leq Me^{\omega\mu} \, \||A(\cdot)\||_{\infty} \cdot |t-s|,$$

that is, the function $\eta \mapsto U(\xi,\eta)$ is locally Lipschitz continuous on $[\xi,\infty)$.

Using (6.2), it follows

$$U(t,s) = \begin{pmatrix} a_{11}(t,s) & a_{12}(t,s) \\ a_{21}(t,s) & a_{22}(t,s) \end{pmatrix},$$

where

$$\begin{aligned}
a_{11}(t,s) &= e^{(s-t)}\cos t\cos s + e^{2(s-t)}\sin t\sin s;\\
a_{12}(t,s) &= -e^{(s-t)}\cos t\sin s + \frac{1}{2}e^{2(s-t)}\sin t\cos s;\\
a_{21}(t,s) &= -e^{(s-t)}\sin t\cos s + e^{2(s-t)}\cos t\sin s;\\
a_{22}(t,s) &= e^{(s-t)}\sin t\sin s + \frac{1}{2}e^{2(s-t)}\cos t\cos s.
\end{aligned}$$

Then from (5.6) we obtain the following approximating formula for $u(\cdot)$:

$$\begin{aligned}
u_1(t) = -\sum_{i=0}^{n-1}\Bigg[a_{11}\left(t,\frac{(2i+1)t}{2n}\right)\left(e^{-\frac{t(i+1)}{n}}-e^{-\frac{ti}{n}}\right)\\
+\frac{1}{2}a_{12}\left(t,\frac{(2i+1)t}{2n}\right)\left(e^{-\frac{2t(i+1)}{n}}-e^{-\frac{2ti}{n}}\right)\Bigg]+R_{1,n}^{(1)}
\end{aligned}$$

and

$$\begin{aligned}
u_2(t) = -\sum_{i=0}^{n-1}\Bigg[a_{21}\left(t,\frac{(2i+1)t}{2n}\right)\left(e^{-\frac{t(i+1)}{n}}-e^{-\frac{ti}{n}}\right)\\
+\frac{1}{2}a_{22}\left(t,\frac{(2i+1)t}{2n}\right)\left(e^{-\frac{2t(i+1)}{n}}-e^{-\frac{2ti}{n}}\right)\Bigg]+R_{2,n}^{(1)},
\end{aligned}$$

where the remainder $R_n^{(1)} = \left(R_{1,n}^{(1)}, R_{2,n}^{(1)}\right)$ satisfies the estimate (5.7) with $\alpha = 1$, $H = Me^{\omega t}\,|||A(\cdot)|||_{\infty}$ and $|||f|||_{[0,t],\infty} \le 2$.

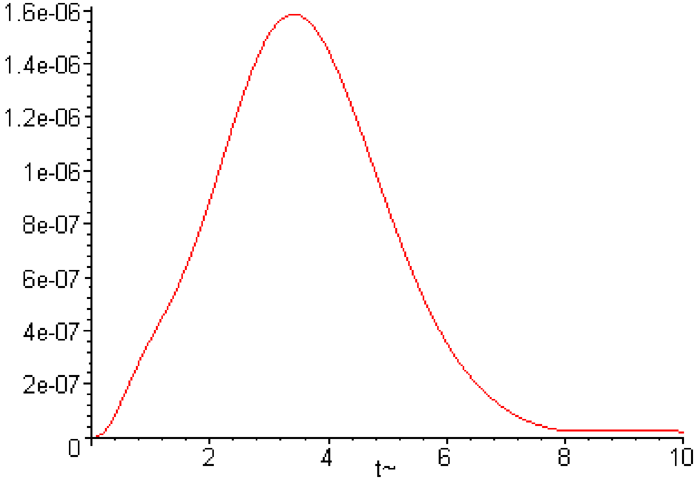

Figure 1: The behaviour of the error $\varepsilon_n(t) := \left\| \left(R_{1,n}^{(1)}, R_{2,n}^{(1)} \right) \right\|_2$ for $n = 200$.

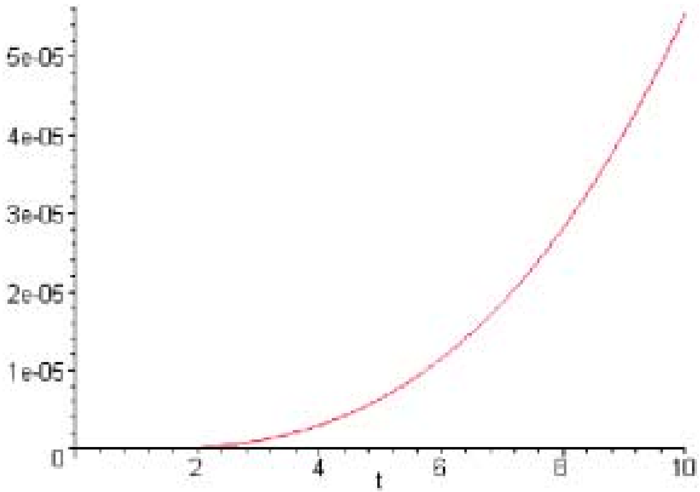

Figure 2: The behaviour of the error $\varepsilon_n(t) := |R_n|$ for $n = 400$.

Figure 1 contains the behaviour of the error $\varepsilon_n(t) := \left\|\left(R_{1,n}^{(1)}, R_{2,n}^{(1)}\right)\right\|_2$ for $n = 200$.

2. Let $X = \mathbb{R}$ and $U(t,s) := \frac{t+1}{s+1}$, $t \geq s \geq 0$. It is clear that the family $\{U(t,s) : t \geq s \geq 0\} \subset \mathcal{L}(\mathbb{R})$ is an exponentially bounded evolution family which solves the Cauchy problem

$$\dot{u}(t) = \frac{1}{t+1}u(t), \quad u(s) = x_s \in \mathbb{R}, \ t \geq s \geq 0.$$

Consider the inhomogeneous Cauchy problem

$$\left\{ \begin{array}{l} \dot{u}(t) = \frac{1}{t+1}u(t) + \cos[\ln(t+1)], \quad t \geq 0 \\ \\ u(0) = 0. \end{array} \right. \tag{6.3}$$

The solution of (6.3) is given by

$$u(t) = \int_0^t \frac{t+1}{\tau+1}\cos(\ln(\tau+1))\,d\tau = (t+1)\sin[\ln(t+1)], \ t \geq 0.$$

From (5.9) we obtain the approximating formula for $u(\cdot)$ as,

$$u(t) = (t+1)\sum_{i=0}^{n-1}\ln\left[\frac{n+ti+t}{n+ti}\right]\cos\left\{\ln\left[1+\frac{(2i+1)t}{2n}\right]\right\} + R_n,$$

where R_n satisfies the estimate (5.10) with $K = M = \omega = \beta = 1$. Indeed,

$$\frac{t+1}{s+1} \leq e^t, \quad \text{for all } t \geq s \geq 0$$

and

$$|\cos[\ln(t+1)] - \cos[\ln(s+1)]| = |t-s| \left| \frac{1}{c+1} \sin[\ln(c+1)] \right| \leq |t-s|$$

for all $t \geq s \geq 0$, where c is some real number between s and t.

The following Figure 2 contains the behaviour of the error $\varepsilon_n(t) := |R_n|$ for $n = 400$.

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