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AN OSTROWSKI TYPE INEQUALITY FOR $p-$NORMS

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Abstract

In this paper, we establish general form of an inequality of Ostrowski type for twice differentiable mappings in terms of L_p-norm, with first derivative absolutely continuous. The integral inequality of similar type already pointed out in literature is a special case of ours. The already established inequality contains a mistake and as a result incorrect consequences and applications. The corrected version of the inequality is pointed out and the inequality is also applied to special means and numerical integration.

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Key words: Ostrowski inequality, Numerical integration, Special means.

Contents

1. Introduction

We establish here the general form of an inequality of Ostrowski type, different to that of Cerone, Dragomir and Roumeliotis [1], for twice differentiable mappings in terms of L_p-norm. The integral inequality of similar type already pointed out by N.S. Barnett, P. Cerone, S.S. Dragomir, J. Roumeliotis and A. Sofo [2], contains a mistake which has already been reported by N.A. Mir and A. Rafiq in their research work [3]. The same mistake has been carried out in their other research article, namely Theorem 20 of [2] and as a result incorrect consequences and applications of this theorem. The corrected form of the theorem is as follows:

Theorem 1.1. *Let* $g : [a,b] \longrightarrow \mathbb{R}$ *be a mapping whose first derivative is absolutely continuous on* $[a,b]$. *If we assume that the second derivative* $g'' \in L_p(a,b), 1 < p < \infty$, *then we have the inequality*

$$\left| \int_a^b g(t)dt - \frac{1}{2}\left[g(x) + \frac{g(a)+g(b)}{2} \right](b-a) + \frac{1}{2}(b-a)\left(x - \frac{a+b}{2} \right) g'(x) \right| \leq \frac{1}{2}\left(\frac{b-a}{2} \right)^{2+\frac{1}{q}} \|g''\|_p \tag{1.1}$$

$$\times \begin{cases} [B(q+1,q+1) + B_{x_1}(q+1,q+1) + \Psi_{x_2}(q+1,q+1)]^{\frac{1}{q}} & \text{for } x \in \left[a, \frac{a+b}{2}\right], \\ [B(q+1,q+1) + B_{x_3}(q+1,q+1) + B_{x_4}(q+1,q+1)]^{\frac{1}{q}} & \text{for } x \in \left(\frac{a+b}{2}, b\right], \end{cases}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, $q > 1$, *and* $B(\cdot,\cdot)$ *is the Beta function of Euler given* *by*

$$B(l,s)=\int_0^1 t^{l-1}(1-t)^{s-1}dt, \quad l,s>0.$$

Further

$$B_r(l,s)=\int_0^r t^{l-1}(1-t)^{s-1}dt$$

is the incomplete Beta function,

$$\Psi_r(l,s)=\int_0^r t^{l-1}(1+t)^{s-1}dt$$

is the real positive valued integral,

$$x_1=\frac{2(x-a)}{b-a}, \quad x_2=1-x_1, \quad x_3=x_1-1, \quad x_4=2-x_1$$

and

$$\|g''\|_p := \left(\int_a^b |g''(t)|^p\, dt\right)^{\frac{1}{p}}.$$

If we assume that $g''\in L_1(a,b)$, then we have

$$\text{(1.2)}\quad \left|\int_a^b g(t)dt-\frac{1}{2}\left[g(x)+\frac{g(a)+g(b)}{2}\right](b-a)\right.$$
$$\left.+\frac{1}{2}(b-a)\left(x-\frac{a+b}{2}\right)g'(x)\right|\le\frac{\|g''\|_1}{8}(b-a)^2,$$

where

$$\|g''\|_1 := \int_a^b |g''(t)|\, dt.$$

2. Main Results

The following theorem is now proved and subsequently applied to numerical integration and special means.

Theorem 2.1. *Let* $g : [a,b] \longrightarrow \mathbb{R}$ *be a mapping whose first derivative is absolutely continuous on* $[a,b]$*. If we assume that the second derivative* $g'' \in L_p(a,b), 1 < p < \infty$, *then we have the inequality*

$$\begin{aligned}(2.1)\quad &\Bigg| \frac{1}{\alpha+\beta}\left(\frac{\alpha}{x-a}\int_a^x g(t)dt + \frac{\beta}{b-x}\int_x^b g(t)dt\right) \\ &- \frac{1}{2}g(x) - \frac{1}{2(\alpha+\beta)}\Bigg[\left(x-\frac{a+b}{2}\right)g(x)\left(\frac{\alpha}{x-a}-\frac{\beta}{b-x}\right) \\ &+ \frac{(b-a)}{2}\left(\frac{\alpha}{x-a}g(a)+\frac{\beta}{b-x}g(b)\right) - (\alpha+\beta)\left(x-\frac{a+b}{2}\right)g'(x)\Bigg]\Bigg| \\ &\le \left(\frac{b-a}{2}\right)^{2+\frac{1}{q}}\|g''\|_p \\ &\times \begin{cases} \left[\left(\frac{\beta}{\alpha+\beta}\frac{1}{b-x}\right)^q B(q+1,q+1) + \left(\frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\right)^q B_{x_1}(q+1,q+1) \right. \\ \qquad \left. + \left(\frac{\beta}{\alpha+\beta}\frac{1}{b-x}\right)^q \Psi_{x_2}(q+1,q+1)\right]^{\frac{1}{q}} \quad \text{for } x \in \left[a, \frac{a+b}{2}\right], \\ \\ \left[\left(\frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\right)^q B(q+1,q+1) + \left(\frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\right)^q B_{x_3}(q+1,q+1) \right. \\ \qquad \left. + \left(\frac{\beta}{\alpha+\beta}\frac{1}{b-x}\right)^q B_{x_4}(q+1,q+1)\right]^{\frac{1}{q}} \quad \text{for } x \in \left(\frac{a+b}{2}, b\right], \end{cases}\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, $q>1$, *and* $B(\cdot,\cdot)$ *is the Beta function of Euler given by*

$$B(l,s)=\int_0^1 t^{l-1}(1-t)^{s-1}dt,l,s>0.$$

Further,

$$B_r(l,s)=\int_0^r t^{l-1}(1-t)^{s-1}dt$$

is the incomplete Beta function,

$$\Psi_r(l,s)=\int_0^r t^{l-1}(1+t)^{s-1}dt$$

is a real positive valued integral,

$$x_1=\frac{2(x-a)}{b-a},\quad x_2=1-x_1,\quad x_3=x_1-1,\quad x_4=2-x_1$$

and

$$\|g''\|_p:=\left(\int_a^b |g''(t)|^p\,dt\right)^{\frac{1}{p}}.$$

If we assume that $g''\in L_1(a,b)$, *then we have*

$$\begin{aligned}(2.2)\quad &\left|\frac{1}{\alpha+\beta}\left(\frac{\alpha}{x-a}\int_a^x g(t)dt+\frac{\beta}{b-x}\int_x^b g(t)dt\right)-\frac{1}{2}g(x)\right.\\ &\qquad -\frac{1}{2(\alpha+\beta)}\left[\left(x-\frac{a+b}{2}\right)g(x)\left(\frac{\alpha}{x-a}-\frac{\beta}{b-x}\right)\right.\end{aligned}$$

$$+\frac{(b-a)}{2}\left(\frac{\alpha}{x-a}g(a)+\frac{\beta}{b-x}g(b)\right)-(\alpha+\beta)\left(x-\frac{a+b}{2}\right)g'(x)\Bigg]\Bigg| \leq \frac{1}{2}\left\|g''\right\|_1\left\|K(x,t)\right\|_\infty,$$

where

$$\left\|g''\right\|_1=\int_a^b\left|g''(t)\right|dt,$$

and

$$\left\|K(x,t)\right\|_\infty=\frac{1}{\alpha+\beta}\max\left(\frac{\alpha}{x-a},\frac{\beta}{b-x}\right)\frac{(b-a)^2}{4}\quad \textit{for } x\in[a,b].$$

Proof. We begin by recalling the following integral equality proved by N.A. Mir and A. Rafiq [3] which is generalization of an integral equality proved by Dragomir and Wang [4].

$$\begin{aligned}(2.3)\quad &\Bigg|\frac{1}{\alpha+\beta}\left(\frac{\alpha}{x-a}\int_a^x g(t)dt+\frac{\beta}{b-x}\int_x^b g(t)dt\right)-\frac{1}{2}g(x)\\ &-\frac{1}{2(\alpha+\beta)}\Bigg[\left(x-\frac{a+b}{2}\right)g(x)\left(\frac{\alpha}{x-a}-\frac{\beta}{b-x}\right)\\ &+\frac{(b-a)}{2}\left(\frac{\alpha}{x-a}g(a)+\frac{\beta}{b-x}g(b)\right)-(\alpha+\beta)\left(x-\frac{a+b}{2}\right)g'(x)\Bigg]\Bigg|\\ &=\frac{1}{2}\left|\int_a^b p(x,t)\left(t-\frac{a+b}{2}\right)g''(t)dt\right|\end{aligned}$$

whose left hand side is equivalent to that of (2.1). From the right hand side of (2.3) we have, by Hölder's inequality, that

$$\left|\int_a^b p(x,t)\left(t-\frac{a+b}{2}\right)g''(t)dt\right|$$
$$\leq \left(\int_a^b |g''(t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_a^b |p(x,t)|^q\left|t-\frac{a+b}{2}\right|^q dt\right)^{\frac{1}{q}}$$
$$= \|g''\|_p\left(\int_a^b |p(x,t)|^q\left|t-\frac{a+b}{2}\right|^q dt\right)^{\frac{1}{q}},$$

and from (2.3) we obtain the inequality

$$\left|\frac{1}{\alpha+\beta}\left(\frac{\alpha}{x-a}\int_a^x g(t)dt+\frac{\beta}{b-x}\int_x^b g(t)dt\right)\right.$$
$$-\frac{1}{2}g(x)-\frac{1}{2(\alpha+\beta)}\left[\left(x-\frac{a+b}{2}\right)g(x)\left(\frac{\alpha}{x-a}-\frac{\beta}{b-x}\right)\right.$$
$$\left.\left.+\frac{(b-a)}{2}\left(\frac{\alpha}{x-a}g(a)+\frac{\beta}{b-x}g(b)\right)-(\alpha+\beta)\left(x-\frac{a+b}{2}\right)g'(x)\right]\right|$$
$$\leq \frac{1}{2}\|g''\|_p\left(\int_a^b |p(x,t)|^q\left|t-\frac{a+b}{2}\right|^q dt\right)^{\frac{1}{q}}. \tag{2.4}$$

From the right hand side of (2.4) we may define

$$
\begin{aligned}
I &:= \int_a^b |p(x,t)|^q \left| t - \frac{a+b}{2} \right|^q dt \\
&= \left(\frac{\alpha}{\alpha+\beta} \cdot \frac{1}{x-a} \right)^q \int_a^x (t-a)^q \left| t - \frac{a+b}{2} \right|^q dt \\
&\quad + \left(\frac{\beta}{\alpha+\beta} \cdot \frac{1}{b-x} \right)^q \int_x^b |t-b|^q \left| t - \frac{a+b}{2} \right|^q dt
\end{aligned} \tag{2.5}
$$

such that we can identify two distinct cases.

(a) For $x \in \left[a, \frac{a+b}{2}\right]$

$$
\begin{aligned}
I_A = &\left(\frac{\alpha}{\alpha+\beta} \frac{1}{x-a} \right)^q \int_a^x (t-a)^q \left(\frac{a+b}{2} - t \right)^q dt \\
&+ \left(\frac{\beta}{\alpha+\beta} \frac{1}{b-x} \right)^q \int_x^{\frac{a+b}{2}} (b-t)^q \left(\frac{a+b}{2} - t \right)^q dt \\
&+ \left(\frac{\beta}{\alpha+\beta} \frac{1}{b-x} \right)^q \int_{\frac{a+b}{2}}^b (b-t)^q \left(t - \frac{a+b}{2} \right)^q dt.
\end{aligned}
$$

Investigating the three separate integrals, we may evaluate as follows:

$$
I_1 = \int_a^x (t-a)^q \left(\frac{a+b}{2} - t \right)^q dt,
$$

making the change of variable $t = a + \left(\frac{b-a}{2}\right) w$, we arrive at

$$I_1 = \left(\frac{b-a}{2}\right)^{2q+1} \int_0^{x_1} w^q (1-w)^q dw,$$
$$= \left(\frac{b-a}{2}\right)^{2q+1} Bx_1(q+1, q+1),$$

where $B_{x_1}(\cdot,\cdot)$ is the incomplete Beta function and $x_1 = \frac{2(x-a)}{b-a}$.

$$I_2 = \int_x^{\frac{a+b}{2}} (b-t)^q \left(\frac{a+b}{2} - t\right)^q dt,$$

making the change of variable $t = \frac{a+b}{2} - \left(\frac{b-a}{2}\right) w$, we obtain

$$I_2 = \left(\frac{b-a}{2}\right)^{2q+1} \int_0^{x_2} w^q (1+w)^q dw = \left(\frac{b-a}{2}\right)^{2q+1} \Psi_{x_2}(q+1, q+1),$$

where

$$\Psi_{x_2} := \int_0^{x_2} w^q (1+w)^q dw$$

and $x_2 = \frac{a+b-2x}{b-a} = 1 - x_1.$

$$I_3 = \int_{\frac{a+b}{2}}^{b} (b-t)^q \left(t - \frac{a+b}{2}\right)^q dt,$$

making the change of variable $t = \frac{a+b}{2} + \left(\frac{b-a}{2}\right) w$, we get

$$I_3 = \left(\frac{b-a}{2}\right)^{2q+1} \int_0^{1} w^q (1-w)^q dw = \left(\frac{b-a}{2}\right)^{2q+1} B(q+1, q+1),$$

where $B(\cdot,\cdot)$ is the Beta function.

We may now write

$$\begin{aligned} I_A &= I_1 + I_2 + I_3 \\ &= \left(\frac{b-a}{2}\right)^{2q+1}\left[\left(\frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\right)^q Bx_1(q+1,q+1) \right. \\ &\qquad + \left(\frac{\beta}{\alpha+\beta}\frac{1}{b-x}\right)^q \Psi_{x_2}(q+1,q+1) \\ &\qquad \left. + \left(\frac{\beta}{\alpha+\beta}\frac{1}{b-x}\right)^q B(q+1,q+1)\right] \end{aligned}$$

for $x \in \left[a, \frac{a+b}{2}\right]$.

(b) For $x \in \left(a, \frac{a+b}{2}\right]$

$$\begin{aligned} I_B = \left(\frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\right)^q \int_a^{\frac{a+b}{2}} (t-a)^q \left(\frac{a+b}{2}-t\right)^q dt \\ + \left(\frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\right)^q \int_{\frac{a+b}{2}}^x (t-a)^q \left(t-\frac{a+b}{2}\right)^q dt \\ + \left(\frac{\beta}{\alpha+\beta}\frac{1}{b-x}\right)^q \int_x^b (b-t)^q \left(t-\frac{a+b}{2}\right)^q dt. \end{aligned}$$

In a similar fashion to the previous case, we have

$$I_4 = \int_a^{\frac{a+b}{2}} (t-a)^q \left(\frac{a+b}{2}-t\right)^q dt.$$

Letting $t = a + \left(\frac{b-a}{2}\right) w$, we obtain

$$I_4 = \left(\frac{b-a}{2}\right)^{2q+1} \int_0^1 w^q (1-w)^q dw = \left(\frac{b-a}{2}\right)^{2q+1} B(q+1, q+1),$$

where $B(\cdot,\cdot)$ is the Beta function.

$$I_5 = \int_{\frac{a+b}{2}}^{x} (t-a)^q \left(t - \frac{a+b}{2}\right)^q dt,$$

making the change of variable $t = \frac{a+b}{2} + \left(\frac{b-a}{2}\right) w$, we arrive at

$$I_5 = \left(\frac{b-a}{2}\right)^{2q+1} \int_0^{x_3} w^q (1-w)^q dw = \left(\frac{b-a}{2}\right)^{2q+1} B_{x_3}(q+1, q+1),$$

where $B_{x_3}(\cdot,\cdot)$ is the incomplete Beta function and $x_3 = x_1 - 1$.

$$I_6 = \int_x^b (b-t)^q \left(t - \frac{a+b}{2}\right)^q dt,$$

making the change of variable $t = b - \left(\frac{b-a}{2}\right) w$, we get

$$I_6 = \left(\frac{b-a}{2}\right)^{2q+1} \int_0^{x_4} w^q (1-w)^q dw = \left(\frac{b-a}{2}\right)^{2q+1} B_{x_4}(q+1, q+1),$$

where $B_{x_4}(\cdot,\cdot)$ is the incomplete Beta function and $x_4 = 2 - x_1$.

$$\begin{aligned} I_B &= I_4 + I_5 + I_6 \\ &= \left(\frac{b-a}{2}\right)^{2q+1} \left[\left(\frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\right)^q B(q+1,q+1) \right. \\ &\qquad + \left(\frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\right)^q B_{x_3}(q+1,q+1) \\ &\qquad \left. + \left(\frac{\beta}{\alpha+\beta}\frac{1}{b-x}\right)^q B_{x_4}(q+1,q+1)\right] \end{aligned}$$

for $x \in (\frac{a+b}{2}, b]$.

Also from (2.5)

$$\begin{aligned} I &= I_A + I_B \\ &= \left(\frac{b-a}{2}\right)^{2q+1} \begin{cases} \left(\frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\right)^q B_{x_1}(q+1,q+1) \\ \quad + \left(\frac{\beta}{\alpha+\beta}\frac{1}{b-x}\right)^q \Psi_{x_2}(q+1,q+1) \\ \quad + \left(\frac{\beta}{\alpha+\beta}\frac{1}{b-x}\right)^q B(q+1,q+1) \text{ for } x \in \left[a, \frac{a+b}{2}\right], \\ \\ \left(\frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\right)^q B(q+1,q+1) \\ \quad + \left(\frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\right)^q B_{x_3}(q+1,q+1) \\ \quad + \left(\frac{\beta}{\alpha+\beta}\frac{1}{b-x}\right)^q B_{x_4}(q+1,q+1) \text{ for } x \in \left[\frac{a+b}{2}, b\right]. \end{cases} \end{aligned}$$

Using (2.4), we obtain the result (2.1). Using the inequality (2.3), we can also state that

$$\begin{aligned}\Bigg| \frac{1}{\alpha+\beta}\left(\frac{\alpha}{x-a}\int_a^x g(t)dt+\frac{\beta}{b-x}\int_x^b g(t)dt\right) \\ -\frac{1}{2}g(x)-\frac{1}{2(\alpha+\beta)}\left[\left(x-\frac{a+b}{2}\right)g(x)\left(\frac{\alpha}{x-a}-\frac{\beta}{b-x}\right)\right. \\ \left.+\frac{(b-a)}{2}\left(\frac{\alpha}{x-a}g(a)+\frac{\beta}{b-x}g(b)\right)-(\alpha+\beta)\left(x-\frac{a+b}{2}\right)g'(x)\right]\Bigg| \\ \le \frac{1}{2}\left\|g''\right\|_1\left\|K(x,t)\right\|_\infty,\end{aligned}$$

where

$$\|K(x,t)\|_\infty = p(x,t)\left(t-\frac{a+b}{2}\right).$$

As it is easy to see that

$$\|K(x,t)\|_\infty = \frac{1}{\alpha+\beta}\cdot\max\left(\frac{\alpha}{x-a},\frac{\beta}{b-x}\right)\cdot\frac{(b-a)^2}{4} \quad \text{for } x\in[a,b],$$

we deduce (2.2). □

Remark 1. *Putting $\alpha = x-a$ and $\beta = b-x$ in (2.1) and (2.2), we get the inequalities (1.1) and (1.2).*

Remark 2. *Simple manipulation of (2.1) will allow for the corrected result of (1.1) and (1.2), owing to a missing factor of $\frac{1}{2}$ in the third term of the original result (1.1) of the Barnett, Cerone, Dragomir, Roumeliotis and Sofo, this will not be done here.*

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