

# ON THE MAXIMUM MODULUS OF POLYNOMIALS. II

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*Received:* 15 February, 2007

*Accepted:* 23 August, 2007

*Communicated by:* N.K. Govil

*2000 AMS Sub. Class.:* 30D15, 41A10, 41A17.

*Key words:* Polynomials, Inequality, Zeros.

*Abstract:* Let  $f(z) := \sum_{\nu=0}^n a_{\nu} z^{\nu}$  be a polynomial of degree  $n$  having no zeros in the open unit disc, and suppose that  $\max_{|z|=1} |f(z)| = 1$ . How small can  $\max_{|z|=\rho} |f(z)|$  be for any  $\rho \in [0, 1)$ ? This problem was considered and solved by Rivlin [4]. There are reasons to consider the same problem under the additional assumption that  $f'(0) = 0$ . This was initiated by Govil [2] and followed up by the present author [3]. The exact answer is known when the degree  $n$  is even. Here, we make some observations about the case where  $n$  is odd.



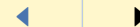
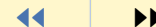
Maximum Modulus of  
Polynomials

M. A. Qazi

vol. 8, iss. 3, art. 72, 2007

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## 1. Introduction

For any entire function  $f$  let

$$M(f; \rho) := \max_{|z|=\rho} |f(z)| \quad (0 \leq \rho < \infty),$$

and denote by  $\mathcal{P}_n$  the class of all polynomials of degree at most  $n$ . If  $f \in \mathcal{P}_n$ , then, applying the maximum modulus principle to the polynomial

$$f^\sim(z) := z^n \overline{f(1/\bar{z})},$$

we see that

$$(1.1) \quad M(f; r) = r^n M(f^\sim; r^{-1}) \geq r^n M(f^\sim; 1) = r^n M(f; 1) \quad (0 \leq r < 1),$$

where equality holds if and only if  $f(z) := cz^n$ ,  $c \in \mathbb{C}$ ,  $c \neq 0$ . For the same reason

$$(1.2) \quad M(f; R) = R^n M(f^\sim; R^{-1}) \leq R^n M(f^\sim; 1) = R^n M(f; 1) \quad (R \geq 1).$$

Rivlin [6] proved that if  $f \in \mathcal{P}_n$  and  $f(z) \neq 0$  for  $|z| < 1$ , then

$$(1.3) \quad M(f; r) \geq M(f; 1) \left( \frac{1+r}{2} \right)^n \quad (0 \leq r < 1),$$

where equality holds if and only if  $f(z) := \sum_{\nu=0}^n c_\nu z^\nu$  has a zero of multiplicity  $n$  on the unit circle, that is, if and only if  $c_0 \neq 0$  and  $|c_1| = |p'(0)| = n|c_0|$ .

The preceding inequality was generalized by Govil [2] as follows.

**Theorem A.** *Let  $f \in \mathcal{P}_n$ . Furthermore let  $f(z) \neq 0$  for  $|z| < 1$ . Then,*

$$(1.4) \quad M(f; r_1) \geq M(f; r_2) \left( \frac{1+r_1}{1+r_2} \right)^n \quad (0 \leq r_1 < r_2 \leq 1).$$

Here again equality holds for polynomials of the form  $f(z) := c(1 + e^{i\gamma}z)^n$ , where  $c \in \mathbb{C}$ ,  $c \neq 0$ ,  $\gamma \in \mathbb{R}$ .



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The next result which is also due to Govil [2] gives a refinement of (1.4) under the additional assumption that  $f'(0) = 0$ .

**Theorem B.** Let  $f(z) := \sum_{\nu=0}^n c_{\nu} z^{\nu} \neq 0$  for  $|z| < 1$ , and let  $c_1 = f'(0) = 0$ . Then for  $0 \leq r_1 < r_2 \leq 1$ , we have

$$(1.5) \quad M(f; r_1) \geq M(f; r_2) \left( \frac{1+r_1}{1+r_2} \right)^n \left\{ 1 - \frac{(1-r_2)(r_2-r_1)n}{4} \left( \frac{1+r_1}{1+r_2} \right)^{n-1} \right\}^{-1}.$$

Improving upon Theorem B, we proved (see [3] or [5, Theorem 12.4.10]) the following result.

**Theorem C.** Let  $f(z) := \sum_{\nu=0}^n c_{\nu} z^{\nu} \neq 0$  for  $|z| < 1$ , and let  $\lambda := c_1/(nc_0)$ . Then

$$(1.6) \quad M(f; r_1) \geq M(f; r_2) \left( \frac{1+2|\lambda|r_1+r_1^2}{1+2|\lambda|r_2+r_2^2} \right)^{\frac{n}{2}} \quad (0 \leq r_1 < r_2 \leq 1).$$

**Note.** It may be noted that  $0 \leq |\lambda| \leq 1$ .

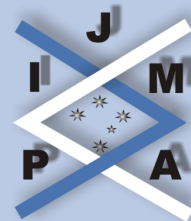
If  $n$  is even, then for any  $r_2 \in (0, 1]$ , and any  $r_1 \in [0, r_2)$ , equality holds in (1.6) for

$$f(z) := c(1+2|\lambda|e^{i\gamma}z + e^{2i\gamma}z^2)^{n/2}, \quad c \in \mathbb{C}, c \neq 0, |\lambda| \leq 1, \gamma \in \mathbb{R}.$$

By an argument different from the one used to prove Theorem C, we obtained in [4] the following refinement of (1.6).

**Theorem D.** Let  $f(z) := \sum_{\nu=0}^n c_{\nu} z^{\nu} \neq 0$  for  $|z| < 1$ , and let  $\lambda := c_1/(nc_0)$ . Then, for any  $\gamma \in \mathbb{R}$ , we have

$$(1.7) \quad |f(r_1 e^{i\gamma})| \geq |f(r_2 e^{i\gamma})| \left( \frac{1+2|\lambda|r_1+r_1^2}{1+2|\lambda|r_2+r_2^2} \right)^{\frac{n}{2}} \quad (0 \leq r_1 < r_2 \leq 1).$$



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Again, (1.7) is not sharp for odd  $n$ . The proof of (1.7) is based on the observation that for  $0 \leq r < 1$ , we have

$$r \Re \frac{f'(r)}{f(r)} = n - \Re \frac{n}{1 - r \varphi(r)} \leq n - \frac{n}{1 + r |\varphi(r)|},$$

where

$$\varphi(z) := \frac{f'(z)}{zf'(z) - nf(z)}$$

is analytic in the closed unit disc, and  $\max_{|z|=1} |\varphi(z)| \leq 1$ . Since  $\varphi(0) = -\lambda$ , a familiar generalization of Schwarz's lemma [7, p. 212] implies that  $|\varphi(r)| \leq (r + \lambda)/(\lambda r + 1)$  for  $0 \leq r < 1$ , and so if  $0 \leq r_1 < r_2 \leq 1$ , then

$$|f(r_2)| = |f(r_1)| \exp \left( \int_{r_1}^{r_2} \Re \frac{f'(r)}{f(r)} dr \right) \leq |f(r_1)| \left( \frac{1 + 2|\lambda|r_2 + r_2^2}{1 + 2|\lambda|r_1 + r_1^2} \right)^{\frac{n}{2}},$$

which readily leads us to (1.7).

It is intriguing that this reasoning works fine for any even  $n$ , and so does the one that was used to prove Theorem C, but somehow both lack the sophistication needed to settle the case where  $n$  is odd. We know that when  $n$  is even, the polynomials which minimize  $|f(r_1)|/|f(r_2)|$  have two zeros of multiplicity  $n/2$  each. However,  $n/2 \notin \mathbb{N}$  when  $n$  is odd, and so the form of the extremals must be different in the case where  $n$  is even.

Q.I. Rahman, who co-authored [4], had communicated with James Clunie about Theorem D years earlier, and had asked him for his thoughts about possible extremals when  $n$  is odd and  $c_1$  is 0. In other words, what kind of a polynomial  $f$  of odd degree  $n$  would minimize  $|f(r)|/|f(1)|$  if

$$f(z) := \prod_{\nu=1}^n (1 + \zeta_\nu z) \quad \left( |\zeta_1| \leq 1, \dots, |\zeta_n| \leq 1; \sum_{\nu=1}^n \zeta_\nu = 0 \right) ?$$



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Generally, one uses a variational argument in such a situation. In a written note, Clunie remarked that, in the case of odd degree polynomials, the condition  $\sum_{\nu=1}^n \zeta_{\nu} = 0$  is much more difficult to work with than it is in the case of even degree polynomials, and proposed to check if

$$(1.8) \quad \frac{|f(r)|}{|f(1)|} \geq \frac{1+r^3}{2} \quad \text{for } 0 \leq r \leq 1 \text{ if } n = 3 \text{ and } f'(0) = 0$$

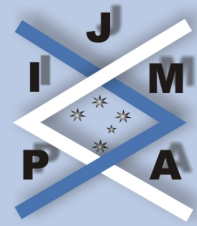
and

$$\frac{|f(r)|}{|f(1)|} \geq \frac{1+r^3}{2} \frac{1+r^2}{2} \quad \text{for } 0 \leq r \leq 1 \text{ if } n = 5 \text{ and } f'(0) = 0.$$

He added that *if above held, it would seem reasonable to conjecture that if  $n = 2m + 1$ ,  $m \in \mathbb{N}$ , and  $f'(0) = 0$ , then*

$$(1.9) \quad \frac{|f(r)|}{|f(1)|} \geq \frac{1+r^3}{2} \left( \frac{1+r^2}{2} \right)^{m-1} \quad \text{for } 0 \leq r \leq 1.$$

We shall see that (1.8) does not hold at least for  $r = 0$ . The same can be said about (1.9).



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## 2. Statement of Results

Let  $\lambda \in \mathbb{C}$ ,  $|\lambda| \leq 1$ . We shall denote by  $\mathcal{P}_{n,\lambda}$  the class of all polynomials of the form  $f(z) := \sum_{\nu=0}^n c_\nu z^\nu$ , not vanishing in the open unit disc, such that  $c_1/(nc_0) = \lambda$ . Thus, if  $f$  belongs to  $\mathcal{P}_{n,\lambda}$ , then

$$f(z) := c_0 \prod_{\nu=1}^n (1 + \zeta_\nu z) \quad \left( |\zeta_1| \leq 1, \dots, |\zeta_n| \leq 1; \sum_{\nu=1}^n \zeta_\nu = n\lambda \right).$$

Let us take any two numbers  $r_1$  and  $r_2$  in  $[0, 1]$  such that  $r_1 < r_2$ . Then by (1.7), for any real  $\gamma$ , we have

$$\frac{|f(r_2 e^{i\gamma})|}{|f(r_1 e^{i\gamma})|} \leq \left( \frac{1 + 2|\lambda|r_2 + r_2^2}{1 + 2|\lambda|r_1 + r_1^2} \right)^{\frac{n}{2}} \quad (0 \leq r_1 < r_2 \leq 1).$$

In addition, we know that the upper bound for  $|f(r_2 e^{i\gamma})|/|f(r_1 e^{i\gamma})|$  given by the preceding inequality is attained if the degree  $n$  is even, and that it is attained for a polynomial which has exactly two distinct zeros, *each of multiplicity  $n/2$  and of modulus 1*. When it comes to the case where  $n$  is odd, this bound is not sharp. What then is the best possible upper bound for  $|f(r_2 e^{i\gamma})|/|f(r_1 e^{i\gamma})|$  when  $n$  is odd; is the bound attained? If the bound is attained, can we say something about the extremals? We shall first show that

$$(2.1) \quad \Omega_{r_1, r_2, \gamma} := \sup \left\{ \frac{|f(r_2 e^{i\gamma})|}{|f(r_1 e^{i\gamma})|} : f \in \mathcal{P}_{n,\lambda} \right\}$$

is attained. For this it is enough to prove that for any  $c \neq 0$  the polynomials

$$\{f \in \mathcal{P}_{n,\lambda} : f(r_1 e^{i\gamma}) = c\}$$

form a *normal family of functions*, say  $\mathcal{F}_c$  (for the definition of a normal family see [1, pp. 210–211]). In order to prove that  $\mathcal{F}_c$  is normal, let  $f(z) := a_0 \prod_{\nu=1}^n (1 + \zeta_\nu z)$ ,



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where  $|\zeta_1| \leq 1, \dots, |\zeta_n| \leq 1$ . Then  $|f(z)| \leq |a_0| 2^n$  for  $|z| = 1$  whereas  $|c| = |f(r_1 e^{i\gamma})| \geq |a_0| (1 - r_1)^n$ . Hence

$$\max_{|z|=1} |f(z)| \leq \frac{2^n}{(1 - r_1)^n} |c|,$$

and so, by (1.2), we have

$$(2.2) \quad \max_{|z|=R>1} |f(z)| \leq \frac{2^n}{(1 - r_1)^n} |c| R^n \quad (f \in \mathcal{F}_c).$$

Since any compact subset of  $\mathbb{C}$  is contained in  $|z| < R$  for some large enough  $R$ , inequality (2.2) implies that the polynomials in  $\mathcal{F}_c$  are uniformly bounded on every compact set. By a well-known result, for which we refer the reader to [1, p. 216], the family  $\mathcal{F}_c$  is normal. Hence  $\Omega_{r_1, r_2, \gamma}$ , defined in (2.1), is attained. This implies that

$$(2.3) \quad \omega_{r_1, r_2, \gamma} := \inf \left\{ \frac{|f(r_1 e^{i\gamma})|}{|f(r_2 e^{i\gamma})|} : f \in \mathcal{P}_{n, \lambda} \right\}$$

is also attained.

Given  $r_1 < r_2$  in  $[0, 1]$  and a real number  $\gamma$ , let  $\mathcal{E} = \mathcal{E}(n; r_1, r_2; \gamma)$  denote the set of all polynomials  $f \in \mathcal{P}_{n, \lambda}$  for which the infimum  $\omega_{r_1, r_2, \gamma}$  defined in (2.3) is attained. Does a polynomial  $f \in \mathcal{P}_{n, \lambda}$  necessarily have all its zeros on the unit circle? We already know that the answer to this question is “yes” for even  $n$ , we have yet to find out if the same holds when  $n$  is odd. The following result contains the answer.

**Theorem 2.1.** For  $\lambda \in \mathbb{C}$ ,  $|\lambda| \leq 1$  let  $\mathcal{P}_{n, \lambda}$  denote the class of all polynomials of the form  $f(z) := \sum_{\nu=0}^n c_\nu z^\nu$ , not vanishing in the open unit disc, such that  $c_1/(nc_0) = \lambda$ . Given  $r_1 < r_2$  in  $[0, 1]$  and a real number  $\gamma$ , let  $\mathcal{E} = \mathcal{E}(n; r_1, r_2; \gamma)$  denote the set of all polynomials  $f \in \mathcal{P}_{n, \lambda}$  for which the infimum  $\omega_{r_1, r_2, \gamma}$  defined in (2.3) is attained. Then, any  $g \in \mathcal{E}$  must have at least  $n - 1$  zeros on the unit circle.





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The theoretical possibility that a polynomial  $g \in \mathcal{E}$  may not have all its  $n$  zeros on the unit circle can indeed occur in the case where  $n$  is odd. This is illustrated by our next result.

**Theorem 2.2.** Let  $f(z) := \sum_{\nu=0}^3 c_{\nu} z^{\nu} \neq 0$  for  $|z| < 1$ , and let  $c_1 = 0$ . Then, for any real  $\gamma$ , we have

$$(2.4) \quad \frac{|f(0)|}{|f(\rho e^{i\gamma})|} \geq \frac{4}{4 + 4\rho^2 + \rho^4} \quad (0 < \rho \leq 1).$$

For any given  $\rho \in (0, 1]$  equality holds in (2.4) for constant multiples of the polynomial

$$f_{\rho}(z) := \left(1 - \frac{\rho + i\sqrt{4-\rho^2}}{4} z e^{-i\gamma}\right) \left(1 - \frac{\rho - i\sqrt{4-\rho^2}}{4} z e^{-i\gamma}\right) \left(1 + \frac{\rho}{2} z\right).$$

*Remark 1.* Inequality (2.4) says in particular that (1.8) does not hold for  $r = 0$ . In (1.8) it is presumed that the lower bound is attained by a polynomial that has all its zeros on the unit circle. Surprisingly, it turns out to be false.

The following result is a consequence of Theorem 2.2. It is obtained by choosing  $\gamma$  such that  $|f(\rho e^{i\gamma})| = \max_{|z|=\rho} |f(z)|$ .

**Corollary 2.3.** Let  $f(z) := \sum_{\nu=0}^3 c_{\nu} z^{\nu} \neq 0$  for  $|z| < 1$ , and let  $c_1 = 0$ . Then

$$(2.5) \quad |f(0)| \geq \frac{4}{4 + 4\rho^2 + \rho^4} \max_{|z|=\rho} |f(z)| \quad (0 < \rho \leq 1).$$

The estimate is sharp for each  $\rho \in (0, 1]$ .



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### 3. An Auxiliary Result

**Lemma 3.1.** For any given  $a \in [0, 1/2]$ ,  $b := \sqrt{1 - a^2}$  and  $\beta \in \mathbb{R}$ , let

$$f_{a,\beta}(z) := (1 + (a + ib)ze^{i\beta}) (1 + (a - ib)ze^{i\beta}) (1 - 2aze^{i\beta}).$$

Then, for any  $\rho \in [0, 1]$  and any real  $\theta$ , we have

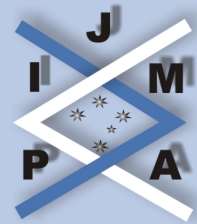
$$|f_{a,\beta}(\rho e^{i\theta})| \leq |f_{a,\beta}(-\rho e^{-i\beta})| = 1 + (1 - 4a^2)\rho^2 + 2a\rho^3.$$

*Proof.* It is enough to prove the result for  $\beta = 0$ . The case  $a = 1/2$  being trivial, let  $a \in (0, 1/2)$ . We have

$$\begin{aligned} |f_{a,0}(\rho e^{i\theta})|^2 &= \left| (1 + a\rho e^{i\theta})^2 + b^2\rho^2 e^{2i\theta} \right|^2 (1 - 4a\rho \cos \theta + 4a^2\rho^2) \\ &= |1 + 2a\rho e^{i\theta} + \rho^2 e^{2i\theta}|^2 (1 - 4a\rho \cos \theta + 4a^2\rho^2) \\ &= |e^{-i\theta} + 2a\rho + \rho^2 e^{i\theta}|^2 (1 - 4a\rho \cos \theta + 4a^2\rho^2) \\ &= \left| (1 + \rho^2) \cos \theta + 2a\rho + i(-1 + \rho^2) \sin \theta \right|^2 \\ &\quad \times (1 - 4a\rho \cos \theta + 4a^2\rho^2) \\ &= \{1 - 2\rho^2 + 4a^2\rho^2 + \rho^4 + (4a\rho + 4a\rho^3) \cos \theta + 4\rho^2 \cos^2 \theta\} \\ &\quad \times (1 - 4a\rho \cos \theta + 4a^2\rho^2) \\ &= \{1 - (1 - 4a^2)\rho^2\}^2 + 4a^2\rho^6 + 4a\rho^3(3 - \rho^2 + 4a^2\rho^2) \cos \theta \\ &\quad + 4(1 - 4a^2)\rho^2 \cos^2 \theta - 16a\rho^3 \cos^3 \theta. \end{aligned}$$

So,  $|f_{a,0}(\rho e^{i\theta})| \leq |f_{a,0}(-\rho)|$  for all real  $\theta$  if and only if

$$a\rho(3 - \rho^2 + 4a^2\rho^2)(1 + \cos \theta) - (1 - 4a^2)(1 - \cos^2 \theta) - 4a\rho(1 + \cos^3 \theta) \leq 0,$$



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that is, if and only if

$$a\rho(3 - \rho^2 + 4a^2\rho^2) - (1 - 4a^2)(1 - \cos\theta) - 4a\rho(1 - \cos\theta + \cos^2\theta) \leq 0.$$

To prove this latter inequality, we may replace  $\cos\theta$  by  $t$ , set

$$A(t) := a\rho(3 - \rho^2 + 4a^2\rho^2) - 1 + 4a^2 - 4a\rho + (1 - 4a^2 + 4a\rho)t - 4a\rho t^2$$

and show that  $A(t) \leq 0$  for  $-1 \leq t \leq 1$ . First we note that

$$A(-1) \leq A(1) = \{-1 - (1 - 4a^2)\rho^2\} a\rho < 0,$$

and so, we may restrict ourselves to the open interval  $(-1, 1)$ .

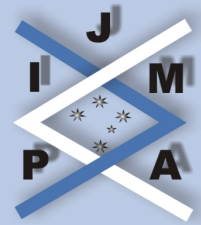
Clearly,  $A'(t)$  vanishes if and only if  $t = (1 - 4a^2 + 4a\rho)/(8a\rho)$  which is inadmissible for  $\rho \leq (1 - 4a^2)/(4a)$ . So, if  $\rho \leq (1 - 4a^2)/(4a)$ , then  $A'(t)$  is positive for all  $t \in (-1, 1)$  since  $A'(0)$  is; and  $A(t) \leq A(1) \leq 0$ .

Now, let  $\rho > (1 - 4a^2)/(4a)$ . Since  $A''(t) = -8a\rho < 0$ , the function  $A$  must have a local maximum at  $t = (1 - 4a^2 + 4a\rho)/(8a\rho)$ . However,

$$\begin{aligned} A\left(\frac{1 - 4a^2 + 4a\rho}{8a\rho}\right) &= a\rho(3 - \rho^2 + 4a^2\rho^2) - 1 + 4a^2 - 4a\rho \\ &\quad + \frac{(1 - 4a^2 + 4a\rho)^2}{8a\rho} - \frac{(1 - 4a^2 + 4a\rho)^2}{16a\rho} \\ &= -\{a\rho + (1 + a\rho^3)(1 - 4a^2)\} \\ &\quad + \frac{(1 - 4a^2)^2 + 16a^2\rho^2 + 8a\rho(1 - 4a^2)}{16a\rho} \\ &= -(1 + a\rho^3)(1 - 4a^2) + \frac{(1 - 4a^2)^2}{16a\rho} + \frac{1}{2}(1 - 4a^2) \end{aligned}$$

$$\begin{aligned}
&= \left\{ -\left(\frac{1}{2} + a\rho^3\right) + \frac{1 - 4a^2}{16a\rho} \right\} (1 - 4a^2) \\
&< -\left(\frac{1}{4} + a\rho^3\right) (1 - 4a^2) \quad \text{since } \rho > \frac{1 - 4a^2}{4a} \\
&< 0.
\end{aligned}$$

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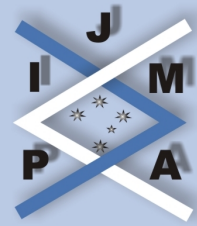
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## 4. Proofs of Theorems 2.1 and 2.2

*Proof of Theorem 2.1.* Let  $g(z) := c_0 \prod_{\nu=1}^n (1 + \zeta_{\nu} z)$ . Suppose, if possible, that  $|\zeta_j| < 1$  and  $|\zeta_k| < 1$ , where  $1 \leq j < k \leq n$ . Now, consider the function

$$\psi(w) := \frac{\{1 + (\zeta_j - w)r_1 e^{i\gamma}\} \{1 + (\zeta_k + w)r_1 e^{i\gamma}\}}{\{1 + (\zeta_j - w)r_2 e^{i\gamma}\} \{1 + (\zeta_k + w)r_2 e^{i\gamma}\}},$$

which is analytic and different from zero in the disc  $|w| < 2\delta$  for all small  $\delta > 0$ . Hence, its minimum modulus in  $|w| < \delta$  cannot be attained at  $w = 0$ . This means that if  $g_w$  is obtained from  $g$  by changing  $\zeta_j$  to  $\zeta_j - w$  and  $\zeta_k$  to  $\zeta_k + w$ , then, for all small  $\delta > 0$ , we can find  $w$  of modulus  $\delta$  such that

$$\left| \frac{g_w(r_1 e^{i\gamma})}{g_w(r_2 e^{i\gamma})} \right| < \left| \frac{g(r_1 e^{i\gamma})}{g(r_2 e^{i\gamma})} \right|.$$

This is a contradiction since  $g_w \in \mathcal{P}_{n,\lambda}$  for  $|w| < \min\{1 - |\zeta_j|, 1 - |\zeta_k|\}$ .  $\square$

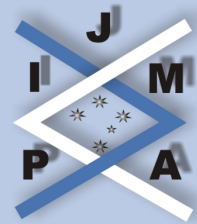
*Proof of Theorem 2.2.* We wish to minimize the quantity  $|f(0)|/|f(\rho e^{i\gamma})|$  over the class  $\mathcal{P}_{3,0}$  of all polynomials of the form

$$f(z) := c_0 \prod_{\nu=1}^3 (1 + \zeta_{\nu} z) \quad \left( |\zeta_1| \leq 1, |\zeta_2| \leq 1, |\zeta_3| \leq 1, \sum_{\nu=1}^3 \zeta_{\nu} = 0 \right).$$

Given  $\rho \in (0, 1]$  and  $\gamma \in \mathbb{R}$ , let

$$m_{\rho,\gamma} := \inf \left\{ \frac{|f(0)|}{|f(\rho e^{i\gamma})|} : f \in \mathcal{P}_{3,0} \right\}.$$

As we have already explained,  $m_{\rho,\gamma}$  is attained, i.e., there exists a cubic  $f^* \in \mathcal{P}_{3,0}$  such that



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$$\frac{|f^*(0)|}{|f^*(\rho e^{i\gamma})|} = m_{\rho,\gamma}.$$

In fact, there is at least one such cubic  $f^*$  with  $f^*(0) = 1$ . By Theorem 2.1, the cubic  $f^*$  must have *at least* two zeros on the unit circle. In other words, if  $f^*(z) := \prod_{\nu=1}^3 (1 + \zeta_\nu z)$ , then at most one of the numbers  $\zeta_1$ ,  $\zeta_2$ , and  $\zeta_3$  can lie in the open unit disc. Thus, only two possibilities need to be considered, namely (i)  $|\zeta_1| = |\zeta_2| = |\zeta_3| = 1$ , and (ii)  $|\zeta_1| = |\zeta_2| = 1$ ,  $0 < |\zeta_3| < 1$ .

*Case (i).* Since  $\zeta_1 + \zeta_2 + \zeta_3 = 0$ , the extremal  $f^*$  could only be of the form  $f^*(z) := 1 + z^3 e^{3i\beta}$ ,  $\beta \in [0, 2\pi/3]$ , and then we would clearly have

$$(4.1) \quad \frac{|f^*(0)|}{|f^*(\rho e^{i\gamma})|} \geq \frac{1}{1 + \rho^3} \quad (0 < \rho \leq 1, \gamma \in \mathbb{R}).$$

*Case (ii).* This time, because of the condition  $\zeta_1 + \zeta_2 + \zeta_3 = 0$ , the extremal  $f^*$  could only be of the form

$$f^*(z) := \{1 + (a + ib)ze^{i\beta}\} \{1 + (a - ib)ze^{i\beta}\} (1 - 2aze^{i\beta}),$$

where  $0 < a < 1/2$ ,  $b = \sqrt{1 - a^2}$  and  $\beta \in \mathbb{R}$ . Then, for any real  $\gamma$  and any  $\rho \in (0, 1]$ , we would, by Lemma 3.1, have

$$(4.2) \quad \frac{|f^*(0)|}{|f^*(\rho e^{i\gamma})|} \geq \min_{0 < a < 1/2} \frac{1}{1 + (1 - 4a^2)\rho^2 + 2a\rho^3} = \frac{4}{4 + 4\rho^2 + \rho^4}.$$

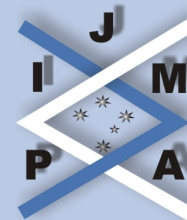
Comparing (4.1) and (4.2), we see that if  $f \in \mathcal{P}_{3,0}$ , then

$$\frac{|f(0)|}{|f(\rho e^{i\gamma})|} \geq \frac{4}{4 + 4\rho^2 + \rho^4} \quad (0 < \rho \leq 1, \gamma \in \mathbb{R}),$$

which proves (2.4). □

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Maximum Modulus of  
Polynomials

M. A. Qazi

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issn: 1443-5756