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# ON MULTIDIMENSIONAL GRÜSS TYPE INEQUALITIES

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ABSTRACT. In this paper we establish some new multidimensional integral inequalities of the Grüss type by using a fairly elementary analysis.

*Key words and phrases:* Multidimensional, Grüss type inequalities, Partial Derivatives, Continuous Functions, Bounded, Identities.

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## 1. INTRODUCTION

The following inequality is well known in the literature as the Grüss inequality [2] (see also [4, p. 296]):

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)\right| \leq (M-m)(N-n),$$

provided that $f,g:[a,b]\to\mathbb{R}$ are integrable on $[a,b]$ and

$$m\leq f(x)\leq M,\quad n\leq g(x)\leq N,$$

for all $x\in[a,b]$, where $m,M,n,N$ are given constants.

Since the appearance of the above inequality in 1935, it has evoked considerable interest and many variants, generalizations and extensions have appeared, see [1, 4] and the references cited therein. The main purpose of the present paper is to establish some new integral inequalities of the Grüss type involving functions of several independent variables. The analysis used in the proofs is elementary and our results provide new estimates on inequalities of this type.

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## 2. STATEMENT OF RESULTS

In what follows, $\mathbb{R}$ denotes the set of real numbers. Let $\Delta = [a,b]\times[c,d]$, $\Delta^0 = (a,b)\times(c,d)$, $\Omega = [a,k]\times[b,m]\times[c,n]$, $\Omega^0 = (a,k)\times(b,m)\times(c,n)$ for $a,b,c,d,k,m,n$ in $\mathbb{R}$. For functions $\alpha$ and $\beta$ defined respectively on $\Delta$ and $\Omega$, the partial derivatives $\frac{\partial^2\alpha(x,y)}{\partial y\partial x}$ and $\frac{\partial^3\beta(x,y,z)}{\partial z\partial y\partial x}$ are denoted by $D_2D_1\alpha(x,y)$ and $D_3D_2D_1\beta(x,y,z)$. Let

$$D = \{x = (x_1,\dots,x_n) : a_i < x_i < b_i \quad (i = 1,2,\dots,n)\}$$

and $\bar{D}$ be the closure of $D$. For a function $e$ defined on $\bar{D}$ the partial derivatives $\frac{\partial e(x)}{\partial x_i}$ $(i = 1,2,\dots,n)$ are denoted by $D_i e(x)$.

First, we give the following notations used to simplify the details of presentation.

$$\begin{aligned} A(D_2D_1f(x,y)) &= A[a,c;x,y;b,d;D_2D_1f(s,t)] \\ &= \int_a^x\int_c^y D_2D_1f(s,t)\,dtds - \int_a^x\int_y^d D_2D_1f(s,t)\,dtds \\ &\quad - \int_x^b\int_c^y D_2D_1f(s,t)\,dtds + \int_x^b\int_y^d D_2D_1f(s,t)\,dtds, \end{aligned}$$

$$\begin{aligned} &B(D_3D_2D_1f(r,s,t)) \\ &= B[a,b,c;r,s,t;k,m,n;D_3D_2D_1f(u,v,w)] \\ &= \int_a^r\int_b^s\int_c^t D_3D_2D_1f(u,v,w)\,dwdvdu - \int_a^r\int_b^s\int_t^n D_3D_2D_1f(u,v,w)\,dwdvdu \\ &- \int_a^r\int_s^m\int_c^t D_3D_2D_1f(u,v,w)\,dwdvdu - \int_r^k\int_b^s\int_c^t D_3D_2D_1f(u,v,w)\,dwdvdu \\ &+ \int_a^r\int_s^m\int_t^n D_3D_2D_1f(u,v,w)\,dwdvdu + \int_r^k\int_s^m\int_c^t D_3D_2D_1f(u,v,w)\,dwdvdu \\ &+ \int_r^k\int_b^s\int_t^n D_3D_2D_1f(u,v,w)\,dwdvdu - \int_r^k\int_s^m\int_t^n D_3D_2D_1f(u,v,w)\,dwdvdu, \end{aligned}$$

$$\begin{aligned} E(f(x,y)) &= E[a,c;x,y;b,d;f] \\ &= \frac{1}{2}[f(x,c)+f(x,d)+f(a,y)+f(b,y)] \\ &\quad - \frac{1}{4}[f(a,c)+f(a,d)+f(b,c)+f(b,d)], \end{aligned}$$

$$\begin{aligned} L(f(r,s,t)) &= L[a,b,c;r,s,t;k,m,n;f] \\ &= \frac{1}{8}[f(a,b,c)+f(k,m,n)] \\ &\quad - \frac{1}{4}[f(r,b,c)+f(r,m,n)+f(r,m,c)+f(r,b,n)] \\ &\quad - \frac{1}{4}[f(a,s,c)+f(k,s,n)+f(a,s,n)+f(k,s,c)] \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{4}\left[f\left(a,b,t\right)+f\left(k,m,t\right)+f\left(k,b,t\right)+f\left(a,m,t\right)\right]\\
&+\frac{1}{2}\left[f\left(a,s,t\right)+f\left(k,s,t\right)\right]+\frac{1}{2}\left[f\left(r,b,t\right)+f\left(r,m,t\right)\right]\\
&+\frac{1}{2}\left[f\left(r,s,c\right)+f\left(r,s,n\right)\right].
\end{aligned}
$$

Our main results are given in the following theorems.

**Theorem 2.1.** *Let $f,g : \Delta \to \mathbb{R}$ be continuous functions on $\Delta$; $D_2D_1f\left(x,y\right)$, $D_2D_1g\left(x,y\right)$ exist on $\Delta^0$ and are bounded, i.e.*

$$
\begin{aligned}
\left\|D_2D_1f\right\|_\infty &= \sup_{(x,y)\in\Delta^0}\left|D_2D_1f\left(x,y\right)\right|<\infty,\\
\left\|D_2D_1g\right\|_\infty &= \sup_{(x,y)\in\Delta^0}\left|D_2D_1g\left(x,y\right)\right|<\infty.
\end{aligned}
$$

*Then*

$$
\begin{aligned}
&\left|\int_a^b\int_c^d f\left(x,y\right)g\left(x,y\right)dydx\right.\\
&\qquad\left.-\frac{1}{2}\int_a^b\int_c^d\left[E\left(f\left(x,y\right)\right)g\left(x,y\right)+E\left(g\left(x,y\right)\right)f\left(x,y\right)\right]dydx\right|\\
&\leq\frac{1}{8}\left(b-a\right)\left(d-c\right)\int_a^b\int_c^d\left(\left|g\left(x,y\right)\right|\left\|D_2D_1f\right\|_\infty+\left|f\left(x,y\right)\right|\left\|D_2D_1g\right\|_\infty\right)dydx.
\end{aligned}
\tag{2.1}
$$

**Theorem 2.2.** *Let $p,q : \Omega \to \mathbb{R}$ be continuous functions on $\Omega$; $D_3D_2D_1p\left(r,s,t\right)$, $D_3D_2D_1q\left(r,s,t\right)$ exist on $\Omega^0$ and are bounded, i.e.*

$$
\begin{aligned}
\left\|D_3D_2D_1p\right\|_\infty &= \sup_{(r,s,t)\in\Omega^0}\left|D_3D_2D_1p\left(r,s,t\right)\right|<\infty,\\
\left\|D_3D_2D_1q\right\|_\infty &= \sup_{(r,s,t)\in\Omega^0}\left|D_3D_2D_1q\left(r,s,t\right)\right|<\infty.
\end{aligned}
$$

*Then*

$$
\begin{aligned}
&\left|\int_a^k\int_b^m\int_c^n p\left(r,s,t\right)q\left(r,s,t\right)dtdsdr\right.\\
&\qquad\left.-\frac{1}{2}\int_a^k\int_b^m\int_c^n\left[L\left(p\left(r,s,t\right)\right)q\left(r,s,t\right)+L\left(q\left(r,s,t\right)\right)p\left(r,s,t\right)\right]dtdsdr\right|
\end{aligned}
\tag{2.2}
$$

$$
\begin{aligned}
&\leq\frac{1}{16}\left(k-a\right)\left(m-b\right)\left(n-c\right)\\
&\qquad\times\int_a^k\int_b^m\int_c^n\left(\left|q\left(r,s,t\right)\right|\left\|D_3D_2D_1p\right\|_\infty+\left|p\left(r,s,t\right)\right|\left\|D_3D_2D_1q\right\|_\infty\right)dtdsdr.
\end{aligned}
$$

**Theorem 2.3.** *Let $f,g:\mathbb{R}^n\to\mathbb{R}$ be continuous functions on $\bar{D}$ and differentiable on $D$ whose derivatives $D_if\left(x\right)$, $D_ig\left(x\right)$ are bounded, i.e.,*

$$
\begin{aligned}
\left\|D_if\right\|_\infty &= \sup_{x\in D}\left|D_if\left(x\right)\right|<\infty,\\
\left\|D_ig\right\|_\infty &= \sup_{x\in D}\left|D_ig\left(x\right)\right|<\infty.
\end{aligned}
$$

*Then*

$$\left|\frac{1}{M}\int_D f(x)\,g(x)\,dx-\left(\frac{1}{M}\int_D f(x)\,dx\right)\left(\frac{1}{M}\int_D g(x)\,dx\right)\right| \leq \frac{1}{2M^2}\int_D\sum_{i=1}^{n}\left(|g(x)|\,\|D_i f\|_\infty+|f(x)|\,\|D_i g\|_\infty\right)E_i(x)\,dx, \tag{2.3}$$

*where*

$$\begin{aligned} M &= \prod_{i=1}^{n}(b_i-a_i)\,, \quad E_i(x)=\int_D |x_i-y_i|\,dy, \\ dx &= dx_1\cdots dx_n, \quad dy=dy_1\cdots dy_n. \end{aligned}$$

## 3. Proof of Theorem 2.1

From the hypotheses we have the following identities (see [6]):

$$f(x,y) = E(f(x,y))+\frac{1}{4}A(D_2D_1f(x,y))\,, \tag{3.1}$$

$$g(x,y) = E(g(x,y))+\frac{1}{4}A(D_2D_1g(x,y))\,, \tag{3.2}$$

for $(x,y)\in\Delta$. Multiplying (3.1) by $g(x,y)$ and (3.2) by $f(x,y)$ and adding the resulting identities, then integrating on $\Delta$ and rewriting we have

$$\begin{aligned} \int_a^b\int_c^d f(x,y)\,g(x,y)\,dydx &= \frac{1}{2}\int_a^b\int_c^d\left(E(f(x,y))\,g(x,y)+E(g(x,y))\,f(x,y)\right)dydx \\ &\quad+\frac{1}{8}\int_a^b\int_c^d\left(A(D_2D_1f(x,y))\,g(x,y)+A(D_2D_1g(x,y))\,f(x,y)\right)dydx. \end{aligned} \tag{3.3}$$

From the properties of modulus and integrals, it is easy to see that

$$|A(D_2D_1f(x,y))| \leq \int_a^b\int_c^d |D_2D_1f(s,t)|\,dtds, \tag{3.4}$$

$$|A(D_2D_1g(x,y))| \leq \int_a^b\int_c^d |D_2D_1g(s,t)|\,dtds. \tag{3.5}$$

From (3.3) – (3.5) we observe that

$$\begin{aligned} &\left|\int_a^b\int_c^d f(x,y)\,g(x,y)\,dydx - \frac{1}{2}\int_a^b\int_c^d\left(E(f(x,y))\,g(x,y)+E(g(x,y))\,f(x,y)\right)dydx\right| \\ &\leq \frac{1}{8}\int_a^b\int_c^d\left(|g(x,y)|\,|A(D_2D_1f(x,y))|+|f(x,y)|\,|A(D_2D_1g(x,y))|\right)dydx \end{aligned}$$

$$\leq \frac{1}{8}\int_a^b\int_c^d\left(|g(x,y)|\int_a^b\int_c^d|D_2D_1f(s,t)|\,dtds\right.$$
$$\left.+|f(x,y)|\int_a^b\int_c^d|D_2D_1g(s,t)|\,dtds\right)dydx$$
$$\leq \frac{1}{8}(b-a)(d-c)\int_a^b\int_c^d\left(|g(x,y)|\,\|D_2D_1f\|_\infty+|f(x,y)|\,\|D_2D_1g\|_\infty\right)dydx,$$

which is the required inequality in (2.1). The proof is complete.

## 4. PROOF OF THEOREM 2.2

From the hypotheses we have the following identities (see [5]):

$$p(r,s,t) = L(p(r,s,t))+\frac{1}{8}B(D_3D_2D_1p(r,s,t)), \tag{4.1}$$

$$q(r,s,t) = L(q(r,s,t))+\frac{1}{8}B(D_3D_2D_1q(r,s,t)), \tag{4.2}$$

for $(r,s,t)\in\Omega$. Multiplying (4.1) by $q(r,s,t)$ and (4.2) by $p(r,s,t)$ and adding the resulting identities, then integrating on $\Omega$ and rewriting we have

$$\begin{aligned}\int_a^k\int_b^m\int_c^n p(r,s,t)\,q(r,s,t)\,dtdsdr &= \frac{1}{2}\int_a^k\int_b^m\int_c^n[L(p(r,s,t))\,q(r,s,t)+L(q(r,s,t))\,p(r,s,t)]\,dtdsdr\\ &\quad+\frac{1}{16}\int_a^k\int_b^m\int_c^n(B(D_3D_2D_1p(r,s,t))\,q(r,s,t)\\ &\qquad+B(D_3D_2D_1q(r,s,t))\,p(r,s,t))\,dtdsdr.\end{aligned}\tag{4.3}$$

From the properties of modulus and integrals, we observe that

$$|B(D_3D_2D_1p(r,s,t))| \leq \int_a^k\int_b^m\int_c^n|D_3D_2D_1p(u,v,w)|\,dwdvdu, \tag{4.4}$$

$$|B(D_3D_2D_1q(r,s,t))| \leq \int_a^k\int_b^m\int_c^n|D_3D_2D_1q(u,v,w)|\,dwdvdu. \tag{4.5}$$

Now, from (4.3) – (4.5) and following the same arguments as in the proof of Theorem 2.1 with suitable changes, we get the required inequality in (2.2).

## 5. PROOF OF THEOREM 2.3

Let $x\in\bar{D}$, $y\in D$. From the $n-$dimensional version of the mean value theorem, we have (see [3]):

$$f(x)-f(y)=\sum_{i=1}^n D_if(c)(x_i-y_i), \tag{5.1}$$

where $c=(y_1+\delta(x_1-y_1),\dots,y_n+\delta(x_n-y_n))$, $0<\delta<1$.

Integrating (5.1) with respect to $y$, we obtain

$$f(x)\,\mathrm{mes}D=\int_D f(y)\,dy+\sum_{i=1}^n\int_D D_if(c)(x_i-y_i)\,dy, \tag{5.2}$$

where $\text{mes}D = \prod_{i=1}^{n} (b_i - a_i) = M$. Similarly, we obtain

$$g(x)\,\text{mes}D = \int_D g(y)\,dy + \sum_{i=1}^{n} \int_D D_i g(c)(x_i - y_i)\,dy. \tag{5.3}$$

Multiplying (5.2) by $g(x)$ and (5.3) by $f(x)$ and adding the resulting identities, then integrating on $D$ and noting that $\text{mes}D > 0$, we have

$$\begin{aligned}\int_D f(x)\,g(x)\,dx = \frac{1}{M}\left(\int_D f(x)\,dx\right)\left(\int_D g(x)\,dx\right) \\ + \frac{1}{2M}\left[\int_D g(x)\left(\sum_{i=1}^{n}\int_D D_i f(c)(x_i - y_i)\,dy\right)dx \right. \\ \left. + \int_D f(x)\left(\sum_{i=1}^{n}\int_D D_i g(c)(x_i - y_i)\,dy\right)dx\right].\end{aligned} \tag{5.4}$$

From (5.4) we observe that

$$\begin{aligned}\left|\frac{1}{M}\int_D f(x)\,g(x)\,dx - \left(\frac{1}{M}\int_D f(x)\,dx\right)\left(\frac{1}{M}\int_D g(x)\,dx\right)\right| \\ \leq \frac{1}{2M^2}\int_D \sum_{i=1}^{n}\left(|g(x)|\,\|D_i f\|_\infty + |f(x)|\,\|D_i g\|_\infty\right) E_i(x)\,dx.\end{aligned}$$

This is the required inequality in (2.3). The proof is complete.

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