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APPROXIMATION OF  $\pi(x)$  BY  $\Psi(x)$

MEHDI HASSANI

Institute for Advanced  
Studies in Basic Sciences  
P.O. Box 45195-1159  
Zanjan, Iran.

*E*Mail: [mmhassany@srttu.edu](mailto:mmhassany@srttu.edu)

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Abstract

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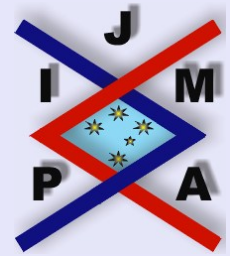


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## Abstract

In this paper we find some lower and upper bounds of the form  $\frac{n}{H_n-c}$  for the function  $\pi(n)$ , in which  $H_n = \sum_{k=1}^n \frac{1}{k}$ . Then, we consider  $H(x) = \Psi(x+1) + \gamma$  as generalization of  $H_n$ , such that  $\Psi(x) = \frac{d}{dx} \log \Gamma(x)$  and  $\gamma$  is Euler constant; this extension has been introduced for the first time by J. Sándor and it helps us to find some lower and upper bounds of the form  $\frac{x}{\Psi(x)-c}$  for the function  $\pi(x)$  and using these bounds, we show that  $\Psi(p_n) \sim \log n$ , when  $n \rightarrow \infty$  is equivalent with the Prime Number Theorem.

**2000 Mathematics Subject Classification:** 11A41, 26D15, 33B15.

**Key words:** Primes, Harmonic series, Gamma function, Digamma function.

I deem it my duty to thank P. Dusart, L. Panaitopol, M.R. Razvan and J. Sándor for sending or bringing me the references, respectively [3], [5, 6], [2] and [7].

Dedicated to Professor J. Rooin on the occasion of his 50th birthday

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# 1. Introduction

As usual, let  $\mathbb{P}$  be the set of all primes and  $\pi(x) = \#\mathbb{P} \cap [2, x]$ . If  $H_n = \sum_{k=1}^n \frac{1}{k}$ , then easily we have:

$$(1.1) \quad \gamma + \log n < H_n < 1 + \log n \quad (n > 1),$$

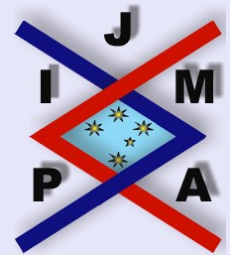
in which  $\gamma$  is the Euler constant. So,  $H_n = \log n + O(1)$  and considering the prime number theorem [2], we obtain:

$$\pi(n) = \frac{n}{H_n + O(1)} + o\left(\frac{n}{\log n}\right).$$

Thus, comparing  $\frac{n}{H_n + O(1)}$  with  $\pi(n)$  seems to be a nice problem. In 1959, L. Locker-Ernst [4] affirms that  $\frac{n}{H_n - \frac{3}{2}}$ , is very close to  $\pi(n)$  and in 1999, L. Panaitopol [6], proved that for  $n \geq 1429$  it is actually a lower bound for  $\pi(n)$ .

In this paper we improve Panaitopol's result by proving  $\frac{n}{H_n - a} < \pi(n)$  for every  $n \geq 3299$ , in which  $a \approx 1.546356705$ . Also, we find same upper bound for  $\pi(n)$ . Then we consider generalization of  $H_n$  as a real value function, which has been studied by J. Sándor [7] in 1988; for  $x > 0$  let  $\Psi(x) = \frac{d}{dx} \log \Gamma(x)$ , in which  $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ , is the well-known gamma function [1]. Since  $\Gamma(x+1) = x\Gamma(x)$  and  $\Gamma(1) = -\gamma$ , we have  $H_n = \Psi(n+1) + \gamma$ , and this relation led him to define:

$$(1.2) \quad \begin{cases} H : (0, \infty) \longrightarrow \mathbb{R}, \\ H(x) = \Psi(x+1) + \gamma, \end{cases}$$



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as a natural generalization of  $H_n$ , and more naturally, it motivated us to find some bounds for  $\pi(x)$  concerning  $\Psi(x)$ . In our proofs, we use the obvious relation:

$$(1.3) \quad \Psi(x+1) = \Psi(x) + \frac{1}{x}.$$

Also, we need some bounds of the form  $\frac{x}{\log x - 1 - \frac{c}{\log x}}$ , which we yield them by using the following known sharp bounds [3], for  $\pi(x)$ :

$$(1.4) \quad \frac{x}{\log x} \left( 1 + \frac{1}{\log x} + \frac{1.8}{\log^2 x} \right) \leq \pi(x) \quad (x \geq 32299),$$

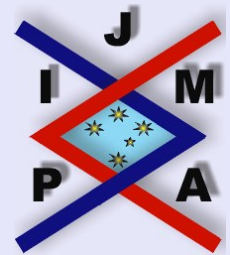
and

$$(1.5) \quad \pi(x) \leq \frac{x}{\log x} \left( 1 + \frac{1}{\log x} + \frac{2.51}{\log^2 x} \right) \quad (x \geq 355991).$$

Finally, using the above mentioned bounds concerning  $\pi(x)$ , we show that  $\Psi(p_n) \sim \log n$ , when  $n \rightarrow \infty$  is equivalent with the Prime Number Theorem. To do this, we need the following bounds [3], for  $p_n$ :

$$(1.6) \quad \log n + \log_2 n - 1 + \frac{\log_2 n - 2.25}{\log n} \leq \frac{p_n}{n} \leq \log n + \log_2 n - 1 + \frac{\log_2 n - 1.8}{\log n},$$

in which the left hand side holds for  $n \geq 2$  and the right hand side holds for  $n \geq 27076$ . Also, by  $\log_2 n$  we mean  $\log \log n$  and base of all logarithms is  $e$ .



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## 2. Bounds of the Form $\frac{x}{\log x - 1 - \frac{c}{\log x}}$

**Lower Bounds.** We are going to find suitable values of  $a$ , in which  $\frac{x}{\log x - 1 - \frac{a}{\log x}} \leq \pi(x)$ . Considering (1.4) and letting  $y = \log x$ , we should study the inequality

$$\frac{1}{y - 1 - \frac{a}{y}} \leq \frac{1}{y} \left( 1 + \frac{1}{y} + \frac{9}{5y^2} \right),$$

which is equivalent with

$$\frac{y^4}{y^2 - y - a} \leq y^2 + y + \frac{9}{5},$$

and supposing  $y^2 - y - a > 0$ , it will be equivalent with

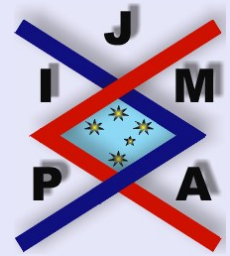
$$\left( \frac{4}{5} - a \right) y^2 - \left( a + \frac{9}{5} \right) y - \frac{9a}{5} \geq 0,$$

and this forces  $\frac{4}{5} - a > 0$ , or  $a < \frac{4}{5}$ . Let  $a = \frac{4}{5} - \epsilon$  for some  $\epsilon > 0$ . Therefore we should study

$$\frac{1}{y - 1 - \frac{\frac{4}{5} - \epsilon}{y}} \leq \frac{1}{y} \left( 1 + \frac{1}{y} + \frac{9}{5y^2} \right),$$

which is equivalent with:

$$(2.1) \quad \frac{25\epsilon y^2 + (25\epsilon - 65)y + (45\epsilon - 36)}{5y^3(5y^2 - 5y + (5\epsilon - 4))} \geq 0.$$



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The equation  $25\epsilon y^2 + (25\epsilon - 65)y + (45\epsilon - 36) = 0$  has discriminant  $25\Delta_1$  with  $\Delta_1 = 169 + 14\epsilon - 155\epsilon^2$ , which is non-negative for  $-1 \leq \epsilon \leq \frac{169}{155}$  and the greater root of it, is  $y_1 = \frac{13-5\epsilon+\sqrt{\Delta_1}}{10\epsilon}$ . Also, the equation  $5y^2 - 5y + (5\epsilon - 4) = 0$  has discriminant  $\Delta_2 = 105 - 100\epsilon$ , which is non-negative for  $\epsilon \leq \frac{21}{20}$  and the greater root of it, is  $y_2 = \frac{1}{2} + \frac{\sqrt{\Delta_2}}{10}$ . Thus, (2.1) holds for every  $0 < \epsilon \leq \min\{\frac{169}{155}, \frac{21}{20}\} = \frac{21}{20}$ , with  $y \geq \max_{0 < \epsilon \leq \frac{21}{20}} \{y_1, y_2\} = y_1$ . Therefore, we have proved the following theorem.

**Theorem 2.1.** For every  $0 < \epsilon \leq \frac{21}{20}$ , the inequality:

$$\frac{x}{\log x - 1 - \frac{\frac{4}{5} - \epsilon}{\log x}} \leq \pi(x),$$

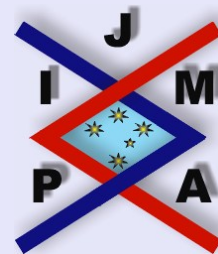
holds for all:

$$x \geq \max \left\{ 32299, e^{\frac{13-5\epsilon+\sqrt{169+14\epsilon-155\epsilon^2}}{10\epsilon}} \right\}.$$

**Corollary 2.2.** For every  $x \geq 3299$ , we have:

$$\frac{x}{\log x - 1 + \frac{1}{4\log x}} \leq \pi(x).$$

*Proof.* Taking  $\epsilon = \frac{21}{20}$  in above theorem, we yield the result for  $x \geq 32299$ . For  $3299 \leq x \leq 32298$ , we check it by a computer; to do this, consider the following program in MapleV software's worksheet:



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restart:
with(numtheory):
for x from 32298 by -1 while
evalf(pi(x)-x/(log(x)-1+1/(4*log(x))))>0
do x end do;

```

Running this program, it starts checking the result from  $x = 32298$  and verify it, until  $x = 3299$ . This completes the proof.  $\square$

**Upper Bounds.** Similar to lower bounds, we should search suitable values of  $b$ , in which  $\pi(x) \leq \frac{x}{\log x - 1 - \frac{b}{\log x}}$ . Considering (1.5) and letting  $y = \log x$ , we should study

$$\frac{1}{y} \left( 1 + \frac{1}{y} + \frac{251}{100y^2} \right) \leq \frac{1}{y - 1 - \frac{b}{y}}.$$

Assuming  $y^2 - y - b > 0$ , it will be equivalent with

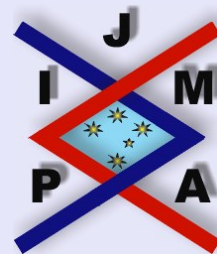
$$\left( \frac{151}{100} - b \right) y^2 - \left( b + \frac{251}{100} \right) y - \frac{251b}{100} \leq 0,$$

which forces  $b \geq \frac{151}{100}$ . Let  $b = \frac{151}{100} + \epsilon$  for some  $\epsilon \geq 0$ . Therefore we should study

$$\frac{1}{y} \left( 1 + \frac{1}{y} + \frac{251}{100y^2} \right) \leq \frac{1}{y - 1 - \frac{151 + \epsilon}{100y}},$$

which is equivalent with:

$$(2.2) \quad \frac{10000\epsilon y^2 + (10000\epsilon + 40200)y + (25100\epsilon + 37901)}{100y^3(100y^2 - 100y - (100\epsilon + 151))} \geq 0.$$



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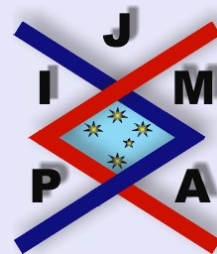


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The quadratic equation in the numerator of (2.2), has discriminant  $40000\Delta_1$  with  $\Delta_1 = 40401 - 17801\epsilon - 22600\epsilon^2$ , which is non-negative for  $-\frac{40401}{22600} \leq \epsilon \leq 1$  and the greater root of it, is  $y_1 = \frac{-201-50\epsilon+\sqrt{\Delta_1}}{100\epsilon}$ . Also, the quadratic equation in denominator of it, has discriminant  $1600\Delta_2$  with  $\Delta_2 = 44 + 25\epsilon$ , which is non-negative for  $-\frac{44}{25} \leq \epsilon$  and the greater root of it, is  $y_2 = \frac{1}{2} + \frac{\sqrt{\Delta_2}}{5}$ . Thus, (2.2) holds for every  $0 \leq \epsilon \leq \min\{1, +\infty\} = 1$ , with  $y \geq \max_{0 \leq \epsilon \leq 1} \{y_1, y_2\} = y_2$ .

Finally, we note that for  $0 \leq \epsilon \leq 1$ , the function  $y_2(\epsilon)$  is strictly increasing and so,

$$6 < e^{\frac{1}{2} + \frac{\sqrt{44}}{5}} = e^{y_2(0)} \leq e^{y_2(\epsilon)} \leq e^{y_2(1)} = e^{\frac{1}{2} + \frac{\sqrt{69}}{5}} < 9.$$

Therefore, we obtain the following theorem.

**Theorem 2.3.** For every  $0 \leq \epsilon \leq 1$ , we have:

$$\pi(x) \leq \frac{x}{\log x - 1 - \frac{151 + \epsilon}{100 \log x}} \quad (x \geq 355991).$$

**Corollary 2.4.** For every  $x \geq 7$ , we have:

$$\pi(x) \leq \frac{x}{\log x - 1 - \frac{151}{100 \log x}}.$$

*Proof.* Taking  $\epsilon = 0$  in above theorem, yields the result for  $x \geq 355991$ . For  $7 \leq x \leq 355991$  it has been checked by computer [5].  $\square$



### 3. Bounds of the Form $\frac{n}{H_n - c}$ and $\frac{x}{\Psi(x) - c}$

#### Theorem 3.1.

(i) For every  $n \geq 3299$ , we have:

$$\frac{n}{H_n - a} < \pi(n),$$

in which  $a = \gamma + 1 - \frac{1}{4 \log 3299} \approx 1.5463567$ .

(ii) For every  $n \geq 9$ , we have:

$$\pi(n) < \frac{n}{H_n - b},$$

in which  $b = 2 + \frac{151}{100 \log 7} \approx 2.77598649$ .

*Proof.* For  $n \geq 3299$ , we have

$$\gamma + \log n \geq a + \log n - 1 + \frac{1}{4 \log n},$$

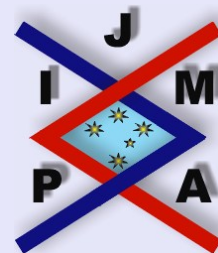
and considering this with the left hand side of (1.1), we obtain  $\frac{n}{H_n - a} < \frac{n}{\log n - 1 + \frac{1}{4 \log n}}$

and this inequality with Corollary 2.2, yields the first part of theorem.

For  $n \geq 9$ , we have

$$b + \log n - 1 - \frac{151}{100 \log n} > 1 + \log n$$

and considering this with the right hand side of (1.1), we obtain  $\frac{n}{\log n - 1 - \frac{151}{100 \log n}} < \frac{n}{H_n - b}$ . Considering this, with Corollary 2.4, completes the proof.  $\square$



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### Theorem 3.2.

(i) For every  $x \geq 3299$ , we have:

$$\frac{x}{\Psi(x) - A} < \pi(x),$$

in which  $A = 1 - \frac{\Psi(3299)}{3298} - \frac{3299}{13192 \log 3299} \approx 0.9666752780$ .

(ii) For every  $x \geq 9$ , we have:

$$\pi(x) < \frac{x}{\Psi(x) - B},$$

in which  $B = 2 + \frac{151}{100 \log 7} - \gamma \approx 2.198770832$ .

*Proof.* Let  $H_x$  be the step function defined by  $H_x = H_n$  for  $n \leq x < n + 1$ . Considering (1.2), we have  $H(x - 1) < H_x \leq H(x)$ .

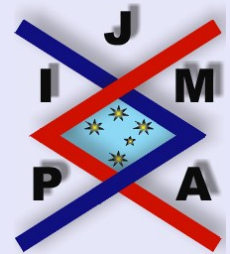
For  $x \geq 3299$ , by considering part (i) of the previous theorem, we have:

$$\pi(x) > \frac{x}{H_x - a} \geq \frac{x}{H(x) - a} = \frac{x}{\Psi(x + 1) + \gamma - a}.$$

Thus, by considering (1.3), we obtain:

$$\pi(x) > \frac{x - 1}{\Psi(x) + \frac{1}{x} + \gamma - a} \geq \frac{x - 1}{\Psi(x) + \frac{1}{3299} + \gamma - a} \geq \frac{x}{\Psi(x) - A},$$

in which  $A = \Psi(3299) - \frac{3299}{3298} \left( \Psi(3299) + \frac{1}{3299} + \gamma - a \right) = 1 - \frac{\Psi(3299)}{3298} - \frac{3299}{13192 \log 3299}$ .



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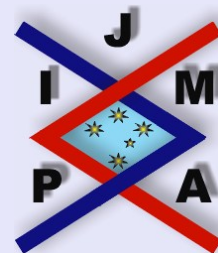
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For  $x \geq 9$ , by considering second part of previous theorem, we obtain:

$$\pi(x) < \frac{x+1}{H_{x+1}-b} < \frac{x}{H(x-1)-b} = \frac{x}{\Psi(x)+\gamma-b} = \frac{x}{\Psi(x)-B},$$

in which  $B = b - \gamma = 2 + \frac{151}{100 \log 7} - \gamma$ , and this completes the proof.  $\square$



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## 4. An Equivalent for the Prime Number Theorem

Theorem 3.2, seems to be nice; because using it, for every  $x \geq 3299$  we obtain:

$$(4.1) \quad \frac{x}{\pi(x)} + A < \Psi(x) < \frac{x}{\pi(x)} + B.$$

Moreover, considering this inequality with (1.4) and (1.5), we yield the following bounds for  $x \geq 355991$ :

$$\frac{\log x}{1 + \frac{1}{\log x} + \frac{2.51}{\log^2 x}} + A < \Psi(x) < \frac{\log x}{1 + \frac{1}{\log x} + \frac{1.8}{\log^2 x}} + B.$$

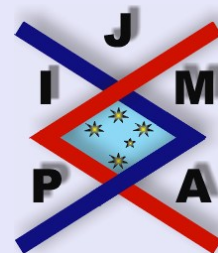
Also, by putting  $x = p_n$ ,  $n^{\text{th}}$  prime in (4.1), for  $n \geq 463$  we yield that:

$$(4.2) \quad \frac{p_n}{n} + A < \Psi(p_n) < \frac{p_n}{n} + B.$$

Considering this inequality with (1.6), for every  $n \geq 27076$  we obtain:

$$\begin{aligned} \log n + \log_2 n + A - 1 + \frac{\log_2 n - 2.25}{\log n} \\ < \Psi(p_n) < \log n + \log_2 n + B - 1 + \frac{\log_2 n - 1.8}{\log n}. \end{aligned}$$

This inequality is a very strong form of an equivalent of the Prime Number Theorem (PNT), which asserts  $\pi(x) \sim \frac{x}{\log x}$  and is equivalent with  $p_n \sim n \log n$  (see [1]). In this section, we have another equivalent as follows:




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**Theorem 4.1.**  $\Psi(p_n) \sim \log n$ , when  $n \rightarrow \infty$  is equivalent with the Prime Number Theorem.

*Proof.* First suppose PNT. Thus, we have  $p_n = n \log n + o(n \log n)$ . Also, (4.2) yields that  $\Psi(p_n) = \frac{p_n}{n} + O(1)$ . Therefore, we have:

$$\Psi(p_n) = \frac{n \log n + o(n \log n)}{n} + O(1) = \log n + o(\log n).$$

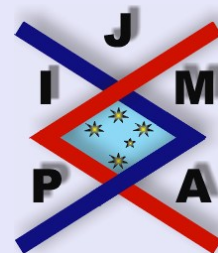
Conversely, suppose  $\Psi(p_n) = \log n + o(\log n)$ . By solving (4.2) according to  $p_n$ , we obtain:

$$n\Psi(p_n) - Bn < p_n < n\Psi(p_n) - An.$$

Therefore, we have:

$$p_n = n\Psi(p_n) + O(n) = n(\log n + o(\log n)) + O(n) = n \log n + o(n \log n),$$

which, this is PNT. □




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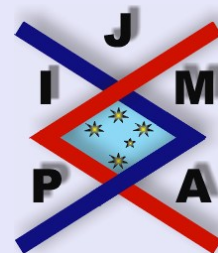
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