Journal of Inequalities in Pure and Applied Mathematics

APPROXIMATION OF $\pi(x)$ BY $\Psi(x)$

MEHDI HASSANI

Institute for Advanced
Studies in Basic Sciences
P.O. Box 45195-1159
Zanjan, Iran.

EMail: mmhassany@srttu.edu

©2000 Victoria University
ISSN (electronic): 1443-5756
065-05

volume 7, issue 1, article 7,
2006.

*Received 07 March, 2005;
accepted 25 August, 2005.*

Communicated by: J. Sándor

Abstract

In this paper we find some lower and upper bounds of the form $\frac{n}{H_n - c}$ for the function $\pi(n)$, in which $H_n = \sum_{k=1}^{n} \frac{1}{k}$. Then, we consider $H(x) = \Psi(x+1) + \gamma$ as generalization of H_n, such that $\Psi(x) = \frac{d}{dx} \log \Gamma(x)$ and γ is Euler constant; this extension has been introduced for the first time by J. Sándor and it helps us to find some lower and upper bounds of the form $\frac{x}{\Psi(x) - c}$ for the function $\pi(x)$ and using these bounds, we show that $\Psi(p_n) \sim \log n$, when $n \to \infty$ is equivalent with the Prime Number Theorem.

2000 Mathematics Subject Classification: 11A41, 26D15, 33B15.
Key words: Primes, Harmonic series, Gamma function, Digamma function.

I deem it my duty to thank P. Dusart, L. Panaitopol, M.R. Razvan and J. Sándor for sending or bringing me the references, respectively [3], [5, 6], [2] and [7].

Dedicated to Professor J. Rooin on the occasion of his 50th birthday

Contents

1. Introduction

As usual, let $\mathbb{P}$ be the set of all primes and $\pi(x) = \#\mathbb{P} \cap [2, x]$. If $H_n = \sum_{k=1}^{n} \frac{1}{k}$, then easily we have:

$$\gamma + \log n < H_n < 1 + \log n \qquad (n > 1), \tag{1.1}$$

in which γ is the Euler constant. So, $H_n = \log n + O(1)$ and considering the prime number theorem [2], we obtain:

$$\pi(n) = \frac{n}{H_n + O(1)} + o\left(\frac{n}{\log n}\right).$$

Thus, comparing $\frac{n}{H_n+O(1)}$ with $\pi(n)$ seems to be a nice problem. In 1959, L. Locker-Ernst [4] affirms that $\frac{n}{H_n - \frac{3}{2}}$, is very close to $\pi(n)$ and in 1999, L. Panaitopol [6], proved that for $n \geq 1429$ it is actually a lower bound for $\pi(n)$.

In this paper we improve Panaitopol's result by proving $\frac{n}{H_n - a} < \pi(n)$ for every $n \geq 3299$, in which $a \approx 1.546356705$. Also, we find same upper bound for $\pi(n)$. Then we consider generalization of H_n as a real value function, which has been studied by J. Sándor [7] in 1988; for $x > 0$ let $\Psi(x) = \frac{d}{dx} \log \Gamma(x)$, in which $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$, is the well-known gamma function [1]. Since $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(1) = -\gamma$, we have $H_n = \Psi(n+1) + \gamma$, and this relation led him to define:

$$\begin{cases} H : (0, \infty) \longrightarrow \mathbb{R}, \\ H(x) = \Psi(x+1) + \gamma, \end{cases} \tag{1.2}$$

as a natural generalization of H_n, and more naturally, it motivated us to find some bounds for $\pi(x)$ concerning $\Psi(x)$. In our proofs, we use the obvious relation:

$$\Psi(x+1) = \Psi(x) + \frac{1}{x}. \tag{1.3}$$

Also, we need some bounds of the form $\frac{x}{\log x - 1 - \frac{c}{\log x}}$, which we yield them by using the following known sharp bounds [3], for $\pi(x)$:

$$\frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{1.8}{\log^2 x}\right) \leq \pi(x) \qquad (x \geq 32299), \tag{1.4}$$

and

$$\pi(x) \leq \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2.51}{\log^2 x}\right) \qquad (x \geq 355991). \tag{1.5}$$

Finally, using the above mentioned bounds concerning $\pi(x)$, we show that $\Psi(p_n) \sim \log n$, when $n \to \infty$ is equivalent with the Prime Number Theorem. To do this, we need the following bounds [3], for p_n:

$$\log n + \log_2 n - 1 + \frac{\log_2 n - 2.25}{\log n} \leq \frac{p_n}{n} \leq \log n + \log_2 n - 1 + \frac{\log_2 n - 1.8}{\log n}, \tag{1.6}$$

in which the left hand side holds for $n \geq 2$ and the right hand side holds for $n \geq 27076$. Also, by $\log_2 n$ we mean $\log \log n$ and base of all logarithms is e.

2. Bounds of the Form $\frac{x}{\log x - 1 - \frac{c}{\log x}}$

Lower Bounds. We are going to find suitable values of a, in which $\frac{x}{\log x - 1 - \frac{a}{\log x}} \leq \pi(x)$. Considering (1.4) and letting $y = \log x$, we should study the inequality

$$\frac{1}{y - 1 - \frac{a}{y}} \leq \frac{1}{y}\left(1 + \frac{1}{y} + \frac{9}{5y^2}\right),$$

which is equivalent with

$$\frac{y^4}{y^2 - y - a} \leq y^2 + y + \frac{9}{5},$$

and supposing $y^2 - y - a > 0$, it will be equivalent with

$$\left(\frac{4}{5} - a\right)y^2 - \left(a + \frac{9}{5}\right)y - \frac{9a}{5} \geq 0,$$

and this forces $\frac{4}{5} - a > 0$, or $a < \frac{4}{5}$. Let $a = \frac{4}{5} - \epsilon$ for some $\epsilon > 0$. Therefore we should study

$$\frac{1}{y - 1 - \frac{\frac{4}{5} - \epsilon}{y}} \leq \frac{1}{y}\left(1 + \frac{1}{y} + \frac{9}{5y^2}\right),$$

which is equivalent with:

$$\frac{25\epsilon y^2 + (25\epsilon - 65)y + (45\epsilon - 36)}{5y^3\left(5y^2 - 5y + (5\epsilon - 4)\right)} \geq 0. \tag{2.1}$$

The equation $25\epsilon y^2 + (25\epsilon - 65)y + (45\epsilon - 36) = 0$ has discriminant $25\Delta_1$ with $\Delta_1 = 169 + 14\epsilon - 155\epsilon^2$, which is non-negative for $-1 \leq \epsilon \leq \frac{169}{155}$ and the greater root of it, is $y_1 = \frac{13-5\epsilon+\sqrt{\Delta_1}}{10\epsilon}$. Also, the equation $5y^2 - 5y + (5\epsilon - 4) = 0$ has discriminant $\Delta_2 = 105 - 100\epsilon$, which is non-negative for $\epsilon \leq \frac{21}{20}$ and the greater root of it, is $y_2 = \frac{1}{2} + \frac{\sqrt{\Delta_2}}{10}$. Thus, (2.1) holds for every $0 < \epsilon \leq \min\{\frac{169}{155}, \frac{21}{20}\} = \frac{21}{20}$, with $y \geq \max_{0<\epsilon\leq\frac{21}{20}} \{y_1, y_2\} = y_1$. Therefore, we have proved the following theorem.

Theorem 2.1. *For every $0 < \epsilon \leq \frac{21}{20}$, the inequality:*

$$\frac{x}{\log x - 1 - \frac{\frac{4}{5}-\epsilon}{\log x}} \leq \pi(x),$$

holds for all:

$$x \geq \max\left\{32299, e^{\frac{13-5\epsilon+\sqrt{169+14\epsilon-155\epsilon^2}}{10\epsilon}}\right\}.$$

Corollary 2.2. *For every $x \geq 3299$, we have:*

$$\frac{x}{\log x - 1 + \frac{1}{4\log x}} \leq \pi(x).$$

Proof. Taking $\epsilon = \frac{21}{20}$ in above theorem, we yield the result for $x \geq 32299$. For $3299 \leq x \leq 32298$, we check it by a computer; to do this, consider the following program in MapleV software's worksheet:

```
restart:
with(numtheory):
for x from 32298 by -1 while
evalf(pi(x)-x/(log(x)-1+1/(4*log(x))))>0
do x end do;
```

Running this program, it starts checking the result from $x = 32298$ and verify it, until $x = 3299$. This completes the proof. □

Upper Bounds. Similar to lower bounds, we should search suitable values of b, in which $\pi(x) \leq \frac{x}{\log x - 1 - \frac{b}{\log x}}$. Considering (1.5) and letting $y = \log x$, we should study

$$\frac{1}{y}\left(1 + \frac{1}{y} + \frac{251}{100y^2}\right) \leq \frac{1}{y - 1 - \frac{b}{y}}.$$

Assuming $y^2 - y - b > 0$, it will be equivalent with

$$\left(\frac{151}{100} - b\right) y^2 - \left(b + \frac{251}{100}\right) y - \frac{251b}{100} \leq 0,$$

which forces $b \geq \frac{151}{100}$. Let $b = \frac{151}{100} + \epsilon$ for some $\epsilon \geq 0$. Therefore we should study

$$\frac{1}{y}\left(1 + \frac{1}{y} + \frac{251}{100y^2}\right) \leq \frac{1}{y - 1 - \frac{\frac{151}{100}+\epsilon}{y}},$$

which is equivalent with:

$$\frac{10000\epsilon y^2 + (10000\epsilon + 40200)y + (25100\epsilon + 37901)}{100y^3\left(100y^2 - 100y - (100\epsilon + 151)\right)} \geq 0. \tag{2.2}$$

The quadratic equation in the numerator of (2.2), has discriminant $40000\Delta_1$ with $\Delta_1 = 40401 - 17801\epsilon - 22600\epsilon^2$, which is non-negative for $-\frac{40401}{22600} \leq \epsilon \leq 1$ and the greater root of it, is $y_1 = \frac{-201-50\epsilon+\sqrt{\Delta_1}}{100\epsilon}$. Also, the quadratic equation in denominator of it, has discriminant $1600\Delta_2$ with $\Delta_2 = 44 + 25\epsilon$, which is non-negative for $-\frac{44}{25} \leq \epsilon$ and the greater root of it, is $y_2 = \frac{1}{2} + \frac{\sqrt{\Delta_2}}{5}$. Thus, (2.2) holds for every $0 \leq \epsilon \leq \min\{1, +\infty\} = 1$, with $y \geq \max_{0\leq\epsilon\leq 1}\{y_1, y_2\} = y_2$. Finally, we note that for $0 \leq \epsilon \leq 1$, the function $y_2(\epsilon)$ is strictly increasing and so,

$$6 < e^{\frac{1}{2}+\frac{\sqrt{44}}{5}} = e^{y_2(0)} \leq e^{y_2(\epsilon)} \leq e^{y_2(1)} = e^{\frac{1}{2}+\frac{\sqrt{69}}{5}} < 9.$$

Therefore, we obtain the following theorem.

Theorem 2.3. *For every $0 \leq \epsilon \leq 1$, we have:*

$$\pi(x) \leq \frac{x}{\log x - 1 - \frac{\frac{151}{100}+\epsilon}{\log x}} \qquad (x \geq 355991).$$

Corollary 2.4. *For every $x \geq 7$, we have:*

$$\pi(x) \leq \frac{x}{\log x - 1 - \frac{151}{100\log x}}.$$

Proof. Taking $\epsilon = 0$ in above theorem, yields the result for $x \geq 355991$. For $7 \leq x \leq 35991$ it has been checked by computer [5]. □

3. Bounds of the Form $\frac{n}{H_n - c}$ and $\frac{x}{\Psi(x) - c}$

Theorem 3.1.

(i) For every $n \geq 3299$, we have:

$$\frac{n}{H_n - a} < \pi(n),$$

in which $a = \gamma + 1 - \frac{1}{4 \log 3299} \approx 1.5463567$.

(ii) For every $n \geq 9$, we have:

$$\pi(n) < \frac{n}{H_n - b},$$

in which $b = 2 + \frac{151}{100 \log 7} \approx 2.77598649$.

Proof. For $n \geq 3299$, we have

$$\gamma + \log n \geq a + \log n - 1 + \frac{1}{4 \log n},$$

and considering this with the left hand side of (1.1), we obtain $\frac{n}{H_n - a} < \frac{n}{\log n - 1 + \frac{1}{4 \log n}}$ and this inequality with Corollary 2.2, yields the first part of theorem.

For $n \geq 9$, we have

$$b + \log n - 1 - \frac{151}{100 \log n} > 1 + \log n$$

and considering this with the right hand side of (1.1), we obtain $\frac{n}{\log n - 1 - \frac{151}{100 \log n}} < \frac{n}{H_n - b}$. Considering this, with Corollary 2.4, completes the proof. □

Theorem 3.2.

(i) *For every* $x \geq 3299$, *we have:*

$$\frac{x}{\Psi(x) - A} < \pi(x),$$

in which $A = 1 - \frac{\Psi(3299)}{3298} - \frac{3299}{13192 \log 3299} \approx 0.9666752780.$

(ii) *For every* $x \geq 9$, *we have:*

$$\pi(x) < \frac{x}{\Psi(x) - B},$$

in which $B = 2 + \frac{151}{100 \log 7} - \gamma \approx 2.198770832.$

Proof. Let H_x be the step function defined by $H_x = H_n$ for $n \leq x < n + 1$. Considering (1.2), we have $H(x - 1) < H_x \leq H(x)$.

For $x \geq 3299$, by considering part (i) of the previous theorem, we have:

$$\pi(x) > \frac{x}{H_x - a} \geq \frac{x}{H(x) - a} = \frac{x}{\Psi(x + 1) + \gamma - a}.$$

Thus, by considering (1.3), we obtain:

$$\pi(x) > \frac{x - 1}{\Psi(x) + \frac{1}{x} + \gamma - a} \geq \frac{x - 1}{\Psi(x) + \frac{1}{3299} + \gamma - a} \geq \frac{x}{\Psi(x) - A},$$

in which $A = \Psi(3299) - \frac{3299}{3298}\left(\Psi(3299) + \frac{1}{3299} + \gamma - a\right) = 1 - \frac{\Psi(3299)}{3298} - \frac{3299}{13192 \log 3299}$.

For $x \geq 9$, by considering second part of previous theorem, we obtain:

$$\pi(x) < \frac{x+1}{H_{x+1} - b} < \frac{x}{H(x-1) - b} = \frac{x}{\Psi(x) + \gamma - b} = \frac{x}{\Psi(x) - B},$$

in which $B = b - \gamma = 2 + \frac{151}{100 \log 7} - \gamma$, and this completes the proof. □

4. An Equivalent for the Prime Number Theorem

Theorem 3.2, seems to be nice; because using it, for every $x \geq 3299$ we obtain:

$$\frac{x}{\pi(x)} + A < \Psi(x) < \frac{x}{\pi(x)} + B. \tag{4.1}$$

Moreover, considering this inequality with (1.4) and (1.5), we yield the following bounds for $x \geq 355991$:

$$\frac{\log x}{1 + \frac{1}{\log x} + \frac{2.51}{\log^2 x}} + A < \Psi(x) < \frac{\log x}{1 + \frac{1}{\log x} + \frac{1.8}{\log^2 x}} + B.$$

Also, by putting $x = p_n$, n^{th} prime in (4.1), for $n \geq 463$ we yield that:

$$\frac{p_n}{n} + A < \Psi(p_n) < \frac{p_n}{n} + B. \tag{4.2}$$

Considering this inequality with (1.6), for every $n \geq 27076$ we obtain:

$$\log n + \log_2 n + A - 1 + \frac{\log_2 n - 2.25}{\log n}$$
$$< \Psi(p_n) < \log n + \log_2 n + B - 1 + \frac{\log_2 n - 1.8}{\log n}.$$

This inequality is a very strong form of an equivalent of the Prime Number Theorem (PNT), which asserts $\pi(x) \sim \frac{x}{\log x}$ and is equivalent with $p_n \sim n \log n$ (see [1]). In this section, we have another equivalent as follows:

Theorem 4.1. *$\Psi(p_n) \sim \log n$, when $n \to \infty$ is equivalent with the Prime Number Theorem.*

Proof. First suppose PNT. Thus, we have $p_n = n \log n + o(n \log n)$. Also, (4.2) yields that $\Psi(p_n) = \frac{p_n}{n} + O(1)$. Therefore, we have:

$$\Psi(p_n) = \frac{n \log n + o(n \log n)}{n} + O(1) = \log n + o(\log n).$$

Conversely, suppose $\Psi(p_n) = \log n + o(\log n)$. By solving (4.2) according to p_n, we obtain:

$$n\Psi(p_n) - Bn < p_n < n\Psi(p_n) - An.$$

Therefore, we have:

$$p_n = n\Psi(p_n) + O(n) = n\big(\log n + o(\log n)\big) + O(n) = n \log n + o(n \log n),$$

which, this is PNT. □

References

[1] M. ABRAMOWITZ AND I.A. STEGUN, *Handbook of Mathematical Functions: with Formulas, Graphs, and Mathematical Tables*, Dover Publications, 1972.

[2] H. DAVENPORT, *Multiplicative Number Theory* (Second Edition), Springer-Verlag, 1980.

[3] P. DUSART, Inégalités explicites pour $\psi(X)$, $\theta(X)$, $\pi(X)$ et les nombres premiers, *C. R. Math. Acad. Sci. Soc. R. Can.*, **21**(2) (1999), 53–59.

[4] L. LOCKER-ERNST, Bemerkungen über die verteilung der primzahlen, *Elemente der Mathematik* ***XIV***, **1** (1959), 1–5, Basel.

[5] L. PANAITOPOL, A special case of the Hardy-Littlewood conjecture, *Math. Reports*, **4(54)**(3) (2002), 265–258.

[6] L. PANAITOPOL, Several approximation of $\pi(x)$, *Math. Inequal. & Applics.*, **2**(3) (1999), 317–324.

[7] J. SÁNDOR, Remark on a function which generalizes the harmonic series, *C. R. Acad. Bulgare Sci.*, **41**(5) (1988), 19–21.