

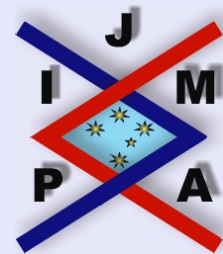
COEFFICIENT INEQUALITY FOR A FUNCTION WHOSE DERIVATIVE HAS A POSITIVE REAL PART

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Abstract

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Abstract

Let \mathcal{R} denote the subclass of normalised analytic univalent functions f defined by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and satisfy

$$\operatorname{Re}\{f'(z)\} > 0$$

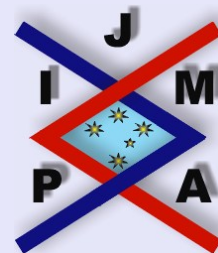
where $z \in \mathcal{D} = \{z : |z| < 1\}$. The object of the present paper is to introduce the functional $|a_2 a_4 - a_3^2|$. For $f \in \mathcal{R}$, we give sharp upper bound for $|a_2 a_4 - a_3^2|$.

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Key words: Fekete-Szegő functional, Hankel determinant, Convex and starlike functions, Positive real functions.

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1. Introduction

Let \mathcal{A} denote the class of normalised analytic functions f of the form

$$(1.1) \quad f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

where $z \in \mathcal{D} = \{z : |z| < 1\}$. In [9], Noonan and Thomas stated that the q th Hankel determinant of f is defined for $q \geq 1$ by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q+1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q+2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

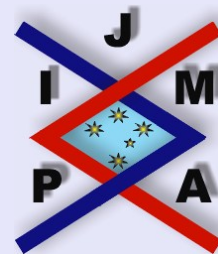
Now, let \mathcal{S} denote the subclass of \mathcal{A} consisting of functions f of the form

$$(1.2) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are univalent in \mathcal{D} .

A classical theorem of Fekete and Szegő [1] considered the Hankel determinant of $f \in \mathcal{S}$ for $q = 2$ and $n = 1$,

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}.$$



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They made an early study for the estimates of $|a_3 - \mu a_2^2|$ when $a_1 = 1$ and μ real. The well-known result due to them states that if $f \in \mathcal{S}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 4\mu - 3, & \text{if } \mu \geq 1, \\ 1 + 2 \exp\left(\frac{-2\mu}{1-\mu}\right), & \text{if } 0 \leq \mu \leq 1, \\ 3 - 4\mu, & \text{if } \mu \leq 0. \end{cases}$$

Hummel [3, 4] proved the conjecture of V. Singh that $|a_3 - a_2^2| \leq \frac{1}{3}$ for the class \mathcal{C} of convex functions. Keogh and Merkes [5] obtained sharp estimates for $|a_3 - \mu a_2^2|$ when f is close-to-convex, starlike and convex in \mathcal{D} .

Here, we consider the Hankel determinant of $f \in \mathcal{S}$ for $q = 2$ and $n = 2$,

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}.$$

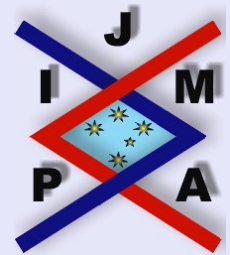
Now, we are working on the functional $|a_2 a_4 - a_3^2|$. In this earlier work, we find a sharp upper bound for the functional $|a_2 a_4 - a_3^2|$ for $f \in \mathcal{R}$. The subclass \mathcal{R} is defined as the following.

Definition 1.1. Let f be given by (1.2). Then $f \in \mathcal{R}$ if it satisfies the inequality

$$(1.3) \quad \operatorname{Re}\{f'(z)\} > 0, \quad (z \in \mathcal{D}).$$

The subclass \mathcal{R} was studied systematically by MacGregor [8] who indeed referred to numerous earlier investigations involving functions whose derivative has a positive real part.

We first state some preliminary lemmas which shall be used in our proof.



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2. Preliminary Results

Let \mathcal{P} be the family of all functions p analytic in \mathcal{D} for which $\operatorname{Re}\{p(z)\} > 0$ and

$$(2.1) \quad p(z) = 1 + c_1z + c_2z^2 + \dots$$

for $z \in \mathcal{D}$.

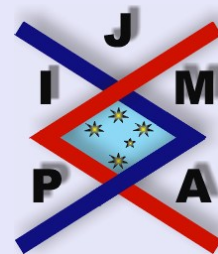
Lemma 2.1 ([10]). *If $p \in \mathcal{P}$ then $|c_k| \leq 2$ for each k .*

Lemma 2.2 ([2]). *The power series for $p(z)$ given in (2.1) converges in \mathcal{D} to a function in \mathcal{P} if and only if the Toeplitz determinants*

$$(2.2) \quad D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, \quad n = 1, 2, 3, \dots$$

and $c_{-k} = \bar{c}_k$, are all nonnegative. They are strictly positive except for $p(z) = \sum_{k=1}^m \rho_k p_0(e^{it_k} z)$, $\rho_k > 0$, t_k real and $t_k \neq t_j$ for $k \neq j$; in this case $D_n > 0$ for $n < m - 1$ and $D_n = 0$ for $n \geq m$.

This necessary and sufficient condition is due to Carathéodory and Toeplitz and can be found in [2].



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3. Main Result

Theorem 3.1. *Let $f \in \mathbb{R}$. Then*

$$|a_2a_4 - a_3^2| \leq \frac{4}{9}.$$

The result obtained is sharp.

Proof. We refer to the method by Libera and Zlotkiewicz [6, 7]. Since $f \in \mathcal{R}$, it follows from (1.3) that

$$(3.1) \quad f'(z) = p(z)$$

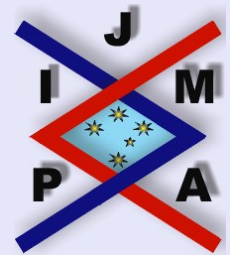
for some $z \in \mathcal{D}$. Equating coefficients in (3.1) yields

$$(3.2) \quad \begin{cases} 2a_2 = c_1 \\ 3a_3 = c_2 \\ 4a_4 = c_3 \end{cases} .$$

From (3.2), it can be easily established that

$$|a_2a_4 - a_3^2| = \left| \frac{c_1c_3}{8} - \frac{c_2^2}{9} \right|.$$

We make use of Lemma 2.2 to obtain the proper bound on $\left| \frac{c_1c_3}{8} - \frac{c_2^2}{9} \right|$. We may assume without restriction that $c_1 > 0$. We begin by rewriting (2.2) for the cases



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$n = 2$ and $n = 3$.

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ c_1 & 2 & c_1 \\ \bar{c}_2 & c_1 & 2 \end{vmatrix} = 8 + 2 \operatorname{Re}\{c_1^2 c_2\} - 2|c_2|^2 - 4c_1^2 \geq 0,$$

which is equivalent to

$$(3.3) \quad 2c_2 = c_1^2 + x(4 - c_1^2)$$

for some x , $|x| \leq 1$. Then $D_3 \geq 0$ is equivalent to

$$|(4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2| \leq 2(4 - c_1^2)^2 - 2|2c_2 - c_1^2|^2;$$

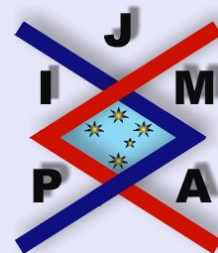
and this, with (3.3), provides the relation

$$(3.4) \quad 4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z,$$

for some value of z , $|z| \leq 1$.

Suppose, now, that $c_1 = c$ and $c \in [0, 2]$. Using (3.3) along with (3.4) we get

$$\begin{aligned} & \left| \frac{c_1c_3}{8} - \frac{c_2^2}{9} \right| \\ &= \left| \frac{c^4}{288} + \frac{c^2(4 - c^2)x}{144} - \frac{(4 - c^2)(32 + c^2)x^2}{288} + \frac{c(4 - c^2)(1 - |x|^2)z}{16} \right| \end{aligned}$$



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and an application of the triangle inequality shows that

$$(3.5) \quad \left| \frac{c_1 c_3}{8} - \frac{c_2^2}{9} \right| \leq \frac{c^4}{288} + \frac{c(4-c^2)}{16} + \frac{c^2(4-c^2)\rho}{144} + \frac{(c-2)(c-16)(4-c^2)\rho^2}{288} = F(\rho)$$

with $\rho = |x| \leq 1$. We assume that the upper bound for (3.5) attains at the interior point of $\rho \in [0, 1]$ and $c \in [0, 2]$, then

$$F'(\rho) = \frac{c^2(4-c^2)}{144} + \frac{(c-2)(c-16)(4-c^2)\rho}{144}.$$

We note that $F'(\rho) > 0$ and consequently F is increasing and $Max_{\rho} F(\rho) = F(1)$, which contradicts our assumption of having the maximum value at the interior point of $\rho \in [0, 1]$. Now let

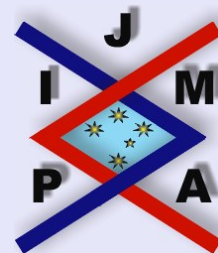
$$G(c) = F(1) = \frac{c^4}{288} + \frac{c(4-c^2)}{16} + \frac{c^2(4-c^2)}{144} + \frac{(c-2)(c-16)(4-c^2)}{288},$$

then

$$G'(c) = \frac{-c(5+c^2)}{36} = 0$$

implies $c = 0$ which is a contradiction. Observe that

$$G''(c) = \frac{-5-3c^2}{36} < 0.$$



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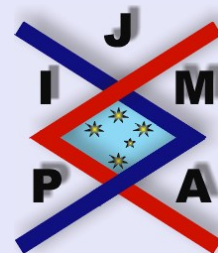
Thus any maximum points of G must be on the boundary of $c \in [0, 2]$. However, $G(c) \geq G(2)$ and thus G has maximum value at $c = 0$. The upper bound for (3.5) corresponds to $\rho = 1$ and $c = 0$, in which case

$$\left| \frac{c_1 c_3}{8} - \frac{c_2^2}{9} \right| \leq \frac{4}{9}.$$

Equality is attained for functions in \mathcal{R} given by

$$f'(z) = \frac{1 + z^2}{1 - z^2}.$$

This concludes the proof of our theorem. □



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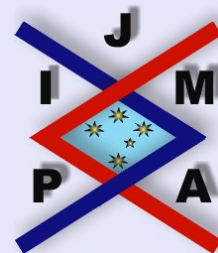
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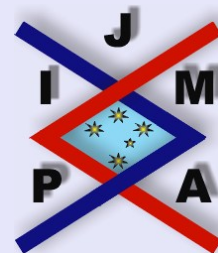
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